Discretising systematically integrable systems

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Question asked time and again:

"How do you find a *good* discretisation?"

Standard (rather unsatisfactory) answer:

"With experience and a little bit of luck"

We need a *systematic discretisation approach*

$\exists$ *Infinitely many* discrete analogues of a given continuous system

However for *integrable* systems the answer is almost unique.

Two important names: **Mickens** and **Hirota**

Mickens discretisation rules

1 *The orders of “discrete” and “differential” derivatives should be equal*

2 *The discrete representations for derivatives must, in general, have nontrivial denominators*

3 *Nonlinear terms must be, in general, replaced by nonlocal discrete representations*

4 *A property that holds for the differential equation should also be present in the discrete model*

An example: discretise the **Riccati** equation

$$x' = ax^2 + 2bx + f$$

Mickens prescription

$$x' \to \frac{x_{n+1} - x_n}{\Delta t}$$

$$x^2 \to x_{n+1} x_n$$

Discrete form

$$x_{n+1} = \frac{(1 + 2b\Delta t)x_n + f\Delta t}{1 - a\Delta t x_n}$$

What about Rule 4?

Integrability by direct linearisation is preserved !

The Hirota method: *bilinearisation* and *gauge invariance*

Riccati example: introduce ansatz

$$x = P/Q$$

Gauge transformation $P \to g(t)P$, $Q \to g(t)Q$ leaves x invariant

Riccati becomes

$$PQ' - QP' = aP^2 + 2bPQ + fQ^2$$

Gauge-invariance $\Rightarrow$ nonlocal discretisation of the quadratic terms

$$\frac{Q_{n+1}P_n - P_{n+1}Q_n}{\Delta t} = aP_nP_{n+1} + b(\alpha Q_{n+1}P_n + \beta P_{n+1}Q_n) + fQ_nQ_{n+1}$$

where $\alpha + \beta = 2$

$$x_{n+1} = \frac{(1 + b\alpha\Delta t)x_n + f\Delta t}{1 - b\beta\Delta t - a\Delta t x_n}$$

Our approach

Discretisation procedure based on

ad hoc linearisation of differential system
and Padé-type approximation of the exponential operator

Example, linear first-order equation

$$x' = \alpha x + \beta$$

with solution

$$x(t) = ce^{\alpha t} - \frac{\beta}{\alpha}$$

Time-discretisation

$$x(t + \Delta t) = ce^{\alpha(t+\Delta t)} - \frac{\beta}{\alpha} = e^{\alpha \Delta t}\left(x(t) + \frac{\beta}{\alpha}\right) - \frac{\beta}{\alpha}$$

Rational approximation of exponential

$$e^{\sigma} = \frac{1 + (\lambda + 1)\sigma}{1 + \lambda\sigma}$$

Finally

$$x_{n+1} = \frac{1 + (\lambda + 1)\alpha\Delta t}{1 + \lambda\alpha\Delta t} x_n + \frac{\Delta t \beta}{1 + \lambda\alpha\Delta t}$$

Second example, Riccati equation

$$x' = (ax + 2b)x + f$$

We find

$$x_{n+1} = \frac{(1 + (\lambda + 1)\Delta t(ax_n + 2b))x_n + f\Delta t}{1 + \lambda\Delta t(ax_n + 2b)}$$

For generic λ not acceptable (violates reversibility)

Taking $\lambda = -1$ we find

$$x_{n+1} = \frac{x_n + f\Delta t}{1 + 2b\Delta t - a\Delta t x_n} \qquad (*)$$

Compare to Hirota result

Equation (*) is obtained from Hirota for $\alpha = 0, \beta = 2$

Different derivation

Mapping

$$x_{n+1} = \frac{x_n + f\Delta t}{1 + 2b\Delta t - a\Delta t x_n} \qquad (*)$$

can be obtained from

$$x' = (ax + 2b)x + f$$

by ansatz

$$x' \to (x_{n+1} - x_n)/\Delta t$$

$$x^2 \to x_{n+1}x_n \quad \text{and} \quad x \to (x_{n+1} + x_n)/2$$

Two applications ($\Delta t \equiv \epsilon$)

The **Lotka-Volterra** system

$$x' = x(\lambda - y) \qquad y' = y(x - \mu)$$

Discrete form

$$\frac{x_{n+1}}{x_n} = \frac{1 + (p+1)(\lambda - y)\epsilon}{1 + p(\lambda - y)\epsilon} \qquad \frac{y_{n+1}}{y_n} = \frac{1 + (q+1)(x - \mu)\epsilon}{1 + q(x - \mu)\epsilon}$$

Freedom on the staggering

$$\frac{x_{n+1}}{x_n} = \frac{1 + (p+1)(\lambda - y_n)\epsilon}{1 + p(\lambda - y_n)\epsilon} \qquad \frac{y_{n+1}}{y_n} = \frac{1 + (q+1)(x_{n+1} - \mu)\epsilon}{1 + q(x_{n+1} - \mu)\epsilon}$$

Take $p = -1$ and $q = 0$

$$\frac{x_{n+1}}{x_n} = \frac{1}{1 - \epsilon\lambda + \epsilon y_n} \qquad \frac{y_{n+1}}{y_n} = 1 - \epsilon\mu + \epsilon x_{n+1}$$

SIR model for epidemic dynamics

$$S' = -SI \qquad I' = -\mu I + SI$$

(Lotka-Volterra with $\lambda = 0$) $\Rightarrow$ discrete form

$$\frac{S_n}{S_{n-1}} = \frac{1 + (p+1)I_n\epsilon}{1 + pI_n\epsilon} \qquad \frac{I_{n+1}}{I_n} = \frac{1 + (q+1)(S_n - \mu)\epsilon}{1 + q(S_n - \mu)\epsilon}$$

Staggering different from that of LV

"Intuitive" discretisation

$$\frac{S_n}{S_{n-1}} = \frac{1 + cI_n}{1 + I_n} \qquad \frac{I_{n+1}}{I_n} = \frac{a + S_n}{1 + bS_n}$$

Same as "systematic" by
rescaling of variables and appropriate definition of a, b, c

A system of coupled Riccatis

$$x' = -x^2 + axy$$

$$y' = -y^2 + bxy$$

Painlevé singularity analysis $\Rightarrow$ 5 integrable cases

i) $a = 0,\ b = n$ (n nonnegative integer)
ii) $a = 1,\ b = 1$
iii) $a = 2,\ b = 2$
iv) $a = 1,\ b = 3$ and its dual $a = 3,\ b = 3$
v) $a = 1,\ b = 2$ and its duals $a = 1,\ b = 5$ and $a = 2,\ b = 5$

Case i) is a special case of the Gambier equation

Discretisation, with $p = 0$

$$x_{n+1} = \frac{x_n}{1 + x_n}$$

$$y_{n+1} = \frac{(1 + bx_{n+1})y_n}{1 + y_n}$$

Special case of the Gambier mapping:

$$x_{n+1} = \frac{\lambda x_n + \mu}{1 + x_n}$$

$$y_{n+1} = \frac{x_n y_n + \sigma}{1 + \nu y_n}$$

with $\sigma = 0$, and specific staggering

For remaining cases again $p = 0$

$$x_{n+1} = \frac{x_n}{1 + x_n - a y_n}$$

$$y_{n-1} = \frac{y_n}{1 - y_n + b x_n}$$

Study integrability with:

singularity confinement & algebraic entropy

Integrable cases found: *exactly* cases (ii) to (v)

The same values a and b lead to integrable for continuous and discrete

Apply our method to the discretisation of **Painlevé equations**

e.g. Painlevé I

$$x'' = x^2 + t$$

Discrete form

$$x_{n+1} + x_n + x_{n-1} = \frac{\alpha n + \beta}{x_n} + 1$$

Was known for 70 years
but only recognised in the 90s

Extend our method to 2nd-order systems

Alas! Not very useful beyond $\mathrm{P_I}$

The Okamoto Hamiltonian formalism for the Painlevé equations

Hamiltonian is related to the τ-function

$$H = (\log \tau)'$$

Write Painlevé equations as Hamiltonian system

Starting with $H(x, p, z)$ and equations of motion

$$f(t)\frac{dx}{dt} = \frac{\partial H}{\partial p}$$

$$f(t)\frac{dp}{dt} = -\frac{\partial H}{\partial x}$$

Eliminating p find equation for x (and vice versa)

Miura transformation

Use Hamiltonian formalism for integrable discretisations

Hamiltonian equations of motion are in general of Riccati type

Ansatz for x:

$$x' \to x_{n+1} - x_n \qquad x^2 \to x_{n+1}x_n \qquad x \to (x_{n+1} + x_n)/2$$

For p, analogous ansatz but down-shifted

$$p' \to p_n - p_{n-1} \qquad p^2 \to p_n p_{n-1} \qquad p \to (p_n + p_{n-1})/2$$

staggering is essential

Discretisation of Painlevé II

Hamiltonian:

$$H(x,p) = \frac{1}{2}p^2 - p\left(x^2 + \frac{t}{2}\right) - \left(\mu + \frac{1}{2}\right)x$$

The equations of motion have the form

$$x' = -x^2 + p - \frac{t}{2} \qquad p' = 2xp + \mu + \frac{1}{2}$$

Eliminating p gives $\mathrm{P_{II}}$ for x

Use ansatz

$$x_{n+1} + x_{n-1} = \frac{x_n(t+2) + \mu + 1/2}{1 - x_n^2}$$

Discrete (autonomous)

Deautonomisation: here take t linear in n

Painlevé III

$$H(x,p) = 2x^2p^2 - p(zx^2 + 2\mu x - z) + \kappa z x$$

with $z = e^t$ and $f(z) = 1$

$$x' = x^2(4p - z) - 2\mu x + z$$

$$p' = -4p^2x + 2p(xz + \mu) - \kappa z$$

Discretisation

$$x_{n+1}x_{n-1} = \frac{x^2(\mu^2 - 1) - 2\mu x z + z^2}{x^2z^2 + 2xz(\mu - 2\kappa) + \mu^2 - 1}$$

with $z = \lambda^n$ we find q-discrete $\mathrm{P}_{\mathrm{III}}$

Painlevé IV

$$H(x,p) = 2xp^2 - p(x^2 + 2tx + \mu) + \kappa x$$

and

$$x' = -x^2 + 2x(2p - t) - \mu$$

$$p' = -2p^2 + 2p(x + t) - \kappa$$

Discretisation

$$(x_{n+1} + x_n)(x_n + x_{n-1}) = \frac{(x^2 - \mu)^2 - 4x^2}{(x + t)^2 - 2\kappa - 1}$$

Deautonomisation $\Rightarrow$ t linear in n

Painlevé V

$$H(x,p) = x(x-1)^2p^2 - p(\nu(x-1)^2 - \mu x(x-1) - zx) + \kappa(x-1)$$

and $z = e^t$

$$x' = 2px^3 - (4p + \nu - \mu)x^2 + (2p + z + 2\nu - \mu)x - \nu$$

$$p' = -p^2(3x^2 - 4x + 1) + p(2(\nu - \mu)x + \mu - 2\nu - z) - \kappa$$

Equation for x is *not* of Riccati type

Introduce auxiliary variable $u = xp$ and eliminate p

$$x' = 2ux^2 - 4ux - (\nu - \mu)x^2 + 2u + (z + 2\nu - \mu)x - \nu$$

$$u' = -\left(x - \frac{1}{x}\right)u^2 + (\nu - \mu)xu - \frac{\nu}{x}u - \kappa x$$

Discretisation $\Rightarrow$ discrete equation for x

but *not* in canonical form

Introduce new variable

$$x_n = \frac{y_n - 1}{y_n + 1}$$

Mapping for y

$$(y_{n+1}y_n - 1)(y_n y_{n-1} - 1)$$
$$= \frac{(y_n^2 - 1)^2(\mu^2 - 4) + 4y_n(y_n - 1)^2(4\kappa + 2\mu\nu - \mu^2) + 16\nu^2 y_n^2}{(zy_n - \mu)^2 - 4}$$

with $z = \lambda^n$ we find q-discrete $\mathrm{P_V}$

Painlevé VI

$$H(x,p) = x(x-1)(x-t)p^2 - p(\nu(x-1)(x-t) + \rho x(x-t) + \mu x(x-1)) + \kappa(x-t)$$

where for the time being we do not care about $f(t)$

$$x' = 2px^3 - (2p(t+1) + \nu + \mu + \rho)x^2 + (2pt + \nu(t+1) + \mu + \rho t)x - \nu t$$

$$p' = -p^2(3x^2 - 2x(t+1) + t) + p(2(\nu + \mu + \rho)x - \mu - \nu(t+1) - \rho t) - \kappa$$

Again introduce $u = xp$ and eliminate p

$$x' = 2ux^2 - (\nu + \mu + \rho)x^2 - 2(t+1)ux + (\nu(t+1) + \mu + \rho t)x + 2ut - \nu t$$

$$u' = -\left(x - \frac{t}{x}\right)u^2 + (\nu + \mu + \rho)xu - \frac{\nu t}{x}u - \kappa x$$

The mapping for x is not in canonical form

$$x_n = \sqrt{t}\frac{1 - y_n}{1 + y_n}$$

The (continuous) independent variable must also be changed

$$t = \left(\frac{1 - s}{1 + s}\right)^2$$

We finally find ($s = \lambda^n$, $\sigma = \rho + \mu$)

$$\frac{(y_{n+1}y_n - s^2)(y_n y_{n-1} - s^2)}{(y_{n+1}y_n - 1)(y_n y_{n-1} - 1)}$$

$$= \frac{(\rho(y_n - s)^2 + \mu(y_n + s)^2)^2 - 4t^{-1}(y_n^2 - s^2)^2}{(\sigma^2 - 4t^{-1})(y_n^2 - 1)^2 + (16\kappa - 4\sigma(\sigma + 2\nu))y_n(y_n - 1)^2 + 16\nu^2 y_n^2}$$

All Painlevé equations could be discretised

An intriguing remark:

only the discrete forms of the "standard" family were obtained

Why? The standard forms are not even the more fundamental!

On the contrary, if we implement full freedom

P_{I}→P_{II}, P_{II}→P_{III}, P_{III}→P_{VI}, P_{IV}→P_{VI} and P_{V},P_{VI} → higher

More important

Where are the other discrete forms of the Painlevé equations?

We must find a different approach

A (not so) short introduction to the QRT mapping

Motivation:

Autonomous limit of Painlevé transcendents $\Rightarrow$ elliptic functions

Angle of attack:

To obtain discrete Painlevé equations

start from mapping with elliptic function solutions

then extend by deautonomisation

Enter QRT mapping

Our strategy for discretisation:

Perform discretisation on autonomous form

require integrability i.e. ask that they be of QRT type

if non-autonomous form is already known, identify it

if not, deautonomise maintaining integrability

Ansatz for x:

$$x'' \to x_{n+1} + x_{n-1} - 2x_n \qquad x \to a_1(x_{n+1} + x_n) + a_2 x_n$$

$$x^2 \to b_1 x_{n+1} x_{n-1} + b_2 x_n (x_{n+1} + x_{n-1}) + b_3 x_n^2$$

Canonical forms of A_1 QRT matrices

(I) $$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad x_{n+1} + x_{n-1} = F(x_n)$$

(II) $$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad x_{n+1} x_{n-1} = F(x_n)$$

(III) $$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad (x_{n+1} + x_n)(x_n + x_{n-1}) = F(x_n)$$

(IV) $A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ $\qquad (x_{n+1}x_n - 1)(x_n x_{n-1} - 1) = F(x_n)$

(V)

$$A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & z \\ 1 & z & 0 \end{pmatrix} \qquad \frac{(x_{n+1} + x_n + z)(x_n + x_{n-1} + z)}{(x_{n+1} + x_n)(x_n + x_{n-1})} = F(x_n)$$

(VI)

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -z-1 & 0 \\ 0 & 0 & z \end{pmatrix} \qquad \frac{(x_{n+1}x_n - z)(x_n x_{n-1} - z)}{(x_{n+1}x_n - 1)(x_n x_{n-1} - 1)} = F(x_n)$$

(VII)

$$A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & -2z \\ 1 & -2z & z^2 \end{pmatrix} \frac{(x_{n+1} - x_n - z)(x_{n-1} - x_n - z) + 4zx_n}{x_{n+1} - 2x_n + x_{n-1} - 2z} = F(x_n)$$

(VIII)

$$A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & z+1/z & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad \frac{(x_{n+1}z + x_n)(x_{n-1}z + x_n) - z^2}{(x_{n+1} + zx_n)(x_{n-1} + zx_n) - 1} = F(x_n)$$

Matrix A_1 for case VI has different structure with $\alpha_1 \neq 0$

Traditionally associated with the Painlevé VI equation

but it is possible to transform to $\alpha_1 = 0$

(VI′)

$$A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & z+1/z & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \frac{(x_{n+1} + zx_n)(zx_n + x_{n-1})}{(zx_{n+1} + x_n)(x_n + zx_{n-1})} = F(x_n)$$

More convenient for degeneration process starting from VIII

Discretising the Painlevé I equation

$$x'' = x^2 + \lambda x + \mu$$

Ansatz

$$ax_{n+1}x_{n-1} + bx_n(x_{n+1} + x_{n-1}) + cx_n^2 + f(x_{n+1} + x_{n-1}) + gx_n + h = 0$$

Constraints

If $a = 0$, $b = 0$ and $c = 0$, mapping becomes linear

If $a = 0$, $b = 0$ and $f = 0$, mapping becomes linear

If $a = 0$ and $b^2h - bfg + cf^2 = 0$, factorisation and mapping becomes linear

If $ac - b^2 = 0$, $ag - 2bf = 0$ and $ah - f^2 = 0$ factorisation into linear first-order mappings

A few selected examples

Two difference P_{I}s

$$x_{n+1}+x_{n-1}=-\frac{h}{x_n+f}$$

$$x_{n+1}+x_n+x_{n-1}=-g-\frac{h}{x_n}$$

A q-P_{I}

$$x_{n+1}x_{n-1}=-gx_n-h$$

Also a P_{I} from the P_{IV} family

$$(x_{n+1}+x_n)(x_n+x_{n-1})=-gx_n-h$$

From the family of $\mathrm{P_{VI}}$, two new equations

$$\left(\frac{x_{n+1}+z_n z_{n+1} x_n}{z_n z_{n+1} x_{n+1}+x_n}\right)\left(\frac{z_n z_{n-1} x_n+x_{n-1}}{x_n+z_n z_{n-1} x_{n-1}}\right)=\frac{x_n z_{n-1} z_n^2 z_{n+1}-1}{x_n-z_{n-1} z_n^2 z_{n+1}}$$

and

$$\left(\frac{x_{n+1}+z_n z_{n+1} x_n}{z_n z_{n+1} x_{n+1}+x_n}\right)\left(\frac{z_n z_{n-1} x_n+x_{n-1}}{x_n+z_n z_{n-1} x_{n-1}}\right)=\frac{1}{z_n}\frac{x_n z_{n-1} z_n z_{n+1}+1}{-x_n+z_{n-1} z_n z_{n+1}}$$

where $z_n = z_0 \lambda^n$

For the difference $E_8^{(1)}$ case

$$\frac{(x_n - x_{n+1} + \zeta_n^2)(x_n - x_{n-1} + \zeta_{n-1}^2) + 4x_n\zeta_n\zeta_{n-1}}{\zeta_{n-1}(x_n - x_{n+1} + \zeta_n^2) + \zeta_n(x_n - x_{n-1} + \zeta_{n-1}^2)} = \zeta_n + \zeta_{n-1} - \frac{x_n + f}{\zeta_n + \zeta_{n-1}}$$

$$\frac{(x_n - x_{n+1} + \zeta_n^2)(x_n - x_{n-1} + \zeta_{n-1}^2) + 4x_n\zeta_n\zeta_{n-1}}{\zeta_{n-1}(x_n - x_{n+1} + \zeta_n^2) + \zeta_n(x_n - x_{n-1} + \zeta_{n-1}^2)} = z_{n+1} + z_n + z_{n-1} - \frac{x_n + f}{z_{n+1} + z_n + z_{n-1}}$$

where $z_n = z_0 + \lambda n$ and $\zeta_n = z_n + z_{n+1}$

And for the multiplicative $E_8^{(1)}$ case

$$\frac{(x_{n+1}\zeta_n + x_n)(x_{n-1}\zeta_{n-1} + x_n) - \phi_n}{(x_{n+1} + \zeta_n x_n)(x_{n-1} + \zeta_{n-1}x_n) - \phi_n/(\zeta_n\zeta_{n-1})} = \frac{x_n - f\zeta_n\zeta_{n-1}}{x_n\zeta_n\zeta_{n-1} - f}$$

$$\frac{(x_{n+1}\zeta_n + x_n)(x_{n-1}\zeta_{n-1} + x_n) - \phi_n}{(x_{n+1} + \zeta_n x_n)(x_{n-1} + \zeta_{n-1}x_n) - \phi_n/(\zeta_n\zeta_{n-1})} = \frac{x_n z_n + g\zeta_n\zeta_{n-1}}{-x_n\zeta_n\zeta_{n-1}/z_n + g}$$

where $z_n = z_0\lambda^n$, $\zeta_n = z_n z_{n+1}$ and $\phi_n = (\zeta_n^2 - 1)(\zeta_{n-1}^2 - 1)$

Results on PII

A contiguity of continuous $\mathrm{P_{VI}}$

$$\frac{z_{n+1}+z_n}{x_n+x_{n+1}}+\frac{z_n+z_{n-1}}{x_n+x_{n-1}}=\frac{2z_n}{x_n+a}+\frac{2z_n}{x_n+b}$$

A $E_8^{(1)}$ example

$$\frac{(x_{n+1}\zeta_n+x_n)(x_{n-1}\zeta_{n-1}+x_n)-\phi_n}{(x_{n+1}+\zeta_n x_n)(x_{n-1}+\zeta_{n-1}x_n)-\phi_n/(\zeta_n\zeta_{n-1})}$$
$$=\frac{x_n^2+dz_n^2x_n-z_n^8-1-fz_n^4}{x_n^2z_n^4+dz_n^2x_n-z_n^4-1/z_n^4-f}$$

Many more exist

Conclusion

The new method is very powerful

It allows a "bottom-up" construction of Painlevé equations

First results of equations described by $E_8^{(1)}$ Weyl group

Not presented results: **linearisable** equations

Remaining tasks

Find full freedom of the new equations

Derive discrete forms of the other Painlevé equations