ON A NEW CLASS OF ERGODIC CONTROL PROBLEMS ARISING FROM THE MULTI-CLASS QUEUES

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Basic model

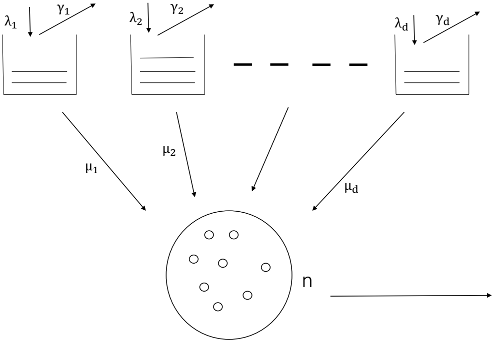

Figure 1: A schematic model of the system

Variant

- Number of server N is finite or $N = o(n)$.
- $N = n$, critical.
- Fluid vs Diffusion scaling

Cost structures: Discounted, long-time average/ergodic, risk-sensitive.

Literature

- Discounted cost: Budhiraja-Ghosh-Liu' 2013(modified $c\mu$ and single server), Atar-Mandelbaum-Reiman' 2004, Atar' 2005 (multi-pool), Dai-Tezcan' 2008.
- Ergodic cost: Budhiraja-Ghosh-Lee' 2011 (finitely many servers), Atar-Giat-Shimkin' 2010 , 2011 (Fluid settings and $c\mu/\theta$ policy)
- Risk-sensitive type cost: Atar-Goswami-Shwartz' 2013, 2014, Atar-Biswas' 2013, Biswas 2013.

Model

- There are $d > 1$ number of customer class.
- Customers of class $i \in \{1, \ldots, d\}$ arrive according to a Poisson process with rate $\lambda_i^n > 0$.
- The service times and patience times of customers are class-dependent and both are assumed to be exponentially distributed, that is, class i customers are served at rate μ_i^n and renege at rate γ_i^n.
- Customers of each class are served in the first-come-first-serve (FCFS) service discipline.

Halfin-Whitt regime

Let $\mathbf{r}_i^n = \frac{\lambda_i^n}{\mu_i^n}$ be the mean offered load of class i customers. The traffic intensity of the n^{th} system is given by $\rho^n = n^{-1} \sum_{i=1}^d \mathbf{r}_i^n$.

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$$\frac{\lambda_i^n}{n} \to \lambda_i > 0\,, \qquad \mu_i^n \to \mu_i > 0\,, \qquad \gamma_i^n \to \gamma_i > 0\,,$$

$$\frac{\lambda_i^n - n\lambda_i}{\sqrt{n}} \to \hat{\lambda}_i\,, \qquad \sqrt{n}(\mu_i^n - \mu_i) \to \hat{\mu}_i\,,$$

$$\frac{\mathbf{r}_i^n}{n} \to \rho_i := \frac{\lambda_i}{\mu_i} < 1\,, \qquad \sum_{i=1}^d \rho_i = 1\,.$$

This implies that

$$\sqrt{n}(1 - \rho^n) \to \hat{\rho} := \sum_{i=1}^d \frac{\rho_i \hat{\mu}_i - \hat{\lambda}_i}{\mu_i} \in \mathbb{R}\,.$$

Let $X_i^n = \{X_i^n(t) : t \geq 0\}$ be the total number of class i customers in the system, $Q_i^n = \{Q_i^n(t) : t \geq 0\}$ the number of class i customers in the queue, and $Z_i^n = \{Z_i^n(t) : t \geq 0\}$ the number of class i customers in service.
Note that Z can be treated as control. We have the following relations

$$X_i^n(t) = X_i^n(0) + A_i^n(\lambda_i^n t) - S_i^n\left(\mu_i^n \int_0^t Z_i^n(s)\,\mathrm{d}s\right) - R_i^n\left(\gamma_i^n \int_0^t Q_i^n(s)\,\mathrm{d}s\right)$$

where A_i^n, S_i^n and R_i^n are all mutually independent rate-1 Poisson processes, for $i = 1, \ldots, d$, and

$$Q_i^n(t) \geq 0\,, \quad Z_i^n(t) \geq 0\,, \quad \text{and} \quad e \cdot Z^n(t) = (e \cdot X^n(t)) \wedge n.$$

Here $e = (1, \ldots, 1)$.

Scaled version

Define

$$\hat{X}_i^n(t) := \frac{1}{\sqrt{n}}(X_i^n(t) - \rho_i nt), \hat{Q}_i^n(t) := \frac{1}{\sqrt{n}} Q_i^n(t)\,,$$
$$\hat{Z}_i^n(t) := \frac{1}{\sqrt{n}}(Z_i^n(t) - \rho_i nt).$$

Then

$$\hat{X}_i^n(t) = \hat{X}_i^n(0) + \ell_i^n t - \mu_i^n \int_0^t \hat{Z}_i^n(s)\,\mathrm{d}s - \gamma_i^n \int_0^t \hat{Q}_i^n(s)\,\mathrm{d}s + \hat{M}_{A,i}^n(t) - \hat{M}_{S,i}^n(t) - \hat{M}_{R,i}^n(t)\,,$$

where $\ell^n = (\ell_1^n, \dots, \ell_d^n)^{\mathsf{T}}$ is defined as $\ell_i^n := \frac{1}{\sqrt{n}}(\lambda_i^n - \mu_i^n \rho_i n)$, and $\hat{M}_{A,i}^n, \hat{M}_{S,i}^n, \hat{M}_{R,i}^n$ are square integrable martingales w.r.t. some suitable filtration filtration.

Cost

Let $r\colon \mathbb{R}^d_+ \to \mathbb{R}_+$ be a given locally Lipschitz function satisfying

$$c_1|x|^m \;\le\; r(x) \;\le\; c_2(1+|x|^m) \qquad \text{for some } m \ge 1\,,$$

for some positive constants c_i, $i = 1, 2$.

Given the initial state $X^n(0)$ and a work-conserving scheduling policy Z^n, we define the diffusion-scaled cost function as

$$J(\hat{X}^n(0), \hat{Z}^n) \;:=\; \limsup_{T\to\infty} \frac{1}{T}\,\mathbb{E}\biggl[\int_0^T r(\hat{Q}^n(s))\,\mathrm{d}s\biggr].$$

Then, the associated cost minimization problem becomes

$$\hat{V}^n(\hat{X}^n(0)) \;:=\; \inf_{Z^n\in\mathfrak{U}^n} J(\hat{X}^n(0), \hat{Z}^n)\,.$$

Defining $r(x,u) = r((e\cdot x)^+ u)$ we can rewrite the control problem as

$$\hat{V}^n(\hat{X}^n(0)) \;=\; \inf\; \tilde{J}(\hat{X}^n(0), \hat{U}^n)\,,$$

where

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We are interested to find the asymptotic of $\hat{V}^n$ as $n\to\infty$ and identify a sequence of policy that is asymptotically optimal.

Standard method

One of the standard ways to do this is to study the analogous controlled diffusion problem and then use the HJB to construct a sequence of optimal control.

The limiting controlled diffusion process

A formal derivation gives

$$\mathrm{d}X_t \;=\; b(X_t, U_t)\,\mathrm{d}t + \Sigma\,\mathrm{d}W_t\,,$$

with initial condition $X_0 \;=\; x$. The drift $b(x,u)\colon \mathbb{R}^d \times \mathcal{S} \to \mathbb{R}^d$ takes the form

$$b(x,u) \;=\; \ell - R\bigl(x - (e\cdot x)^+ u\bigr) - (e\cdot x)^+ \Gamma u\,,$$

with

$$\ell \;:=\; (\ell_1,\dots,\ell_d)^{\mathsf{T}}, \quad R \;:=\; \operatorname{diag}[\mu_i]\,, \quad \text{and} \quad \Gamma \;:=\; \operatorname{diag}[\gamma_i]\,.$$

The control U_t lives $\mathcal{S} = \{u \in \mathbb{R}^d_+, \sum u_i = 1\}$, $W(t)$ is a d-dimensional standard Wiener process independent of the initial condition $X_0 = x$, and the covariance matrix is given by

$$\Sigma\Sigma^{\mathsf{T}} \;=\; \operatorname{diag}(2\lambda_1,\dots,2\lambda_d)\,.$$

Analogous cost

Define $\tilde{r} \colon \mathbb{R}^d_+ \times \mathbb{R}^d_+ \to \mathbb{R}_+$ by

$$\tilde{r}(x, u) \;:=\; r((e \cdot x)^+ u)\,,$$

where r is the same function as earlier. Then we define

$$J(x, U) \;:=\; \limsup_{T \to \infty} \frac{1}{T}\, \mathbb{E}^U_x \left[\int_0^T \tilde{r}(X_t, U_t)\, \mathrm{d}t \right], \quad U \in \mathfrak{U}\,.$$

We obtain the ergodic control problem

$$\varrho_*(x) \;=\; \inf_{U \in \mathfrak{U}} J(x, U)\,.$$

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What's new??????

A new class of ergodic control problem

We consider controlled diffusion

$$\mathrm{d}X_t = b(X_t, U_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t\,,$$

that satisfies standard assumptions like locally Lipschitz coefficients, non-degeneracy etc. Let where $a := \sigma\sigma^{\mathsf{T}}$. We denote the generator by

$$L^u f(x) := \frac{1}{2} a^{ij}(x)\,\partial_{ij} f(x) + b^i(x,u)\,\partial_i f(x)\,, \quad u \in \mathbb{U}\,.$$

Different type of assumption

Assumption

For some open set $\mathcal{K} \subset \mathbb{R}^d$, the following hold:

(i) *The running cost $\tilde{r}$ is inf-compact on $\mathcal{K}$.*

(ii) *There exist inf-compact functions $\mathcal{V} \in \mathcal{C}^2(\mathbb{R}^d)$ and $h \colon \mathbb{R}^d \times \mathbb{U} \to \mathbb{R}$, such that*

$$L^u\mathcal{V}(x) \le 1 - h(x,u) \qquad \forall\,(x,u) \in \mathcal{K}^c \times \mathbb{U}\,,$$

$$L^u\mathcal{V}(x) \le 1 + \tilde{r}(x,u) \qquad \forall\,(x,u) \in \mathcal{K} \times \mathbb{U}\,.$$

Without loss of generality we assume that $\mathcal{V}$ and h are nonnegative.

For admissible control U, define

$$\varrho_U(x) := \limsup_{T\to\infty} \frac{1}{T} \mathbb{E}_x^U\left[\int_0^T \tilde{r}(X_s, U_s)\,\mathrm{d}s\right],$$

Assumption

There exists $U \in \mathfrak{U}$ such that $\varrho_U(x) < \infty$ for some $x \in \mathbb{R}^d$.

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There exists $U \in \mathfrak{U}$ such that $\varrho_U(x) < \infty$ for some $x \in \mathbb{R}^d$.

Definition

Let $\tilde{h}\colon \mathbb{R}^d \times \mathbb{U} \to \mathbb{R}$ be some continuous inf-compact function, locally Lipschitz in its first argument, satisfying

$$r(x,u) \;\le\; \tilde{h}(x,u) \le \frac{k_0}{2}\bigl(1 + h(x,u)\,\mathbb{I}_{\mathcal{K}^c}(x) + r(x,u)\,\mathbb{I}_{\mathcal{K}}(x)\bigr),$$

for some positive constant $k_0 \ge 2$.

Example

Definition

A square matrix R is said to be an M-matrix if it can be written as $R = sI - N$ for some $s > 0$ and nonnegative matrix N with property that $\rho(N) \leq s$, where $\rho(N)$ denotes the spectral radius of N.

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A square matrix R is said to be an M-matrix if it can be written as $R = sI - N$ for some $s > 0$ and nonnegative matrix N with property that $\rho(N) \leq s$, where $\rho(N)$ denotes the spectral radius of N.

Let $\Gamma = [\gamma^{ij}]$ be a given diagonal matrix with positive entries. Let $\ell \in \mathbb{R}^d$ and R be a non-singular M-matrix. Define

$$b(x,u) \;:=\; \ell - R(x - (e\cdot x)^{+}u) - (e\cdot x)^{+}\Gamma u\,,$$

with $u \in \mathcal{S} := \{u \in \mathbb{R}^d_+ :\; e\cdot u = 1\}$. Assume that $e^{\mathsf{T}}R \;\geq\; 0^{\mathsf{T}}$. We consider the following controlled diffusion in $\mathbb{R}^d$:

$$\mathrm{d}X_t \;=\; b(X_t, U_t)\,\mathrm{d}t + \Sigma\,\mathrm{d}W_t\,,$$

where Σ is a constant matrix such that $\Sigma\Sigma^{\mathsf{T}}$ is invertible.

Let $\tilde{r} \colon \mathbb{R}^d \times \mathcal{S} \to [0, \infty)$ be locally Lipschitz with polynomial growth and

$$c_1[(e \cdot x)^+]^m \;\le\; \tilde{r}(x, u) \;\le\; c_2\bigl(1 + [(e \cdot x)^+]^m\bigr)\,,$$

for some $m \ge 1$ and positive constants c_1 and c_2.

Proposition

Let b and $\tilde{r}$ be given as mentioned above. Then our assumptions are satisfied.

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Proposition

Let b and $\tilde{r}$ be given as mentioned above. Then our assumptions are satisfied.

In Dieker-Gao' 2013 it is shown that every constant control is a stable control.

Result

For $\varepsilon > 0$ let

$$\tilde{r}_\varepsilon(x,u) \;:=\; \tilde{r}(x,u) + \varepsilon \tilde{h}(x,u)\,.$$

Then by (Arapostathis-Borkar-Ghosh: *Ergodic control of diffusion processes*, 2012)

Theorem

There exists a unique function $V^\varepsilon \in \mathcal{C}^2(\mathbb{R}^d)$ with $V^\varepsilon(0) = 0$, which is bounded below in $\mathbb{R}^d$, and solves the HJB

$$\min_{u\in\mathbb{U}} \bigl[L^u V^\varepsilon(x) + r_\varepsilon(x,u)\bigr] \;=\; \varrho_\varepsilon\,,$$

where ϱ_ε is the optimal ergodic cost w.r.t. $\tilde{r}_\varepsilon$.

Our result

Theorem

Let V^{ε}, ϱ_{ε}, and v_{ε}, for $\varepsilon > 0$, be as above where v_{ε} is a measurable selector. Then

(a) *The function V^{ε} converges to some $V_* \in \mathcal{C}^2(\mathbb{R}^d)$, uniformly on compact sets, and $\varrho_{\varepsilon} \to \varrho_*$, as $\varepsilon \searrow 0$, and V_* satisfies*

$$\min_{u \in \mathbb{U}} \bigl[L^u V_*(x) + r(x,u)\bigr] \;=\; \varrho_* \,.$$

Also, any limit point v_* (in the topology of Markov controls) as $\varepsilon \searrow 0$ of the set $\{v_{\varepsilon}\}$ satisfies

$$L^{v_*} V_*(x) + r(x, v_*(x)) \;=\; \varrho_* \quad \text{a.e. in } \mathbb{R}^d \,.$$

Theorem (Theorem Cont.)

(b) *A stationary Markov control v is optimal for the ergodic control problem relative to $\tilde{r}$ if and only if it satisfies*

$$H(x, \nabla V_*(x)) \;=\; b(x, v(x)) \cdot \nabla V_*(x) + r(x, v(x)) \quad a.e.\ in\ \mathbb{R}^d\,, \tag{1}$$

where

$$H(x, p) \;:=\; \min_{u \in \mathbb{U}} \left[b(x,u) \cdot p + r(x,u) \right].$$

Moreover, for an optimal $v \in \mathfrak{U}_{\mathrm{SM}}$, we have

$$\lim_{T \to \infty} \frac{1}{T}\, \mathbb{E}_x^v \left[\int_0^T r(X_s, v(X_s))\, \mathrm{d}s \right] \;=\; \varrho_* \qquad \forall x \in \mathbb{R}^d\,.$$

We can also give a stochastic representation for V_* and under some convexity assumption on cost we show that the convergence of optimal control can be obtained in pointwise sence.

Spatial truncation technique

Let $v_0 \in \mathfrak{U}_{\mathrm{SSM}}$ be any control such that $\int_{\mathbb{R}^d} \tilde{r} \mathrm{d}\pi_{v_0} < \infty$ where π_{v_0} denotes the ergodic occupation measure corresponding to v_0.

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We fix the control v_0 on the complement of the ball $\bar{B}_l$ and leave the parameter u free inside. In other words for each $l \in \mathbb{N}$ we define

$$b_l(x,u) \;:=\; \begin{cases} b(x,u) & \text{if } \;(x,u) \in \bar{B}_l \times \mathbb{U}\,, \\ b(x, v_0(x)) & \text{otherwise}, \end{cases}$$

$$\tilde{r}_l(x,u) \;:=\; \begin{cases} \tilde{r}(x,u) & \text{if } \;(x,u) \in \bar{B}_l \times \mathbb{U}\,, \\ \tilde{r}(x, v_0(x)) & \text{otherwise}. \end{cases}$$

We consider the family of controlled diffusions, parameterized by $l \in \mathbb{N}$, given by

$$\mathrm{d}X_t \;=\; b_l(X_t, U_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t\,,$$

with associated running costs $\tilde{r}_l(x, u)$.

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$$\mathrm{d}X_t \;=\; b_l(X_t, U_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t\,,$$

with associated running costs $\tilde{r}_l(x, u)$. We note that for fixed l the controlled dynamics are uniformly stable.

Theorem

Then for each $l \in \mathbb{N}$ there exists a solution V^l in $W^{2,p}_{loc}(\mathbb{R}^d)$, for any $p > d$, with $V^l(0) = 0$, of the HJB equation

$$\min_{u \in \mathbb{U}} \left[L_l^u V^l(x) + r_l(x,u) \right] \;=\; \varrho_l\,,$$

where L_l corresponds to the diffusion with truncated drift. Moreover, the following hold:

(i) *ϱ_l is non-increasing in l;*

(ii) *.... some addition estimate of the growth of V^l uniformly in l.*

Theorem

Then for each $l \in \mathbb{N}$ *there exists a solution* V^l *in* $W^{2,p}_{loc}(\mathbb{R}^d)$*, for any* $p > d$*, with* $V^l(0) = 0$*, of the HJB equation*

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Theorem

$V^l \to V_*$ *and* $\varrho_l \to \varrho_*$ *as* $l \to \infty$*.*

Asymptotic optimality

Proving asymptotic lower bound is relatively simpler. To establish the upper bound we need to construct a scheduling policy that achieves ϱ_* asymptotically. We use special truncation technique for this construction.

Conclusion and comments

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- We obtain a preemptive scheduling policy that is asymptotically optimal. How about a non-preemptive one?
- Multi-pool server case is also a interesting question.
- How about G/G/N+G????

Thank you