The QRT mapping and beyond

A. Ramani (Ecole Polytechnique)

with

B. Grammaticos (IMNC, Paris VII & XI U.)
K.M. Tamizhmani (Pondicherry U.)
R. Willox (Tokyo U.)

THE QRT MAPPING

Two families: symmetric and asymmetric

3×3 matrices, A_0 and A_1 and vectors X_n, Y_n

$$A_i = \begin{pmatrix} \alpha_i & \beta_i & \gamma_i \\ \delta_i & \epsilon_i & \zeta_i \\ \kappa_i & \lambda_i & \mu_i \end{pmatrix} \quad \text{and} \quad X_n = \begin{pmatrix} x_n^2 \\ x_n \\ 1 \end{pmatrix}, \ Y_n = \begin{pmatrix} y_n^2 \\ y_n \\ 1 \end{pmatrix}$$

$$K(x_n, y_m) = \frac{\widetilde{Y}_m A_0 X_n}{\widetilde{Y}_m A_1 X_n}$$

Equations : $K(x_n, y_n) = K(x_{n+1}, y_n) = K(x_{n+1}, y_{n+1}) = \ldots = \mathcal{K}$

besides trivial solution $x_{n+1} = x_n$, $y_{n+1} = y_n$

Asymmetric (generic case) mapping

$$x_{n+1} = \frac{f_1(y_n) - x_n f_2(y_n)}{f_2(y_n) - x_n f_3(y_n)}$$

$$y_{n+1} = \frac{g_1(x_{n+1}) - y_n g_2(x_{n+1})}{g_2(x_{n+1}) - y_n g_3(x_{n+1})}$$

f_i (resp. g_i) polynomials, in general quartic, of y (resp. x)

Symmetric mapping: if *both* A_0 and A_1 symmetric then $g_i \equiv f_i$

$$w_{m+1} = \frac{f_1(w_m) - w_{m-1} f_2(w_m)}{f_2(w_m) - w_{m-1} f_3(w_m)}$$

with identification $x_n \to w_{2n}$, $y_n \to w_{2n+1}$

Integration of QRT mapping

$$\widetilde{Y}_n A_0 X_m - \mathcal{K}\, \widetilde{Y}_n A_1 X_m = 0 \qquad \Rightarrow$$

$$\alpha x^2 y^2 + xy(\beta x + \delta y) + \gamma x^2 + \kappa y^2 + \epsilon xy + \zeta x + \lambda y + \mu = 0$$

Through homographic transformations $x = H_x(\xi),\ y = H_y(\eta)$

(same for x and y iff symmetric case)

$$\xi^2 \eta^2 + \Gamma(\xi^2 + \eta^2) + E\xi\eta + 1 = 0$$

Elliptic functions of modulus k: $\xi = \sqrt{k}\,\mathrm{sn}(z)$, $\eta = \sqrt{k}\,\mathrm{sn}(z \pm q)$

$$k^2 + \Big(\Gamma + \frac{1}{\Gamma} - \frac{E^2}{4\Gamma}\Big)k + 1 = 0, \quad q \text{ such that } \Gamma k\, \mathrm{sn}^2(q) + 1 = 0$$

Finally $\xi_n = \sqrt{k}\,\mathrm{sn}(z_0 + 2nq)$, $\eta_n = \sqrt{k}\,\mathrm{sn}(z_0 + (2n+1)q)$

Symmetric case: $w = H_w(\omega) \to \omega_n = \sqrt{k}\,\mathrm{sn}(z_0 + nq)$

Reinterpretation of the asymmetric QRT

Invariance condition $K(x_n, y_n) = K(x_{n+1}, y_n) = K(x_{n+1}, y_{n+1})$

Introduce w even in *asymmetric* case : $x_n \to w_{2n}$, $y_n \to w_{2n+1}$

$$\frac{\widetilde{W}_{2n+1} A_0 W_{2n}}{\widetilde{W}_{2n+1} A_1 W_{2n}} = \frac{\widetilde{W}_{2n+1} A_0 W_{2n+2}}{\widetilde{W}_{2n+1} A_1 W_{2n+2}} = \frac{\widetilde{W}_{2n+3} A_0 W_{2n+2}}{\widetilde{W}_{2n+3} A_1 W_{2n+2}} = \ldots = \mathcal{K}$$

$$\frac{\widetilde{W}_{2n+1} A_0 W_{2n+2}}{\widetilde{W}_{2n+1} A_1 W_{2n+2}} = \frac{\widetilde{W}_{2n+2} \widetilde{A}_0 W_{2n+1}}{\widetilde{W}_{2n+2} \widetilde{A}_1 W_{2n+1}} \qquad \Rightarrow$$

$$\forall m \quad \frac{\widetilde{W}_{m+1} B_0(m) W_m}{\widetilde{W}_{m+1} B_1(m) W_m} = \mathcal{K} \quad \text{with } B_i(2m) = A_i,\ B_i(2m+1) = \widetilde{A}_i$$

$$K(m; w_m, w_{m+1}) = \mathcal{K}$$

here $K(m; *, *)$ has period 2 (unless $\widetilde{A}_i = A_i$, symmetric QRT)

How to find period 2 invariants?

Start from discrete Painlevé equation: for instance d-$\mathrm{P_I}$

$$w_{n+1} + w_n + w_{n-1} = 1 + \frac{z_n}{w_n}, \qquad z_n = \alpha n + \beta + (-1)^n \gamma$$

Discard "secular" term $\alpha = 0 \to z_n$ periodic function of period 2

Invariant

$$K = w_n w_{n-1}(w_n + w_{n-1} - 1) - z_{n-1} w_n - z_n w_{n-1}$$

In QRT parlance:

$$A_0 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & -\beta + \gamma \\ 0 & -\beta - \gamma & 0 \end{pmatrix} \quad A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Invariants with coefficients of period three and higher

General approach:

Start from a discrete Painlevé equation

Discard the secular dependence on n

$\rightarrow$ mapping, autonomous up to the periodicity of its coefficients

Use canonical expressions of the autonomous forms of the dPs

matrix A_1 is fixed and we focus only on A_0

First case

$$x_{n+1} + x_{n-1} = 1 + \frac{z_n}{x_n}$$

with $z_n = \alpha n + \beta + \gamma j^n + \delta j^{2n}$ where $j = e^{2i\pi/3}$

Put $\alpha = 0 \rightarrow z$ is periodic with period three, i.e. $z_{n+2} = z_{n-1}$

Invariant

$$K = x_n^2 x_{n-1}^2 - x_n x_{n-1}(x_n + x_{n-1}) + a_n x_n x_{n-1} + b_n x_n + c_n x_{n-1} + d_n$$

Start from K and demand that $K(n) \equiv \mathcal{K}$ leads to the equation

$$a_n = 1 - z_n - z_{n-1} + z_{n+1}, \quad b_n = z_{n-1}, \quad c_n = z_n, \quad d_n = z_n z_{n-1}$$

provided $z_{n+2} = z_{n-1}$, i.e. z_n of period three

A mapping with period 5 coefficients

$$x_{n+1}x_{n-1} = \frac{x_n - a_n}{x_n - 1}$$

Postulate invariant

$$K_n = \alpha_n x_n x_{n-1} + \beta_n x_n + \delta_n x_{n-1} + \epsilon_n + \frac{\gamma_n x_n + \zeta_n}{x_{n-1}} + \frac{\kappa_n x_{n-1} + \lambda_n}{x_n} + \frac{\mu_n}{x_n x_{n-1}}$$

Ask that conservation $K_{n+1} = K_n$ lead to equation

$$\beta_n = -(\alpha_n + \alpha_{n+1}),\ \delta_n = -(\alpha_n + \alpha_{n-1}),\ \gamma_n = \alpha_{n+1},\ \kappa_n = \alpha_{n-1}$$

$$\zeta_n = -(a_n\alpha_{n+1} + \alpha_{n+2}),\ \lambda_n = -(a_{n+1}\alpha_{n+2} + \alpha_{n+3}),\ \mu_n = a_n\alpha_{n+2},\ \epsilon_n = 0$$

Period 5 solution (c is a constant)

$$\alpha_{n+5} = \alpha_n \quad \text{and} \quad a_n = c\alpha_{n-2}$$

Extend the mapping

$$x_{n+1}x_{n-1} = b_n \frac{x_n - a_n}{x_n - 1}$$

We find

$$a_n = \frac{c}{b_{n-1}b_n^2 b_{n+1}}$$

and for b

$$b_{n-1}b_n = b_{n+2}b_{n+3}$$

Period 6

Interesting limit $a_n = 0$ (with $y = 1/x$)

$$y_{n+1}y_{n-1} = g_n(y_n - 1)$$

Again period 6.

From elliptic discrete Painlevé equations

Period 12

$$\frac{x_{n-1}-(\phi_{n-1}+\omega_n)^2}{x_{n-1}-(\phi_{n-1}-\omega_n)^2}\,\frac{x_n-(\phi_{n-1}+\phi_{n+1}-\omega_n)^2}{x_n-(\phi_{n-1}+\phi_{n+1}+\omega_n)^2}\,\frac{x_{n+1}-(\phi_{n+1}+\omega_n)^2}{x_{n+1}-(\phi_{n+1}-\omega_n)^2}=1$$

where $\phi_{n+3}=\phi_n$ and $\omega_{n+4}=\omega_n$

and period 20

$$\frac{x_{n-1}-(\phi_n+\phi_{n-2}+\omega_n)^2}{x_{n-1}-(\phi_n-\phi_{n-2}+\omega_n)^2}\,\frac{x_n-(\phi_n-\phi_{n-2}-\phi_{n+2}+\omega_n)^2}{x_n-(\phi_n+\phi_{n-2}+\phi_{n+2}+\omega_n)^2}\times\frac{x_{n+1}-(\phi_n+\phi_{n+2}+\omega_n)^2}{x_{n+1}-(\phi_n-\phi_{n+2}+\omega_n)^2}=1$$

where $\phi_{n+5}=\phi_n$ and $\omega_{n+2}=-\omega_n$

12=3×4, 20=4×5 but also 30=2×3×5, 14=2×7 and “genuine 8”

Is there a limit to the length of periods? No!

Consider the mapping

$$x_{kn} + x_{kn-2} = \frac{d}{x_{kn-1}} \qquad (*)$$

$$x_{kn+1} + x_{kn-1} = x_{kn} + 1 + \frac{d-c}{x_{kn}}$$

$$x_{kn+2} + x_{kn} = \frac{d}{x_{kn+1}}$$

$$x_{kn+3} + x_{kn+1} = -x_{kn+2} + 1 + \frac{c}{x_{kn+2}}$$

followed by

$$x_{kn+2j+2} + x_{kn+2j} = -x_{kn+2j+1} + 1 + \frac{d}{x_{kn+2j+1}}$$

$$x_{kn+2j+3} + x_{kn+2j+1} = -x_{kn+2j+2} + 1 + \frac{c}{x_{kn+2j+2}} \qquad j = 1, \ldots, p$$

$$x_{kn+2j+3} + x_{kn+2j+1} = -x_{kn+2j+2} + 1 + \frac{c}{x_{kn+2j+2}} \qquad j = 1, \ldots, p$$

At $j = p$: 3-point mapping around the point $kn + 2p + 2$
Next equation

$$x_{kn+2p+4} + x_{kn+2p+2} = \frac{d}{x_{kn+2p+3}}$$

i.e. like $(*)$ where $n \to n + 1$ for $k = 2p + 4$
and the pattern repeats indefinitely

General form of the system

$$x_{m+1} + x_{m-1} = \alpha_m x_m + \beta_m + \frac{\gamma_m}{x_m}$$

where $\alpha_m, \beta_m, \gamma_m$ are periodic coefficients with period $2p + 4$

Integrating mappings wth periodic coefficients

Expect their solution to be given in terms of elliptic functions

Mapping with ternary symmetry (z_n has period 3)

$$x_{n+1} + x_{n-1} = 1 + \frac{z_n}{x_n}$$

Start with the integral

$$K = x_n^2 x_{n-1}^2 - x_n x_{n-1}(x_n + x_{n-1}) + x_n x_{n-1}(1 - z_n - z_{n-1} + z_{n+1}) \\ + z_{n-1} x_n + z_n x_{n-1} + z_n z_{n-1}$$

Introduce homographic transformation for x

$$x_m = \frac{\alpha_m y_m + \beta_m}{\gamma_m y_m + \delta_m}$$

where $\alpha, \beta, \gamma, \delta$ must have period 3

Bring invariant $K(y_{3n+j}, y_{3n+1+j}) - k = 0$ to canonical form

for $j = 0, 1, 2$

Two transformations are necessary (one for each of the arguments)

By the usual miracles of integrability:
all three invariants can be brought to canonical form
the solution can be expressed in terms of elliptic functions
the moduli of the elliptic functions involved are the same

Solution
$y_{3n} = \sqrt{\kappa}\,\mathrm{sn}(\omega_n)$, $y_{3n+1} = \sqrt{\kappa}\,\mathrm{sn}(\omega_n + p)$, $y_{3n+2} = \sqrt{\kappa}\,\mathrm{sn}(\omega_n + p + q)$
$y_{3n+3} = \sqrt{\kappa}\,\mathrm{sn}(\omega_n + p + q + r)$, which means that $\omega_{n+1} = \omega_n + p + q + r$

where $\mathrm{sn}(\omega)$ elliptic sine of modulus κ

In general the three steps p, q, r are different

A distinctive difference compared to the asymmetric QRT mapping

Going still beyond

Hirota, Kimura and Yahagi: systems with *biquartic* invariants

$$(x_n x_{n+1} - 1)(x_n x_{n-1} - 1) = \frac{(x_n - a)(x_n - 1/a)(x_n^2 - 1)}{p^2 x_n^2 - 1}$$

Invariant

$$\begin{aligned}(x_n x_{n-1} - 1)^2 K = &\Big((x_n - x_{n-1})^2 - p^2 (x_n x_{n-1} - 1)^2\Big) \\ &\times \Big((x_n + x_{n-1} - a - 1/a)^2 - p^2 (x_n x_{n-1} - 1)^2\Big)\end{aligned}$$

HKY 2nd-order mappings from 3rd-order ones
From q-Painlevé equations with $q = -1$
Reductions of Adler-Bobenko-Suris lattices
From non-QRT mappings
Using folding transformations

HKY mappings can have periodic coefficients

Start from asymmetric q-$\mathrm{P_V}$

$$(x_n y_n - 1)(x_n y_{n-1} - 1) = \frac{(x_n - a)(x_n - b)(x_n - c)(x_n - d)}{(pq^n x_n - 1)(rq^n x_n - 1)}$$

$$(x_{n+1} y_n - 1)(x_n y_n - 1) = \frac{(y_n - 1/a)(y_n - 1/b)(y_n - 1/c)(y_n - 1/d)}{(sq^n y_n - 1)(tq^n y_n - 1)}$$

Take $q = -1$, rescale, go to symmetric case

$$(x_{n+1} x_n - 1)(x_n x_{n-1} - 1) = \frac{(x_n^2 - \kappa x_n + 1)(x_n^2 - 1)}{\alpha x_n^2 + \beta i^n x_n - 1}$$

Period 4

$$(x\bar{x} - 1)^2 K = \Big((x - \bar{x})^2 - \alpha(x\bar{x} - 1)^2\Big)\Big((x + \bar{x} - \kappa)^2 - \alpha(x\bar{x} - 1)^2\Big)$$
$$+2\beta i^n (x\bar{x} - 1)\Big(\alpha(x\bar{x} - 1)^2(\bar{x} + ix) - ((\bar{x} + ix)^2 - 2i)(\bar{x} - ix) + \kappa(\bar{x}^2 + ix^2 - i - 1)\Big)$$
$$+\beta^2(-1)^n(x\bar{x} - 1)^2((\bar{x} + ix)^2 - 2i)$$

How to get HKY mappings with periodic coefficients?

Start from a q-Painlevé equation and take q as a root of unity
(For $q = 1$, back to a QRT)
For $q = -1$ asymmetric QRT if even-odd periodicity is allowed
Otherwise HKY mapping with periodic coefficients

q-Painlevé I

$$x_{n+1}x_{n-1} = aq^n \frac{1 - x_n}{x_n^2} \quad \text{with} \quad q = -1$$

Invariant

$$K = \frac{x_{n-1}^4 x_n^4 + 2a(-1)^n x_{n-1}^2 x_n^2 (x_n - x_{n-1}) + a^2 (x_n - x_{n-1} - 1)^2}{x_{n-1}^2 x_n^2}$$

Another example

$$x_{n+1}x_{n-1} = \frac{x_n - a_n}{x_n - 1}$$

with

$$a_{n+3}a_{n-3} = a_{n+2}a_{n-2}$$

where

$$\log a_n = n \log q + p + rk^n + sk^{2n} + tk^{3n} + uk^{4n} \quad (k^5 = 1)$$

Taking $q = -1$ introduces even-odd periodicity
$\rightarrow$ HKY mapping (period 10)

$$\begin{aligned}\bar{x}^2 x^2 K_n = {} & \alpha_n x^4 \bar{x}^4 + x^3 \bar{x}^3 (\beta_n x + \beta'_n \bar{x} + \gamma_n) + \bar{x}^2 x^2 (\delta_n x^2 + \delta'_n \bar{x}^2 + \epsilon_n x + \epsilon'_n \bar{x}) \\ & + x\bar{x}(\zeta_n x^3 + \zeta'_n \bar{x}^3 + \eta_n x^2 + \eta'_n \bar{x}^2 + \theta_n x + \theta'_n \bar{x} + \kappa_n) \\ & + \lambda_n x^4 + \lambda'_n \bar{x}^4 + \mu_n x^3 + \mu'_n \bar{x}^3 + \nu_n x^2 + \nu'_n \bar{x}^2 + \xi_n x + \xi'_n \bar{x} + \rho_n\end{aligned}$$

A final example ($z_n = c(-1)^n$)

$$\frac{(x_{n+1}x_n - z_n)(x_n x_{n-1} - z_{n-1})}{(x_{n+1}x_n - 1)(x_n x_{n-1} - 1)} = \frac{x_n^4 + 2sz_n x_n^2 - z_n^2}{x_n^4 + 2px_n^2 + 1}$$

Invariant

$$\begin{aligned}
(x_{n+1}x_n - 1)^2(x_{n+1}x_n - z_n)^2 K = (1-z_n)^4(x_{n+1}^2 + x_n^2)^2 \\
-4(x_{n+1}x_n - 1)^2(x_{n+1}x_n - z_n^2)^2(1 - p^2 + s^2) \\
+4(1-z_n)^2\big(x_{n+1}^2 x_n^2((p - sz_n)x_{n+1}^2 + (p + sz_n)x_n^2)\big) \\
-8(1-z_n)^2 x_{n+1}x_n z_n\big((p-s)x_{n+1}^2 + (p+s)x_n^2)\big) \\
+4(1-z_n)^2 z_n((pz_n - s)x_{n+1}^2 + (pz_n + s)x_n^2) \\
+4(1-z_n)^2\big(2(p^2 + s^2 z_n)x_{n+1}^3 x_n^3 - (p^2(1+z_n)^2 + 4s^2 z_n)z_n x_{n+1}^2 x_n^2\big) \\
+4(1-z_n)^2(2p^2 z_n^2 + 2s^2 z_n)x_{n+1}x_n
\end{aligned}$$

From HKY to QRT mappings (and back)

Apparently non-QRT mappings

$$x_{n+1} = i\ x_{n-1}\ \frac{(x_n + i\alpha)(x_n + i/\alpha)}{(x_n + \alpha)(x_n + 1/\alpha)}$$

Invariance condition $K(x_{n+1}, x_n) = iK(x_n, x_{n-1})$ with

$$\begin{aligned} x_n x_{n-1} K(x_n, x_{n-1}) = x_n^2 x_{n-1}^2 + x_n x_{n-1}(\alpha + 1/\alpha)(x_n + i x_{n-1}) \\ +(x_n^2 - x_{n-1}^2) + (\alpha + 1/\alpha)(x_n - i x_{n-1}) + 1 \end{aligned}$$

Invariant $L(x_n, x_{n-1}) = K(x_n, x_{n-1})^4$

Change of variables
$x_{4n} = y_{4n}$, $x_{4n+1} = iy_{4n+1}$, $x_{4n+2} = i/y_{4n+2}$, $x_{4n+3} = 1/y_{4n+3}$

$$y_{n+1}y_{n-1} = \frac{y_n^2 + i(-1)^n(\alpha + 1/\alpha)y_n - 1}{y_n^2 + (\alpha + 1/\alpha)y_n + 1}$$

QRT-type mapping with periodic coefficients!
Invariant

$$\begin{aligned} y_n y_{n-1} M(y_n, y_{n-1}) = {} & y_n^2 y_{n-1}^2 + y_n y_{n-1}(\alpha + 1/\alpha)(y_n + y_{n-1}) \\ & + (y_n^2 + y_{n-1}^2) - i(-1)^n(\alpha + 1/\alpha)(y_n - y_{n-1}) + 1 \end{aligned}$$

Re-interpret as a “standard” asymmetric QRT mapping

Last integrable case

$$X_{n+1} = \sqrt{i}\ X_{n-1}\ \frac{X_n^2 - i}{X_n^2 - 1}$$

Invariant: start from biquartic $K(X_n, X_{n-1})$
with invariance condition $K(X_{n+1}, X_n) = iK(X_n, X_{n-1})$
$\rightarrow$ Invariant $L = K(X_n, X_{n-1})^4$ of degree 16

$$X_n^2 X_{n-1}^2 K(X_n, X_{n-1}) = X_n^4 X_{n-1}^4 - 2X_n^2 X_{n-1}^2 (X_n^2 + iX_{n-1}^2) \\ +X_n^4 - X_{n-1}^4 - 2(X_n^2 - iX_{n-1}^2) + 1$$

Mapping obtained with folding transformation

$$x_{n+1} = i\ x_{n-1}\ \frac{(x_n - i)^2}{(x_n - 1)^2}$$

Introduce X by $x = X^2$ (folding transformation)
and take the square root of the mapping, change of variables
$X_{8n} = Y_{8n}$, $X_{8n+1} = \sqrt{i}\,Y_{8n+1}$, $X_{8n+2} = \sqrt{i}/Y_{8n+2}$,
$X_{8n+3} = -1/Y_{8n+3}$, $X_{8n+4} = -Y_{8n+4}$, $X_{8n+5} = -\sqrt{i}\,Y_{8n+5}$,
$X_{8n+6} = -\sqrt{i}/Y_{8n+6}$, $X_{8n+7} = 1/Y_{8n+7}$

$$Y_{n+1}Y_{n-1} = \frac{Y_n^2 - i(-1)^n}{Y_n^2 - 1}$$

HKY type but with periodic coefficients!

Invariant

$$Y_n^2Y_{n-1}^2M(Y_n, Y_{n-1}) = Y_n^4Y_{n-1}^4 - 2Y_n^2Y_{n-1}^2(Y_n^2 + Y_{n-1}^2) \\ +(Y_n^4 + Y_{n-1}^4) + 2i(-1)^n(Y_n^2 - Y_{n-1}^2) + 1$$

A digression on correspondences and their integrability

The invariant can define a 2-2 correspondence

Evolution: start with an invariant condition e.g.

$$\alpha x^2 y^2 + \beta xy(x+y) + \gamma(x^2+y^2) + \epsilon xy + \zeta(x+y) + \mu = 0$$

For given x we solve for y (more than one solutions)
Iterate: inject values of y and solve for x
(again, more than one solutions, only one being previous x)

Question:
is the evolution defined by the 2-2 correspondence integrable?

Integrable 2-2 correspondences do exist!

Simplest example: the invariant of the QRT mapping

Start with initial values x_n, y_n and compute $\mathcal{K} \equiv K(x_n, y_n)$

Obtain the u's from $K(u, y_n) = \mathcal{K}$
$\rightarrow$two solutions: x_n (obviously) and $u = x_{n+1}$ (from conservation)

Set of values $\{x_n, x_{n+1}\}$

Start form $u = x_{n+1}$ and from $K(x_{n+1}, v) = \mathcal{K}$
$\rightarrow$two solutions: y_n (obviously) and $v = y_{n+1}$ (from conservation)
Starting from $u = x_n$ and from $K(x_n, v) = \mathcal{K}$
$\rightarrow$two solutions: y_n (again) and $v = y_{n-1}$

Set of values $\{y_{n-1}, y_n, y_{n+1}\}$

At the next step, only four solutions for x, namely $\{x_{n-1}, x_n, x_{n+1}, x_{n+2}\}$

Number of images grows linearly with the number of iterations

According to Veselov's criterion this correspondence is integrable

Can we extend this result to the case of periodic coeficients?

Invariant (3.4) in case of ternary freedom

$$K(n;x,y) = y^2x^2 - yx(y+x) + yx(1-z_n-z_{n-1}+z_{n+1}) + z_{n-1}y + z_nx + z_nz_{n-1}$$

Initial conditions $x = x_{n-1}$, $y = x_n$
From x and y compute the value of conserved quantity $\mathcal{K}$

From invariant relation $K(n+1; y, u) = \mathcal{K}$ solve for u
→two solutions, but none coincides with x_n

Using these solutions, obtain the v from $K(n+2; u, v) = \mathcal{K}$
→ four distinct solutions

Next from $K(n+3; v, w) = \mathcal{K}$, obtain 8 values for w

Exponential growth of the number of solutions? Not true!

At $K(n+4; w, \omega) = \mathcal{K}$ we find only 12 distinct ω's

(From 8 values for w find 16 solutions, but only 12 distinct)

At $K(n+5; \omega, \psi) = \mathcal{K}$ we find 18 distinct values (instead of 32)

The number of distinct values grows polynomially (cubic growth!)

For number of iterations of $3n-2$, $3n-1$ and $3n$ ($n \geq 1$)

number of distinct solutions given by $n^2(n+1)$, $n(n+1)^2$ and $(n+1)^3$

For ternary freedom
the invariant relation, as a 2-2 correspondence, is integrable
(by Veselov's criterion)

The result is not specific to the ternary freedom.

Analysis of the case of coefficients of period 5

Again polynomial growth (quintic)

For number of iterations
$5n-4$, $5n-3$, $5n-2$, $5n-1$ and $5n$

the number of distinct solutions are
$n^4(n+1)$, $n^3(n+1)^2$, $n^2(n+1)^3$ and $n(n+1)^4$ and $(n+1)^5$

Again the polynomial growth is an indication of integrability

No rigorous proof but we expect
correspondences obtained from the invariant relation of a mapping solved in terms of elliptic functions involving k different steps to have a number of distinct solutions growing with the number of iterations as a polynomial of degree k

How about HKY mappings and their invariants?

Prototypical case of invariant curve

$$\left((x-y)^2 - p^2(xy-1)^2\right)\left((x+y-b)^2 - p^2(xy-1)^2\right) - K(xy-1)^2 = 0$$

already studied (constant coefficients)

From initial condition (x, y)

obtain number of images 4, 13, 40, 121, 364, 1093, ...

Exponential growth (recursion relation $N_{n+1} = 3N_n + 1$)

Clear indication of nonintegrability

True for HKY mappings with constant coefficients

Since mappings with periodic coefficients grow faster, we expect

nonitegrability also for HKY mappings with periodic coefficients

But an exception does exist!

HKY mappings with biquadratic "pre-invariant"

Start from a QRT-like invariant $K(x_n, x_{n-1})$

Instead of QRT invariance condition $K(x_n, x_{n-1}) = K(x_n, x_{n+1})$
introduce $K(x_n, x_{n+1}) = SK(x_n, x_{n-1})$
where S is an involution

In practice we find simple relations
$K(x_n, x_{n+1}) = -K(x_n, x_{n-1})$ or $K(x_n, x_{n+1}) = 1/K(x_n, x_{n-1})$

HKY invariant M in terms of K:
$M = K^2$ or $M = K + 1/K$ respectively

Example

$$x_{n+1}x_{n-1} + x_n^2 + ax_n(x_{n+1} + x_{n-1}) + b = 0$$

biquadratic pre-invariant

$$K = \frac{2x_n x_{n-1} + a(x_n^2 + x_{n-1}^2) - ab}{2ax_n x_{n-1} + x_n^2 + x_{n-1}^2 + b}$$

invariance condition $K(x_n, x_{n+1}) = -K(x_n, x_{n-1})$ and $M = K^2$

Correspondence defined by $M = \mathcal{M}$ leads to

number of distinct solutions growing as $(n+1)^2$ rather than 4^n

HKY mappings with pre-invariants
$\rightarrow$ associated correspondences expected to be integrable

(For HKY mappings with biquadratic pre-invariants we do not know any extension to forms with periodic coefficients)

Conclusions

QRT mappings can have periodic coeffcients

HKY mappings can have periodic coefficients, too

All these mappings are integrable. Solution in elliptic functions

Interesting result: more than one steps

off-shoot: integrability of a large family of correspondences

From QRT mappings to d-Painlevé equations by deautonomisation

With adequate autonomisation
from d-Painlevé equations back to (extensions of) QRT mappings