Marginalization for rare event simulation in switching diffusions

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Overview

- Critical but rare events occur in
  1. insurance
  2. nuclear engineering
  3. air traffic control
  4. communication networks etc.
- Goal
  1. Estimate probability of the rare event
  2. Estimate conditional probability of the event given certain thresholds.
- Formulation
  1. Model the dynamics as a stochastic process.
  2. Model the rare event as a hitting event of the process to a particular critical subset of the state space before a specified final time.
- Evaluation
  1. Simulate the trajectories.
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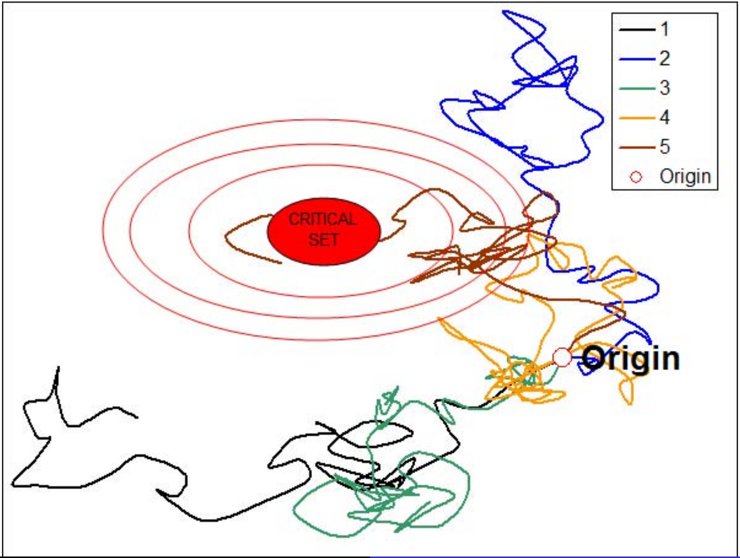
Graphical Representation



Model

Let $X(t) \in \mathbb{R}^d$ and $\theta(t) \in \mathbb{M} = \{1, \ldots, m\}$ where $(X(t), \theta(t))_{t \geq 0}$ is strongly Markov and given by

$$dX(t) = b(X(t))1_{\theta(t)}dt + dW(t)$$
$$P[\theta(t+h) = j \mid X(t) = x, \theta(t) = i] = \delta_{ij} + \lambda_{ij}(x)h + o(h)$$
$$(X(0), \theta(0)) \sim \eta$$

where W is a Wiener process on $\mathbb{R}^d$, $b : \mathbb{R}^d \to \mathbb{R}^{d \times m}$ a matrix valued measurable function, $\Lambda(x) := (\lambda_{ij}(x))_{m \times m}$ the rate matrix, 1_i the ith unit vector in $\mathbb{R}^m$ and η a probability distribution defined on $\mathbb{R}^d \times \mathbb{M}$.

Simulation

Monte-Carlo Simulation: Most of the realizations of the underlying process never reach the critical set.

Importance Sampling: Trajectories are simulated under an equivalent but different probability measure.

Splitting Method: The state space is divided into a sequence of sub-levels, the particle needs to pass before it reaches the rare target.

We exploit the special structure of hybrid processes to improve a standard splitting method by a marginalization technique.

Wonham Filter

Wonham Filter: $\pi^i(t) := P(\theta(t) = i \mid \mathcal{F}^X(t)),\ i = 1, \ldots, m.$

THEOREM

The diffusion process $(X(t), \pi(t))$ is the solution of

$$
\begin{aligned}
dX(t) &= b(X(t))\pi(t)dt + d\tilde{W}(t), \quad \tilde{W} \text{ is a Weiner process on } \mathbb{R}^d \\
d\pi(t) &= \Lambda(X(t))^*\pi(t)dt + (D(\pi(t)) - \pi(t)\pi^*(t))b(X(t))^* d\tilde{W}(t) \\
\pi^i(0) &= \frac{d\eta(\cdot, i)}{d\sum_i \eta(\cdot, i)}(X(0)) \\
X(0) &\sim \sum_i \eta(\cdot, i).
\end{aligned}
$$

Advantage: There is no need to sample the finite set of modes, which can be a tricky issue, especially if some modes have very small probability. Indeed, there are degraded non-nominal modes under which hitting the critical region is very easy, but switching to these modes has a small probability.

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Rewriting the probability

- Critical set $:= B = B_n \subset B_{n-1} \subset \cdots \subset B_0 = \mathbb{R}^d$.
- $T_k := \inf\{t \geq 0 : X(t) \in B_k\}$ non decreasing $\mathcal{F}^X$-stopping times.
- $S_k := T_k \wedge T$
- $\mathcal{X}_k : [S_{k-1}, S_k] \to \mathbb{R}^d$, by $\mathcal{X}_k(t) = X(t)$.
- $\bar{\theta}_k := \theta(S_k)$ & $\bar{\pi}_k := \pi(S_k)$.
- Consider g_p such that $\{T_k \leq T\} \equiv \{\prod_{p=0}^k g_p(\mathcal{X}_p) = 1\}$.
- $\langle \gamma_k, \varphi \rangle := E[\varphi(\mathcal{X}_k, \bar{\theta}_k) \prod_{p=0}^k g_p(\mathcal{X}_p)], \quad \varphi \in L^\infty(\mathcal{E} \times \mathbb{M})$.
- $\langle \Gamma_k, F \rangle := E[F(\mathcal{X}_k, \bar{\pi}_k) \prod_{p=0}^k g_p(\mathcal{X}_p)], \quad F \in L^\infty(\mathcal{E} \times \mathcal{P}(\mathbb{M}))$.
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Variance Reduction

Let $\mathcal{U}_k^j f(e,\theta) := E[f(\mathcal{X}_j) \mid \mathcal{X}_k = e, \bar{\theta}_k = \theta]$ and
$\bar{\mathcal{U}}_k^j f(e,\pi) := E[f(\mathcal{X}_j) \mid \mathcal{X}_k = e, \bar{\pi}_k = \pi]$

Lemma

Let $A \in \sigma(\mathcal{X}_0, \ldots, \mathcal{X}_{k-1}, \mathcal{X}_k)$. We have
(i) $E[\mathcal{U}_k^j f(\mathcal{X}_k, \bar{\theta}_k) \mid \mathcal{X}_k, \mathcal{X}_{k-1}, \ldots, \mathcal{X}_0] = E[\bar{\mathcal{U}}_k^j f(\mathcal{X}_k, \bar{\pi}_k) \mid \mathcal{X}_k, \mathcal{X}_{k-1}, \ldots, \mathcal{X}_0]$
(ii) $Var[\mathcal{U}_k^j f(\mathcal{X}_k, \bar{\theta}_k) \mid I_A] \geq Var[\bar{\mathcal{U}}_k^j f(\mathcal{X}_k, \bar{\pi}_k) \mid I_A]$.

$$\begin{aligned} Var[\mathcal{U}_k^j f(\mathcal{X}_k, \bar{\theta}_k) \mid I_A] &= E[Var[\mathcal{U}_k^j f(\mathcal{X}_k, \bar{\theta}_k) \mid \mathcal{X}_k, \mathcal{X}_{k-1}, \ldots, \mathcal{X}_0, I_A] \mid I_A] \\ &\quad + Var[E[\mathcal{U}_k^j f(\mathcal{X}_k, \bar{\theta}_k) \mid \mathcal{X}_k, \mathcal{X}_{k-1}, \ldots, \mathcal{X}_0, I_A] \mid I_A] \\ Var[\bar{\mathcal{U}}_k^j f(\mathcal{X}_k, \bar{\pi}_k) \mid I_A] &= E[Var[\bar{\mathcal{U}}_k^j f(\mathcal{X}_k, \bar{\pi}_k) \mid \mathcal{X}_k, \mathcal{X}_{k-1}, \ldots, \mathcal{X}_0, I_A] \mid I_A] \\ &\quad + Var[E[\bar{\mathcal{U}}_k^j f(\mathcal{X}_k, \bar{\pi}_k) \mid \mathcal{X}_k, \mathcal{X}_{k-1}, \ldots, \mathcal{X}_0, I_A] \mid I_A] \end{aligned}$$

Variance Reduction

- **N** Sample size
- **Γ_k^N** Generate iid samples by solving SDE and compute empirical distributions.
- **γ_k^N** Generate iid samples by simulating hybrid process to compute.
- **Sampling** Fix any splitting algorithm / particle approximation scheme

THEOREM

$\mathcal{N}(a,b)$ = *Normal distribution: mean a and variance b. Then*

- $\lim_{N\to\infty} \sqrt{N}\left(\frac{\langle \Gamma_n^N, 1\rangle}{\langle \Gamma_n, 1\rangle} - 1\right) \sim \mathcal{N}(0, \bar{V}_n)$.
- $\lim_{N\to\infty} \sqrt{N}\left(\frac{\langle \gamma_n^N, 1\rangle}{\langle \gamma_n, 1\rangle} - 1\right) \sim \mathcal{N}(0, V_n)$.

Similar to: F. Cérou, P. Del Moral, F. Le Gland and P. Lezaud (2006)

THEOREM

$$\bar{V}_n \leq V_n.$$

Proof

The expressions of $\bar{V}_n$ involves $Var[P[T_n \leq T \mid \mathcal{X}_k, \bar{\pi}_k] \mid T_k \leq T]$ for $k = 0, 1, \ldots, n$.
And the expressions of V_n involves $Var[P[T_n \leq T \mid \mathcal{X}_k, \bar{\theta}_k] \mid T_k \leq T]$ for $k = 0, 1, \ldots, n$.
We rewrite
$P[T_n \leq T \mid \mathcal{X}_k, \bar{\pi}_k] = E[I_{[0,T]} \circ \alpha(\mathcal{X}_n) \mid \mathcal{X}_k, \bar{\pi}_k] = \bar{\mathcal{U}}_k^n f(\mathcal{X}_k, \bar{\pi}_k)$,
where $f = I_{[0,T]} \circ \alpha$, a bounded measurable map.
Similarly, we rewrite $P[T_n \leq T \mid \mathcal{X}_k, \bar{\theta}_k] = \mathcal{U}_k^n f(\mathcal{X}_k, \bar{\theta}_k)$
Thus by using the Lemma we get the result. □

Conclusion

There are some schemes leading a less asymptotic variance. But the use of Wonham filter does not restrict one from using those schemes. In fact the use of Wonham filter enables further reduction in asymptotic variance for any choice of particle approximation scheme.

Thank you!