Generation of semi-discrete integrable systems including defect models

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- Discrete Integrable Models: Rational & Trigonometric class
- Unifying Origin: Ancestor model
- Underlying unified algebra
- Alternative appoach- Future direction
- Integrable models with defect: Defect Toda chain

Examples of Discrete Integrable Models
(satisfying Yang-Baxter Equation (YBE))

I. Rational class
1. *Isotropic XXX spin chain* (quantum)
2. *Toda chain*
3. Several Exact-*Lattice versions of NLS model*

$$i\psi(x,t)_t - \psi(x,t)_{xx} + 2(\psi^\dagger(x.t)\psi(x,t))\psi(x,t) = 0,$$

4. *t-j Model* (two component fermionic model) (quantum)
5. Hubbard model (Electron model with spin) (quantum)

etc.

II. Trigonometric class (q-deformed)

1. *Anisotropic XXZ spin chain* (quantum)
2. *Exact lattice versions* of
2.i). *Sine-Gordon model*

$$u(x,t)_{tt} - u(x,t)_{xx} = m^2 \sin u(x,t),$$

2.ii) *Liouville model*

$$u(x,t)_{tt} - u(x,t)_{xx} = e^{2\alpha u(x,t)},$$

2.iii) *Derivative NLS model* (DNLS)

$$i\psi(x,t)_t - \psi(x,t)_{xx} + 4i(\psi^\dagger(x.t)\psi(x,t))\psi_x(x,t) = 0,$$

3. Massive Thirring model (bosonic)
(2-component field $\psi = (\psi^1, \psi^2)$

$$\mathsf{L} = \int dx \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - \frac{1}{2} g j^\mu j_\mu \quad j^\mu = \bar{\psi}\gamma^\mu \psi,$$

4. *Relativistic Toda chain*
5. Ablowitz-Ladik model

Beautiful common properties of Integrability

1) Hamiltonian and higher conserved operators:

$$H, \; C_n \; n = 1, 2, \ldots, \; \{C_n, C_m\} = 0$$

Secret behind Liouville Integrability

2) Models linked to a representative Lax operator: $U^j(\lambda), \; j = [1, N]$ (2×2-matrix), at each lattice site , λ- spectral parameter.

Defininig a global operator

$$, \; T_N(\lambda) = \prod_{j=1}^{N} U^j(\lambda),$$

gives Lax Equation

$$T_{j+1}(\lambda) = U^j(\lambda) T_j(\lambda)$$

Then for

$$\tau(\lambda) = (tr\ T_N(\lambda)),\ \ln \tau(\lambda) = \sum_n C_n \lambda^{-n}$$

3) U^j must satisfy classical **Yang-Baxter equation**(YBE) (by defining a PB: { , } →)

$$\{U^j(\lambda), \otimes U^k(\mu)\} = \delta_{jk}[r(\lambda - \mu), U^j(\lambda), \otimes U^k(\mu)]$$

while *Commuting at different sites* $j \neq k$, (Ultralocality!) 1) $r(\lambda)$-matrix (4×4 matrix (rational or Trigonometric functions of λ,)) fixed

$$r(\lambda) = \begin{pmatrix} a(\lambda) & & & \\ & & b(\lambda) & \\ & b(\lambda) & & \\ & & & a(\lambda) \end{pmatrix},$$

where $a(\lambda) = b(\lambda) = \frac{1}{\lambda}$- for r_{rat},
$a(\lambda) = \cot(\lambda), b(\lambda) = \frac{1}{\sin(\lambda})$ - for r_{trig}
Then *same YBE is satisfied also by the global operator*

(Hopf algebra property):

$$\{T_N(\lambda), \otimes T_N(\mu)\} = [r(\lambda - \mu), T_N(\lambda), \otimes T_N(\mu)]$$

4) Hence (since RHS is a commutator!)

$$\{tr(T_N(\lambda)), \otimes tr(T_N(\mu))\} = 0$$

and since

$$\ln(tr\ T_N(\lambda)) = \sum_n C_n \lambda^{-n}$$

generates Conserved Charges, we get Liouville Integrability condition:

$$\{C_n, C_m\} = 0$$

Therefore, for given Lax operator

- $U^j(\lambda)$

$\longrightarrow\ T_N(\lambda) \longrightarrow\ \tau(\lambda) = (tr\ T_N(\lambda))\ \longrightarrow\ \ln \tau(\lambda) = \sum_n C_n \lambda^{-n},\ n = 1, 2, \ldots$

All conserved charges C_n (including Hamiltonian) are known!

- Integrable Hierarchy of semi-discrete equations (for all models):

$$q^j_{t_n} = \{q^j \ , H\}, \ H = C_n, \ n = 1, 2, \ldots$$

Note: higher time t_n with higher scaling dimention n !

Each integrable model can be given by its own Lax operator $U^j(\lambda)$ like $U^j_{Toda}(\lambda)$, $U^j_{lNLS}(\lambda)$, $U^j_{XXX}(\lambda)$, $U^j_{AL}(\lambda)$ etc.

(Apparantly all *Different* !).

- Different Lax operators satisfy the same YBE

$$\{U^j(\lambda), \otimes U^k(\mu)\} = \delta_{jk}[r(\lambda - \mu), U^j(\lambda), \otimes U^k(\mu)]$$

with same r-matrix of *Rational* $r_{rat}(\lambda)$ or *Trigonometric* $r_{trig}(\lambda)$ type.

Examples of representative Lax operators (associated with $r_{rat}(\lambda)$)

Rational Class:

I.

$$U^j_{XXX}(\lambda) = \begin{pmatrix} (\lambda + \sigma^3_j) & \sigma^-_j \\ \sigma^+_j & (\lambda - \sigma^3_j) \end{pmatrix}, \quad (1)$$

Giving the Hamiltonian :

$$H = \sum_{n}^{N} (\sigma_n^1 \sigma_{n+1}^1 + \sigma_n^2 \sigma_{n+1}^2 + \sigma_n^3 \sigma_{n+1}^3), \qquad (2)$$

II.

$$U^j_{Toda-chain}(\lambda) = \begin{pmatrix} \lambda + p_j & -e^{q_j} \\ e^{-q_j} & 0 \end{pmatrix},$$

giving the Model Hamiltonian:

$$H = \sum_i \left(\frac{1}{2} p_i^2 + e^{(q_i - q_{i+1})} \right)$$

Using PB $\{q_i, p_j\} = \delta_{ij}$ derive the Toda chain equation

III. Lattice NLS model (not AL !)

$$U^j_{lNLS}(\lambda) = \begin{pmatrix} (\lambda + s + N_j) & q_j^* N_j^{1/2} \\ N_j^{1/2} q_j & (\lambda - (s + N_j)) \end{pmatrix}, \quad N = (s - q_j^* q_j) \qquad (3)$$

IV. *Simple lattice NLS model* (SlNLS) (Kundu-Ragnisco) Lax operator

$$U_n(\lambda) = \begin{pmatrix} -\frac{i}{\lambda} + N(n) & -i\psi^*(n) \\ i\psi(n) & 1 \end{pmatrix}, \quad N(n) = 1 + \psi^*(n)\psi(n)$$

$$H = \sum_k (\psi^*(k+1)\psi(k-1) - (N(k) + N(k+1))\phi(k+1)\psi(k) + 3N(k)^3)$$

Using PB $\{\psi(k), \psi^*(j)\} = \delta_{kj}$ derive the lattice NLS equation

Trigonometric Class ($q = ei\alpha$ models- with $r_{trig}(\lambda)$):
I. XXZ-spin chain (quantum)

$$U^j_{XXZ}(\xi) = i[\sin(\lambda + \frac{\alpha}{2}\sigma^3\sigma^3_j) + \sin\alpha(\sigma^+\sigma^-_j + \sigma^-\sigma^+_j)].$$

generating the Hamiltonian

$$H = J\sum_n^N (\sigma^1_n\sigma^1_{n+1} + \sigma^2_n\sigma^2_{n+1} + \cos\alpha\sigma^3_n\sigma^3_{n+1}),$$

II. Lattice sine-Gordon model

$$U^j_{lSG}(\lambda) == \begin{pmatrix} g(u_j)\ e^{ip_j\Delta} & m\Delta\sin(\lambda-\alpha u_n) \\ m\Delta\sin(\lambda+\alpha u_n) & e^{-ip_n\Delta}\ g(u_n)) \end{pmatrix}, \tag{4}$$

where $g(u_j)$- trigonometric function of u_j

III. Lattice Liouville model (LLM)

$$U^j_{LLM}(\lambda) = \begin{pmatrix} e^{p_j\Delta}\ f(u_j) & \Delta e^{\alpha(\lambda+u_j)} \\ \Delta e^{\alpha(-\lambda+u_j)} & f(u_j)\ e^{-p_j\Delta} \end{pmatrix} \tag{5}$$

IV. Lattice DNLS model (lDNLS) [Kundu-Basumallick]

$$U^{(ldnls)}_n(\xi) == \begin{pmatrix} \frac{1}{\xi}q^{-N_n} - \frac{i\xi\Delta}{4}\ q^{N_n+1}) & A^\dagger_n \\ A_n & \frac{1}{\xi}q^{N_n} + \frac{i\xi\Delta}{4}\ q^{-(N_n+1)} \end{pmatrix} \tag{6}$$

where $A_n, A^\dagger_n$ are q-bosons

(bosonized as $A_n = \psi_n\sqrt{\frac{\sin(\alpha N_n)}{N_n\sin\alpha}},\ N_n = \psi^\dagger_n\psi_n$

IV. Lattice massive Thirring model (lMTM) [Kundu-Basumallick]

$$U_n^{(lMTM)}(\xi) = U_n^{(ldnls)}(\xi, A^{(1)})U_n^{(ldnls)}(\xi, A^{(2)})$$

$A^{(1)}, A^{(2)}$ two-component MTM field.
V. Relativistic Toda chain ($q = e^{i\alpha}$-deformed Toda)

$$U^j_{rtoda}(\lambda) = \begin{pmatrix} q^{(p-\lambda)} - q^{-(p_j-\lambda)} & \alpha e^{q_j} \\ -\alpha e^{-q_j} & 0 \end{pmatrix}, \tag{7}$$

yielding Hamiltonian

$$H = \sum_i \left(\cosh 2\alpha p_i + \alpha^2 \cosh \alpha (p_i + p_{i+1}) e^{(q_i - q_{i+1})}\right), \tag{8}$$

using canonical PB $\{q_i, p_j\} = \delta_{ij}$ the Equation for RTC is derived. IV. Ablowitz-Ladik model:

$$U^j_{AL}(\xi) = \begin{pmatrix} \xi^{-1} & \tilde{b}^\dagger_{qj} \\ \tilde{b}_{qj} & \xi \end{pmatrix}, \quad \xi = e^{i\alpha\lambda}$$

yielding Hamiltonian (discrete NLS equation).

$$H = \sum_n b_n^\dagger (b_{n+1} - b_{n-1}) + \ln(1 + b_n^\dagger b_n), \tag{9}$$

with q-bosonic type operators:

$$\{b_m, b_n^\dagger\} = \hbar(1 - b_n^\dagger b_n)\, \delta_{m,n}.$$

producing a discrete-NLS equation (But Trig class!)

Note

Different Integrable discrete models have diffrent representative Lax operators U^j

However is it really true?

A closer look shows:

Known Lax operators are only different *Reductions* , *Realizations* , *Reprsentations* of an Ancestor Lax operator (or its (trigonometric) q-deformation) !

Ancestor Lax operator (rational) [Kundu'PRL]

$$U_r^j(\lambda) = \begin{pmatrix} c_1(\lambda + S_j^3) + c_2 & S_j^- \\ S_j^+ & c_3(\lambda - S_j^3) - c_4 \end{pmatrix}, \tag{10}$$

$\mathbf{S}_j$ satisfy Ancestor-algebra (*spin* -like algebra):

$$[S_j^-, S_j^+] = m^+ S_j^3 + m^-, \ [S_j^\pm, S_j^3] = \pm S_j^\pm$$

$m^\pm(c_i)$ dependent (independent) arbitrary parameters including zero-values.

Note: 1) $U_r^j(\lambda)$ *dependent linearly on* λ ($*$-this fact will be important!)

2) Ancestor algebra algebra is derived from YBE, hence Integrability guaranteed!

Generation of Known rational models from $U_r^j(\lambda)$:

1) XXX-spin chain:

$c_1 = c_3 = 1, \ \ c_2 = c_4 = 0,$ gives $m^+ = 2, m^- = 0$ with spin -$\frac{1}{2}$ reprsnt.

$\mathbf{S}_j \to \sigma_j$ with Anc-algebra $\to$ spin-algebra and generates

$U_r^j(\lambda) \to U_{XXX}^j(\lambda)$

2) Discrete NLS model (Korepin)

Reduction as above, but

spin fields $\mathbf{S}_j$ realised through Bosons $[q_j, \ q_k^\dagger] = \delta_{jk}$ as HPT:

$$S_j^+ = q_j(s - q_j^\dagger q_j)^{1/2}, \ S_j^3 = 2s - q_j^\dagger q_j$$

gives $U^j_r(\lambda) \to U^j_{lNLS}(\lambda)$
2a) NLS field model obtained at $q_j \to \Delta q(x), /\Delta \to 0$ gives: $U^j_{lNLS}(\lambda) = 1 + \Delta U_{NLS}(\lambda)$

$$U_{NLS}(\lambda) = \begin{pmatrix} \lambda & q(x) \\ q^\dagger(x) & -\lambda \end{pmatrix},$$

AKNS form!
3) *Simple lattice NLS model* (Kundu-Ragnisco)
Degenerate reduction $c_2 = c_3 = 0$ with $m^+ = 0, m^- = 1$ realization

$$S^3_n = s - N(n),\ S^+_n = -i\psi^*(n),\ S^-_n = -i\psi(n)$$

Note: However Lax operator of AL model (more popular lNLS) does not give above $U_{NLS}(\lambda)$
4) t-J model
A similar reduction as XXX but with Higher-rank Representation ($su(3)$), (realized through fermions: $(c, c^\dagger)$ with constraint) gives $U^j_r(\lambda) \to U^j_{tJ}(\lambda)$
Similar construction also for Hubbard model 5) Toda chain:

$c_1 = 1, c_2 = c_3 = c_4 = 0 \rightarrow m^+ = m^- = 0$ Anc. field Realized through canonical oprs. $[q_j,\ p_k] = \delta_{jk}$ as

$$S^{\pm}{}_j = e^{\pm q_j},\ S^3{}_j = p_j,$$

generates $U_r^j(\lambda) \rightarrow U^j_{Toda-chain}(\lambda)$

Trigonometric Ancestor Lax operator $U_t^j(\lambda)$ ($q = e^{i\alpha}$ deformation of rational $U_r^j(\lambda)$)
(Kundu'PRL)

$$U_t^j(\lambda) = \begin{pmatrix} c_1^+ q^{(S_j^3+\lambda)} + c_1^- q^{-(S_j^3-\lambda)} & 2\sin\alpha S q_j^- \\ 2\sin\alpha S q_j^+ & c_2^+ q^{-(S_j^3-\lambda)} + c_2^- q^{(S_j^3-\lambda)} \end{pmatrix}, \quad (11)$$

with $Sq_j^{\pm}$ satisfying quantum-deformed Ancestor algebra (generalized quantum algebra)

$$[S^3, Sq^{\pm}] = \pm Sq^{\pm}, \quad [Sq^+, Sq^-] = \frac{1}{2\sin\alpha}\left(M^+ \sin(2\alpha S^3) - iM^- \cos(2\alpha S^3)\right). \quad (12)$$

$c_1^{\pm}, c_2^{\pm}$ arbitrary parameters (including 0-values) and $M^{\pm}$ dependent on them.

bf Generation of known trigonometric class of models

1) Reduction $M^- = 0, M^+ = 2$ gives known quantum algebra $U_q(su(2))$

$$[S^3, S_q^{\pm}] = \pm S^{\pm}, [S_q^+, S_q^-] = \frac{1}{2\sin\alpha}\sin(2\alpha S^3)[\tag{13}$$

For 1.i) spin $\frac{1}{2}$ representation through Pauli matrices: $U_t^j(\lambda) \to U_{XXZ}^j(\lambda)$-xxz **spin chain** and

1.ii) *Lattice sine-Gordon model* (LSG)

with canonical: $\{u_j, p_k\} = \delta_{jk}$, realization of Anc. field:

$$S_j^+ = g(u_j)\ e^{ip_j\alpha}, S_j^3 = u_j$$

with

$$g(u_j) = \left[1 + \frac{1}{2}m^2\Delta^2\cos 2\alpha(u_j + \frac{1}{2})\right]^{\frac{1}{2}}$$

satisfying known quantum algebra gives $\sigma^1\ U_t^j(\lambda) \to U_{lSG}^j(\lambda)$ of **Lattice sine-Gordon** model .

3) **Lattice Liouville model**: For $M^+ = i, M^- = 1$ and a similar bosonic realization:

$$Sq_j^+ = f(u_j)\ e^{ip_j \alpha}, S_j^3 = u_j$$

with

$$f(u_j) = [1 + \Delta^2 e^{\alpha(2u_j+i)}]^{\frac{1}{2}}$$

an exponential function of u_j. satisfying an algebra

$$[S^3, Sq_j^\pm] = \pm Sq_j^\pm, \quad [Sq_j^+, Sq_j^-] = \frac{1}{2\sin\alpha} e^{2\alpha u_j}.weget$$

$\sigma^1\ U_t^j(\lambda) \to U_{LLM}^j(\lambda)$
4) Relativistic Toda chain: at $M^\pm = 0$ reducing algebra to

$$[S^3, Sq_j^\pm] = \pm Sq_j^\pm, \quad [Sq_j^+, Sq_j^-] = 0$$

and bosonic realization

$$Sq_j^\pm = \alpha e^{\pm q_j},\ S_j^3 = e^{(\alpha p_j} \qquad (14)$$

reduces $U_t^j(\lambda) \to U_{rtoda}(\lambda)$

5. **Ablowitz-Ladik model** Through a q-boson type realization (Macfarlane):

$$[\tilde{b}_{qj}, \tilde{b}^\dagger_{qk}] = \hbar(1 - \tilde{b}^\dagger_{qj}\tilde{b}_{qj})\, \delta_{j,k}, \quad \hbar = 1 - q^{-2}$$

reduces to $U_t^j(\lambda) \to U^j_{AL}(\xi)$,

6) Derivative lNLS model:

For $M^+ = 2\sin\alpha,\ M^- = 2i\cos\alpha$ and q-boson realization

$$S_q^+ = -A,\ S_q^- = A^\dagger,\ S^3 = -N, \tag{15}$$

q-Ancestor algebra reduces to q-boson algebra

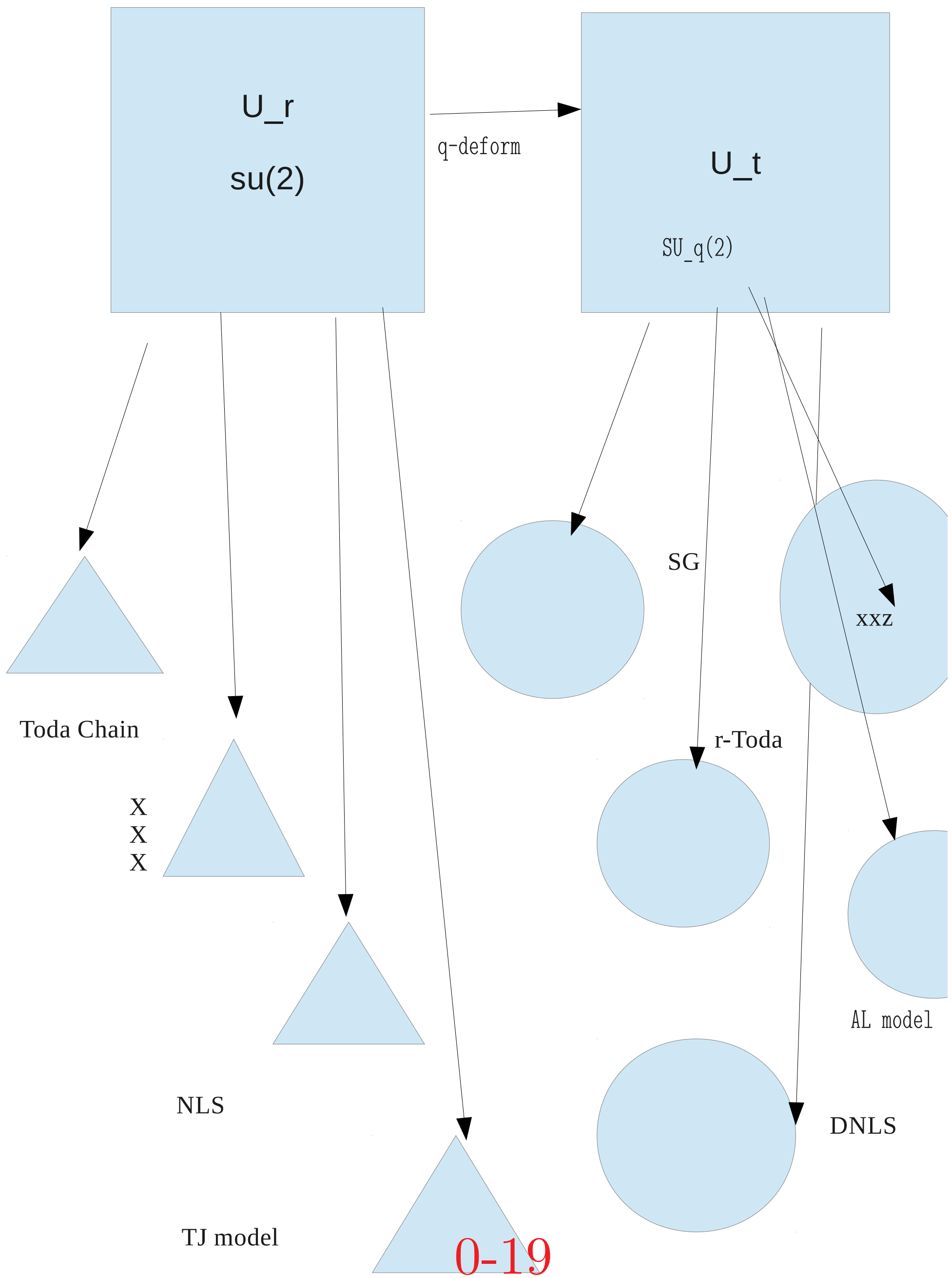
$$[A, N] = A, \quad [A^\dagger, N] = -A^\dagger, \quad [A, A^\dagger] = \frac{\cos(\alpha(2N+1}{\cos\alpha} \tag{16}$$

with $U_t^j \to U^j_{lDNLS}$ gives wellknown Derivative NLS equation at the continuum limit

7) Discrete Massive Thirring model :

Fusion of two U^j_{lDNLS} yiels U^j_{lMTM} etc.
Conclusion *All known integrable models* satisfying YBE (including quantum models) are only different realizations (representaions) of an Ancestor rational model with Lax operator: $U^j_r(\lambda)$ *having linear dependence on* λ.
Or its q-deformation: $U^j_t(\lambda)$.

)RIGIN of INTEGGABLE MODELS from ANCESTOR MODEL



Question

Can one go beyond this Lax operator for constructing *New Models*?

Seems to have No answer in the literature!!

bf Challenging problem

Use discretized Lax operators of higher powers in λ (with higher scaling dimension!

possibly: time t_n- Lax operator $V_n(x, t, \lambda) \rightarrow V_i^j(\lambda)$ for $(t_n, n = 2, 3, ...)$

Do we known any of them?

- find correponding new Ancestor Lax matrix $V_i^j{}_r, V_i^j{}_t$
- producing new families of models (similar to U_r^j, U_t^j)

Note Our successful so far!

NLS field model and Landau Lifshits equation

Known (1+1)-dim NLS model:

$$U_{NLS}(\lambda) = \begin{pmatrix} \lambda & q(x) \\ q^\dagger(x) & -\lambda \end{pmatrix},$$

linear in λ (with scaling dim 1) Hence We can generate infinite number of conserved charges $C_n,\ n = 1, 2, \ldots$:

$$C_1 = \int dx (q^* q)$$

$$C_2 = i \int dx (q_x^* q - q^* q_x)$$

$$C_3 = \int dx (q_x^* q_x + |q|^4)$$

$C_3 = H$ gives known NLS Equation

$$i q_t + q_{xx} + 2|q|^2 q = 0, \quad t = t_2$$

Same Eq can be obtained by NLS Lax pair: (U, V) :

$$U_t - V_x + [U, V] = 0$$

Hence we know an alternative Lax operator with higher scaling dimention $U^{(2)}(\lambda) = V$!
WE define Lax equation (adding another space y)

$$\Phi_y(y, x, t, \lambda) = U^{(2)}(\lambda) \Phi(y, x, t, \lambda),$$

$$U^{(2)}(\lambda) = i \begin{pmatrix} 2\lambda^2 - q^* q & 2\lambda q - i q_x, \\ -2\lambda q^* + i q_x^*, & -2\lambda^2 + q^* q \end{pmatrix}, \quad (17)$$

with λ^2 (scaling dimension 2)

Hence We can generate *New* infinite number of conserved charges $C_n^{(2)},\ n = 1, 2, \ldots$

$$C_1^{(2)} = i \int dy(q_x^* q - q^* q_x),$$

$$C_2^{(2)} = \int dy i q_y^* q + q_x^* q_x + (q^* q)^2 + cc),$$

$$C_3^{(2)} = \int dy q_y^* q_x,$$

$$C_4^{(2)} = \int dy(i q_{xy}^* q_x + q_y^* q_y - i|q|^2 (q^* q_y - q_y^* q)$$

$$-2|q|^2\ q_x^* q_x + ({q^*}^2 q_x^2 + {q_x^*}^2 q^2)),$$

Note :

1) $U^{(2)}(\lambda)$ satisfies YBE with rational $r_{rat}(\lambda)$ matrix

2) *Novel PB*: (not like NLS!)

$$\{q(y), q^*_x(y^{'})\}_{(2)} = \delta(y - y^{'}), \ \{q(y), q^*(y^{'})\}_{(2)} = 0,$$

3) Using $H = C_4^{(2)}$ and above PB we derive New integrable quasi-(2+1)-dim NLS:

$$iq_t + q_{xx} - q_{yy} + 2iq(j^x - j^y) = 0,$$

with $j^x = q^*_x q - q^* q_x, j^y = q^*_y q - q^* q_y,$

Aim: 1) To find discretised $U^{(2)}(x, y, \lambda) \to U^{(2)}(i, j, \lambda)$

2) Find corresponding Ancestor model $U^{(2)}(i, j, \lambda)_{rat}, U^{(2)}(i, j, \lambda)_{trig},$

3) Generate new ancestor algebras (new quantum algebra) and new Integrable models (like 2D spin model) satisfying YBE)

New 2D Landau-Lifshits eqaution : 1) Standard

$$i\mathbf{S}_t = [\mathbf{S}, \mathbf{S}_{xx}], \mathbf{S} = \vec{S} \cdot \vec{\sigma}$$

standard Lax opr (scaling dim 1)

$$U(\lambda) = i\lambda\mathbf{S}$$

2) Scal dim 2 Lax opr:

$$U^{(2)}(\lambda) = i(2\lambda^2\mathbf{S} + \lambda\mathbf{S}\mathbf{S}_x)$$

New LL eqn.

$$i\mathbf{S}_t = [\mathbf{S}_x, \mathbf{S}_y] + \mathbf{S}\mathbf{S}_{xy}]$$

(without further details)

II. Integrable Toda chain

1. *Standard* 2. *With defect site* (New !)

Generation of infinite Charges from Lax operator Global Lax operator (Monodromy matrix)

$$T_N(\lambda) = \prod_n^N U_n(\lambda)$$

$$T_{n+1}(\lambda) = U_n(\lambda) T_n(\lambda)\,,\ T_n = \begin{pmatrix} \Psi_n^1, & \Psi_n^{2*} \\ \Psi_n^2, & \Psi_n^{1*} \end{pmatrix}$$

Lax equation

$$\Psi_{n+1}^a = \sum_a U_n^{ab} \Psi_n^a,\ a, b = 1, 2$$

Conserved charges: $trT_N(\lambda) = \tau(\lambda) \sim$

Hence $\ln \tau(\lambda) \sim \ln(\frac{\Psi_N^1}{\Psi_1^1}(\lambda)) = I(\lambda)$ generates conserved charges!

Defining

$$\psi_n(\lambda) = \frac{\Psi_{n+1}^1}{\Psi_n^1}(\lambda),$$

get generating function:

$$I(\lambda) \equiv \ln \prod_{n}^{N} \psi_n(\lambda) = \sum_{n}^{N} \ln\psi_n(\lambda) = -\sum_{j=1} c_j \lambda^{-j}.$$

Therefore obtain $c_j,\ j = 1, 2, \ldots N$ for any model Knowing Lax opereator U^j

Toda chain

$$U_n^{11} = (\lambda + p_n),\ U_n^{12} = -e^{q_n},\ U_n^{21} = e^{-q_n},\ \ U_n^{22} = 0,$$

Hence obtain

$$\psi_n(\lambda) = \lambda + p_n + -e^{q_n} \sum_{j=1} \Gamma_n^{-j} \lambda^{-j},$$

where

$$\Gamma_n(\lambda) = \frac{\Psi_n^2}{\Psi_n^1}(\lambda)$$

determined from discrete Riccati eqn.

$$\Gamma_{n+1}(\lambda) = \frac{U_n^{21} + U_n^{22}\Gamma_n(\lambda)}{U_n^{11} + U_n^{12}\Gamma_n(\lambda)}$$

We get

$$\Gamma_n^{-1} = e^{-q_{n-1}}, \; \Gamma_n^{-2} = -e^{-q_{n-1}} p_{n-1}, \tag{18}$$

From explicit above solutions of Γ_n^{-j} and conserved quantities for Toda chain:

$$\begin{aligned} c_1 &= -\sum_{-N}^{N} p_n, \quad c_2 = \sum_{-N}^{N} \frac{1}{2} p_n^2 + \sum_{-N+1}^{N} e^{q_n - q_{n-1}}, \\ c_3 &= -\sum_{-N}^{N} \frac{1}{3} p_n^3 + \sum_{-N+1}^{N} p_{n-1} e^{q_n - q_{n-1}}, \end{aligned} \tag{19}$$

Using PB relations

$$\{q_j(t), p_k(t)\} = \delta_{jk}, \; \{q_j(t), q_k(t)\} = \{p_j(t), p_k(t)\} = 0. \tag{20}$$

from $c_2 = H$ obtain the Toda chain Eq.:

$$\dot{q}_n = e^{q_{n+1}-q_n} - e^{q_n - q_{n-1}}$$

and from c_3 etc. the Toda chain Hierarchy!

Integrable Toda chain with a defect point

- Real crystals show defects
- However defects in general spoil Integrability
- Aim therefore is to treat defects preserving Integrability

We are able to formulate Toda chain with a at site $n_0 = 0$

Our approach

at $n < n_0$ Toda chain described by U^n,
while at $n > n_0$ by another copy $\widetilde{U}_n$
bridged by a Bäcklund transformation (BT) $L_0(t, \lambda)$ frozen at n_0

$$\widetilde{\Psi}_{n_0}(t, \lambda) = L_0(t, \lambda)\Psi_{n_0}(t, \lambda)\,, \qquad (21)$$

depending on Lax pair across the defect point $n_0 = 0$

$$L_{n+1} \quad = \quad \widetilde{U}^n L_n U^{n-1}\,, \qquad (22)$$

$$\dot{L}_n \quad = \quad \widetilde{V}_n L_n - L_n V_n \,. \qquad (23)$$

- The generating functions for Toda charges (as shown above)

$$I^{right}_{bulk}(\lambda) = \sum_n c_n \lambda^{-n}, \quad I^{left}_{bulk}(\lambda) \sum_n \tilde{c}_n \lambda^{-n}$$

nolonger conserved

$$\partial_t \left(I^{left}_{bulk}(\lambda) + I^{right}_{bulk}(\lambda) \right) \neq 0$$

- Defect contribution needs to be included:

$$I(\lambda) = I^{left}_{bulk}(\lambda) + I^{right}_{bulk}(\lambda) + I_{defect}(\lambda)\,, \qquad (24)$$

should give $I_t(\lambda) = 0$. **How to determine $I_{defect}(\lambda)$**
Since defect can be considered as a boundary, using $\dot{\Phi}_n = V_n \Phi_n$ we get

$$\partial_t \left(I^{left}_{bulk}(\lambda) + I^{right}_{bulk}(\lambda) \right) - (\widetilde{V}^{11}_{n_0} + \widetilde{V}^{12}_{n_0} \widetilde{\Gamma}_{n_0}(\lambda)) + (V^{11}_{n_0} + V^{12}_{n_0} \Gamma_{n_0}(\lambda))$$
$$= - \left[\ln(L^{11}_0 + L^{12}_0 \Gamma_{n_0}) \right]_t$$

using BT relation, where $\Gamma_n(\lambda)$ already determined And hence

$$I_{defect}(\lambda) = \left[\ln(L_0^{11} + L_0^{12}\Gamma_{n_0}(\lambda)\right]$$

Representing BT as

$$L_n = \begin{pmatrix} \lambda + \alpha_n & \beta_n \\ \gamma_n & 1 \end{pmatrix}, \tag{25}$$

using U^n_{Toda} solve from BT relation for the matrix elements

$$\begin{aligned} \beta_n &= -e^{q_n}, \quad \gamma_n = e^{-\widetilde{q}_{n-1}} \\ \alpha_n &= p_n + e^{\widetilde{q}_n - q_n}, \quad \alpha_{n+1} = \widetilde{p}_n + \ e^{\widetilde{q}_n - q_n}, \end{aligned} \tag{26}$$

Hence we can derive defect contribution $I_{defect}(k),\ k = 1, 2, \ldots$ to all conserved charges:

$$I_{defect}(\lambda) = -\sum_k I_{defect}(k)\lambda^{-k} \quad = \quad \ln(L^{11}{}_n + L^{12}{}_n\Gamma_n(\lambda))|_{n=0}, \tag{27}$$

$L^{12}{}_n = \beta_n,\ L^{11}{}_n = \alpha_n,\ \Gamma_n(\lambda)$ solved from Riccati eqn. expanded in powers of λ^{-k}
Explicitly,

$$\begin{aligned} I_{defect}(1) &= -\alpha_0 = -(p_0 + e^{\widetilde{q_0} - q_0}) \\ I_{defect}(2) &= \frac{1}{2}\alpha_0^2 + e^{q_0 - q_{-1}}, \end{aligned} \tag{28}$$

etc. Taking

$$H \equiv C_2 = c_2 + \tilde{c}_2 + I_{defect}(2),$$

with

$$\tilde{c}_2 = \tilde{H}_{toda} = \sum_{n=1}^{N} \frac{1}{2}\tilde{p}_n^2 + \sum_{n=2}^{N} e^{\tilde{q}_n - \tilde{q}_{n-1}},$$

$$c_2 = H_{toda} = \sum_{n=-N}^{-1} \frac{1}{2}p_n^2 + \sum_{n=-N+1}^{-1} e^{q_n - q_{n-1}},$$

$$I_{defect}(2) = \alpha_0^2 + e^{\tilde{q}_1}\gamma_0 - e^{-q_{-1}}\beta_0,$$

using

$$\{q_0, p_0\} = \{\tilde{q}_0, \tilde{p}_0\} = 1, \ \{\tilde{p}_0, q_0\} = 0$$

etc. we derive Hamilton Eqn

$$\dot{q}_j = \{q_j, H\}, \ \dot{p}_j = \{p_j, H\},$$

for all $j = 1, N$. At defect site $j = 0$ we get

$$\begin{aligned} \dot{q}_0 &= 2\alpha_0, \ \dot{p}_0 = 2\alpha_0 \ e^{\widetilde{q_0} - q_0} - e^{q_0 - q_{-1}} \\ \dot{\tilde{q}}_0 &= 0, \ \dot{\tilde{p}}_0 = -2\alpha_0 \ e^{\widetilde{q_0} - q_0} \end{aligned} \tag{29}$$

with contributions from both sides.

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Thank You