Probabilistic Juggling

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Outline

1. Solitary infallible juggler
2. Markov chain on set partitions
3. Superhuman juggler
4. Errant juggler with a partner

Multivariate Juggling Markov Chain (MJMC)

- Data: **Siteswap notation**
  - h: the maximum height the juggler can throw.
  - ℓ: the number of balls he juggles.
  - $k = h - \ell$: the number of empty spaces he has to throw the balls in
  - x – a probability distribution on $\{0, \dots, k\}$.
- State space: $St_{h,k} \subsetneq \{\bullet, \circ\}^h$
- Configurations $B = (b_1, b_2, \dots, b_h) \in St_{h,k}$

Example: $h = 8,\ k = \ell = 4$

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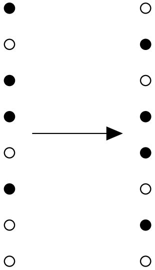

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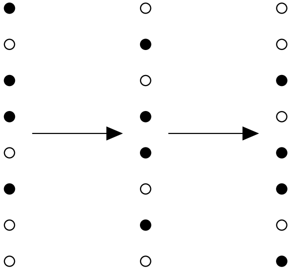

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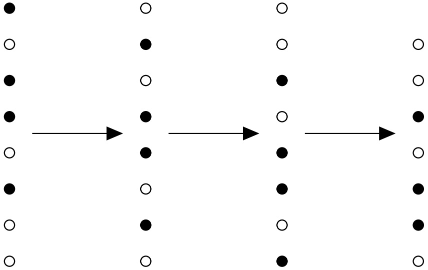

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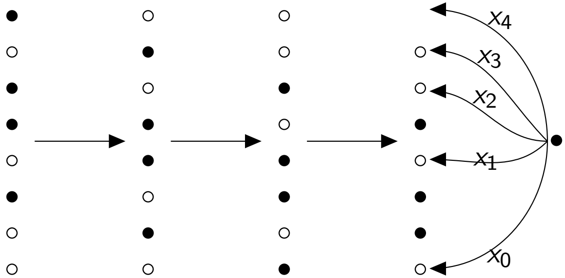


History

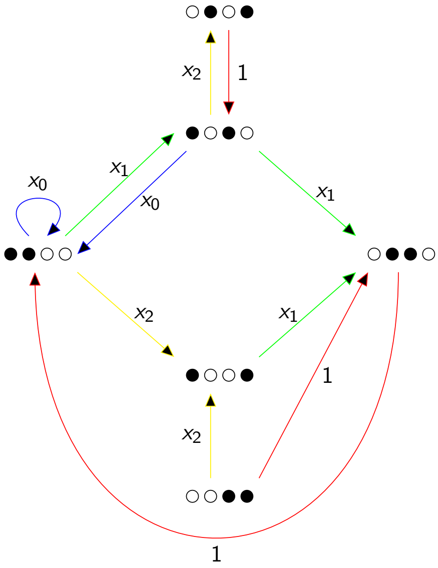
1994 *Juggling sequences* Buhler, Eisenbud, Graham and Wright (AMM)

1996- Connections to other combinatorial structures

2005 *Juggling probabilities* Warrington (AMM) x_i uniform

2012 Juggler's exclusion process on $\mathbb{Z}$, Leskelä and Varpanen

MJMC on $St_{4,2}$



MJMC on $St_{4,2}$

Basis: (gravity $\longleftarrow$)

$$(\bullet\bullet\circ\circ,\ \bullet\circ\bullet\circ,\ \bullet\circ\circ\bullet,\ \circ\bullet\bullet\circ,\ \circ\bullet\circ\bullet,\ \circ\circ\bullet\bullet)$$

Transition matrix:

$$\begin{pmatrix} x_0 & x_1 & x_2 & 0 & 0 & 0 \\ x_0 & 0 & 0 & x_1 & x_2 & 0 \\ 0 & x_0 & 0 & x_1 & 0 & x_2 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Unnormalised Stationary distribution:

$\bullet\bullet\circ\circ$	$\bullet\circ\bullet\circ$	$\bullet\circ\circ\bullet$	$\circ\bullet\bullet\circ$	$\circ\bullet\circ\bullet$	$\circ\circ\bullet\bullet$
1	x_1+x_2	x_2	$(x_1+x_2)^2$	$x_2(x_1+x_2)$	x_2^2

Results about the MJMC

Proposition (A, Bouttier, Corteel, Nunzi, '14)

If $x_i > 0$ for all $i \in \{0, \ldots, k\}$, then the MJMC is irreducible and aperiodic.

Warrington's proof of this was incomplete!

Results about the MJMC

For $B \in St_{h,k}$, define $e_i(B) = \#\{j > i : b_j = \circ\}$ and

$$\Delta(B) = \prod_{\substack{i \in \{1,\dots,h\} \\ b_i = \bullet}} (1 + e_i(B))$$

Theorem (Warrington '05)

If x is uniform, the stationary probability distribution of $B \in St_{h,k}$ is

$$\pi(B) = \frac{\Delta(B)}{\left\{ {h+1 \atop k+1} \right\}}$$

Stirling Numbers of the Second Kind

- $\left\{ {n \atop j} \right\}$ is the number of ways to partition an n-set into j parts.

$$\left\{ {n+1 \atop j} \right\} = j\left\{ {n \atop j} \right\} + \left\{ {n \atop j-1} \right\}; \quad \left\{ {n \atop 0} \right\} = \delta_{n,0}.$$

- For example $\left\{ {4 \atop 2} \right\} = 7$, since $\{1,2,3,4\}$ can be partitioned as

$$123|4,\ 3|124,\ 2|134,\ 1|234,\ 12|34,\ 13|24,\ 23|14$$

- The stationary probability distribution on $St_{4,2}$ becomes

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$\frac{9}{25}$	$\frac{6}{25}$	$\frac{3}{25}$	$\frac{4}{25}$	$\frac{2}{25}$	$\frac{1}{25}$

Stationary distribution of the MJMC

Theorem (A, Bouttier, Corteel, Nunzi, '14)

The stationary distribution π of the MJMC is given by

$$\pi(B) = \frac{1}{Z_{h,k}} \prod_{\substack{i \in \{1,\dots,h\} \\ b_i = \bullet}} \left(x_{E_i(B)} + \cdots + x_k\right),$$

where $E_i(B) = \#\{j < i | b_j = \circ\}$, *and*
$Z_{h,k} \equiv Z_{h,k}(x_0, \dots, x_k)$ *is the normalisation factor.*

Back to example

$\bullet\bullet\circ\circ$	$\bullet\circ\bullet\circ$	$\bullet\circ\circ\bullet$
$(x_0+x_1+x_2)^2$	$(x_0+x_1+x_2)(x_1+x_2)$	$(x_0+x_1+x_2)x_2$
$\circ\bullet\bullet\circ$	$\circ\bullet\circ\bullet$	$\circ\circ\bullet\bullet$
$(x_1+x_2)^2$	$x_2(x_1+x_2)$	x_2^2

Normalisation factor $Z_{h,k}$

- Let $y_m = \sum_{j=m}^{k} x_j$ for $m = 0, \ldots, k$.
- Let $h_n(z_0, \ldots, z_k)$ be the **complete homogeneous symmetric polynomial** of degree n, e.g.,

$$h_2(z_0, z_1, z_2) = z_0^2 + z_0 z_1 + z_0 z_2 + z_1^2 + z_1 z_2 + z_2^2.$$

Lemma (A, Bouttier, Corteel, Nunzi, '14)

The normalisation factor can be written as

$$Z_{h,k} = h_\ell(y_0, y_1, \ldots, y_k).$$

Special cases

Corollary (A, Bouttier, Corteel, Nunzi, '14)

$$Z_{h,k}(1,1,\dots,1,1) = \left\{ {h+1 \atop k+1} \right\},$$

$$Z_{h,k}(q^k, q^{k-1},\dots,q,1) = \left\{ {h+1 \atop k+1} \right\}_q,$$

$$Z_{h,k}(1,q,\dots,q^{k-1},q^k) = q^{k(h-k)} \left\{ {h+1 \atop k+1} \right\}_{1/q},$$

$$Z_{h,k}((1-q),(1-q)q,\dots,(1-q)q^{k-1},q^k) = \binom{h}{k}_q.$$

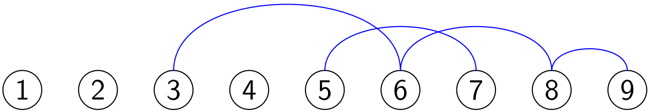
Enriched Multivariate Juggling Markov Chain (EMJMC)

- Data:
  - $H = h + 1$: Cardinality of the set $\{1, \ldots, H\}$
  - $K = k + 1$: Number of parts of the H-set
  - x – a probability distribution on $\{0, \ldots, K - 1\}$.
- State space: $\mathcal{S}(H, K)$
- $\sigma \in \mathcal{S}(H, K)$ written in increasing order of block maxima, e.g.,

$$24|156|37 \in \mathcal{S}(7, 3).$$

Example:

$1|2|4|57|3689 \to 1|3|46|2578|9 \to 2|35|1467|8|9 \to ?$



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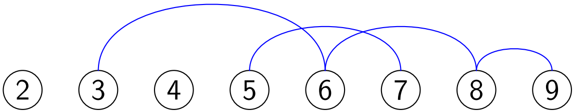

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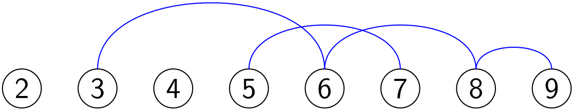

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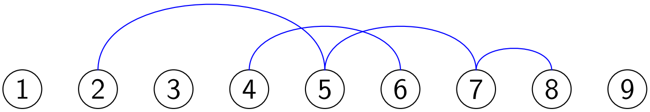

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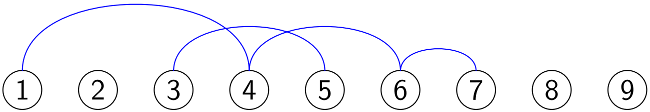

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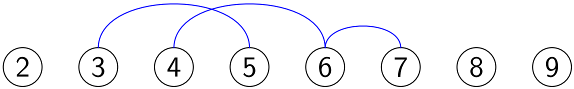

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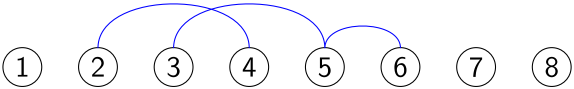

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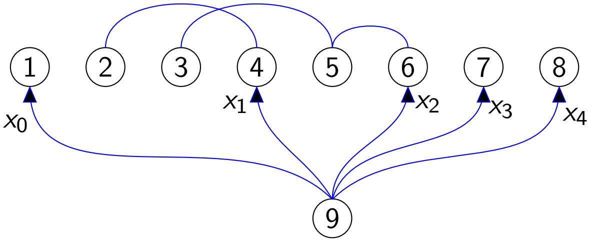


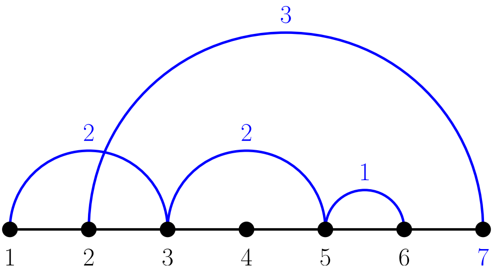
Results about the EMJMC

Proposition (A, Bouttier, Corteel, Nunzi, '14)

If $x_i > 0$ for all $i \in \{0, \dots, k\}$, then the EMJMC is irreducible and aperiodic.

Arches

- (s, t) is an **arch** of $\sigma \in \mathcal{S}(H, K)$ if s and t are nearest neighbours in a block.
- $C_\sigma(s, t)$ be the number of blocks containing at least one element in $\{s, s+1, \ldots, t-1, t\}$.
- Example with $4|1356|27 \in \mathcal{S}(7, 3)$



Results about the EMJMC

Theorem (A, Bouttier, Corteel, Nunzi, '14)

The stationary distribution π of the EMJMC is given by

$$\hat{\pi}(\sigma) = \frac{1}{\hat{Z}_{H,K}} \prod_{(s,t) \text{ arch of } \sigma} x_{K - C_\sigma(s,t)}$$

where $\hat{Z}_{H,K} = Z_{h,k}(x_0, \ldots, x_k)$ is the normalisation factor.

Corollary (Warrington, '05)

When x is uniform, the stationary distribution on $\mathcal{S}(H, K)$ is uniform.

Lumping

- Proof for EMJMC by verifying the master equation
- Proof for MJMC by **lumping/projection** from EMJMC as follows:

Lumping

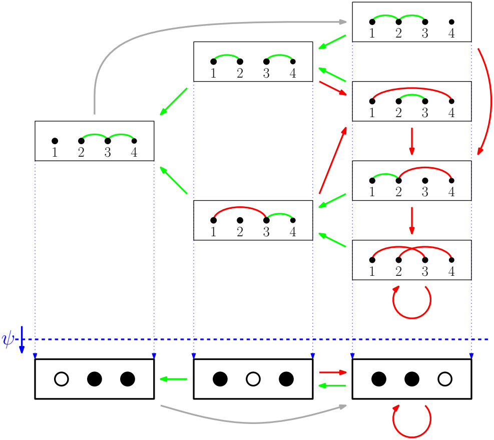
- Proof for EMJMC by verifying the master equation
- Proof for MJMC by **lumping/projection** from EMJMC as follows:
- Define $\psi : \mathcal{S}(H, K) \to St_{h,k}$ so that $b_i = \circ$ iff i is a block maximum in σ.

Full chain on $\mathcal{S}(4,2)$

Red arrows and arches have probability x_0

Green arrows and arches have probability x_1

Gray arrows have probability $\to x_0 + x_1 = 1$



Unbounded Multivariate Juggling Markov Chain (UMJMC)

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Unbounded Multivariate Juggling Markov Chain (UMJMC)

- ℓ balls, which can be thrown to arbitrary heights.
- State space $St^{(\ell)} \subsetneq \{\bullet, \circ\}^{\mathbb{N}}$
- x – a probability distribution on $\mathbb{N}$
- Formally, states are infinite words in $\bullet$ and $\circ$ with exactly ℓ occurences of $\bullet$.
- $T_i(A) \in St^{(\ell)}$ is the word obtained by replacing the $(i+1)$-th occurrence of $\circ$ in A by $\bullet$.
- The transition probability from $A = a_1 a_2 a_3 \cdots$ to B is

$$P_{A,B} = \begin{cases} 1 & \text{if } a_1 = \circ \text{ and } B = a_2 a_3 \cdots, \\ x_i & \text{if } a_1 = \bullet \text{ and } B = T_i(a_2 a_3 \cdots), \\ 0 & \text{otherwise.} \end{cases}$$

Results about UMJMC

Proposition (A, Bouttier, Corteel, Nunzi, '14)

If x_0 and infinitely many x_i's are nonzero, then the UMJMC is irreducible and aperiodic.

Theorem (A, Bouttier, Corteel, Nunzi, '14)

The unique invariant measure (up to constant of proportionality) of the UMJMC is given by

$$w(B) = \prod_{i \in \mathbb{N},\, b_i = \bullet} y_{E_i(B)}$$

where $B = b_1 b_2 b_3 \cdots \in \mathrm{St}^{(\ell)}$, $E_i(B) = \#\{j < i | b_j = \circ\}$ *and* $y_m = \sum_{j=m}^{\infty} x_j$.

More about the invariant measure

Theorem

The invariant measure of the UMJMC is finite if and only if

$$\sum_{i=0}^{\infty} i\, x_i < \infty,$$

in which case its total mass reads

$$Z^{(\ell)} = h_\ell(y_0, y_1, y_2, \ldots).$$

Further, the UMJMC is positive recurrent if and only if the above holds. In that case, there is a unique stationary probability distribution, and the chain started from any initial state converges to it in total variation as time tends to infinity.

Remarks

- Both the solitary and superhuman juggling chains can be interpreted in terms of natural Markov chains on **integer partitions**.
- The superhuman juggling chain can also be naturally generalised to have an infinite number of balls (IMJMC).
- The conditions for the existence of a probability measure and positive recurrence in the IMJMC are **identical** to the UMJMC.

Multivariate Annihilation Juggling Markov Chain (MAJMC)

- Data:
  - h: the maximum height the juggler can throw.
  - z – a probability distribution on $\{0, \ldots, h\}$, with $a \equiv z_0$.
- State space: $St_h = \{\bullet, \circ\}^h$

Example: $h = 8$

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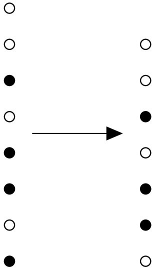

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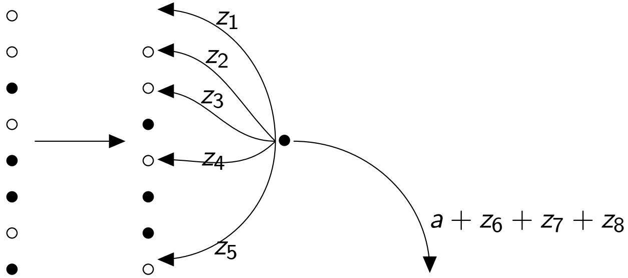
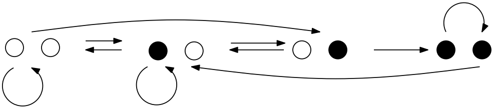


MAJMC for $h = 2$



Transition matrix in the basis (●●, ○●, ●○, ○○)

$$P = \begin{pmatrix} z_1 & 0 & z_2 + a & 0 \\ z_1 & 0 & z_2 + a & 0 \\ 0 & z_1 & z_2 & a \\ 0 & z_1 & z_2 & a \end{pmatrix},$$

Stationary **probability** distribution

$$\left(z_1^2,\ z_1 (z_2 + a),\ (z_1 + z_2)(z_2 + a), a (z_2 + a)\ \right)$$

Results for the MAJMC

Theorem (A, Bouttier, Corteel, Nunzi, '14)

The stationary probability distribution of the annihilation model is

$$\Pi(B) = \prod_{\substack{i=1 \\ b_i=\bullet}}^{h} \left(z_1 + \cdots + z_{e_i(B)+1}\right) \prod_{j=1}^{k} (z_{j+1} + \cdots + z_h + a),$$

where $e_i(B) = \#\{j : i < j \leq h,\ b_j = \circ\}$. *Moreover,*

$$\sum_B \Pi(B) = (z_1 + \cdots + z_h + a)^h = 1$$

Convergence to stationarity

Theorem

For any initial probability distribution η over St_h, the distribution at time h is equal to the stationary distribution, namely

$$\eta P^h = \Pi.$$

In particular, the only eigenvalues of the transition matrix P are 1 (with multiplicity 1) and 0.

Thank you for your attention!