Integrable discrete systems, an introduction

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with

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Plan of the Course

1. General introduction: who cares about integrability?
2. Integrable continuous systems: from Newton to Kruskal.
3. Detecting integrability: the Painlevé approach.
4. A prelude to discrete integrability: Hirota's creations.
5. The paradigmatic discrete systems: QRT maps.
6. The discovery of singularity confinement.
7. Complexity and algebraic entropy.
8. Confinement and complexity for multidimensional systems.
9. The Painlevé equations, discretised.
10. The Okamoto-Sakai approach for Painlevé equations.

Integrability: no definition here
(Arguments towards a ‘working’ definition)

Integrability, integral, differential equations

Poincaré’s definition:
integrate a differential equation is to find for the general solution a finite expression, in a finite number of functions.
(Singlevaluedness)

Integrability is a rare phenomenon
The typical dynamical system is nonintegrable
Study of a generic system only with computers

Integrable systems can be studied in detail
Algebraic and analytic methods available

But Integrability is structurally unstable

(pertinence of integrable systems?)

Calogero: "Integrable systems are both universal and widely applicable"

Novikov: "Scientists do not believe that the laws of nature are to be expressed by arbitrarily chosen equations"

Segur:

Mathematics is the study of abstract strustures and relationships
Physics is the study of the structure of our universe
Sciences is the search for structure which, when found, is encoded in laws.

Who cares about integrability?

Newton: equations of motion of two gravitating bodies

$$V = \frac{1}{|\vec{x}_1 - \vec{x}_2|}$$

*Super*integrable system
(Laplace-Runge-Lenz vector)

Study of differential equations in the complex domain
(Why?)

From local solutions to global results

critical point ('branch point'):
multivaluedness

Critical singularities of a linear ODE are *fixed*: the solution af any linear ODE defines a function

Every linear ODE is integrable

Can we define new functions from *nonlinear* equations?
Difficulty:
Movable critical singularities!

Fuchs and Painlevé: first order equation without movable critical singularities

Riccati equation

$$w' = aw^2 + bw + c$$

Linearizable ($w = F/G$), no new functions

Painlevé

Second order equations without critical movable singularities

$$w'' = f(w', w, z)$$

with f polynomial in w', rational in w and analytic in z

Six equations that define new functions

$$w'' = 6w^2 + z$$

$$w'' = 2w^3 + zw + a$$

$$w'' = \frac{w'^2}{w} - \frac{w'}{z} + \frac{1}{z}(aw^2 + b) + cw^3 + \frac{d}{w}$$

$$w'' = \frac{w'^2}{2w} + \frac{3w^3}{2} + 4zw^2 + 2(z^2 - a)w - \frac{b^2}{2w}$$

etc.

Painlevé transcendents.

Kowalevskaya: study of integrability of a heavy spinning top

$$A\frac{dp}{dt} = (B - C)qr + Mg(\gamma y_0 - \beta z_0)$$

$$B\frac{dq}{dt} = (C - A)pr + Mg(\alpha z_0 - \gamma x_0)$$

$$C\frac{dr}{dt} = (A - B)pq + Mg(\beta x_0 - \alpha y_0)$$

$$\frac{d\alpha}{dt} = \beta r - \gamma q$$

$$\frac{d\beta}{dt} = \gamma p - \alpha r$$

$$\frac{d\gamma}{dt} = \alpha q - \beta p$$

Integrals $\alpha^2 + \beta^2 + \gamma^2 = 1$

$$Ap^2 + Bq^2 + Cr^2 - 2Mg(\alpha x_0 + \beta y_0 + \gamma z_0) = K_1$$

$$A\alpha p + B\beta q + C\gamma r = K_2$$

Fourth integral only for:

Spherical: $A = B = C$ with integral $px_0 + qy_0 + rz_0 = K$

Euler: $x_0 = y_0 = z_0$ with integral $A^2p^2 + B^2q^2 + C^2r^2 = K$

Lagrange: $A = B$ and $x_0 = y_0 = 0$ with integral $Cr = K$
and
Kowalevskaya: $A = B = 2C$ and $z_0 = 0$ with integral

$$[C(p+iq)^2 + Mg(x_0 + iy_0)(\alpha + i\beta)]$$
$$[C(p-iq)^2 + Mg(x_0 - iy_0)(\alpha - i\beta)] = K$$

Result obtained with singularity analysis methods

Solitary waves

Korteweg-de Vries equation
Propagation of long, one-dimensional, small amplitude, surface gravity waves in a shallow water channel

$$\frac{\partial \eta}{\partial \tau} = \frac{3}{2}\sqrt{\frac{g}{h}}\frac{\partial}{\partial \xi}\left(\frac{\eta^2}{2} + \frac{2\alpha\eta}{3} + \frac{\sigma}{3}\frac{\partial^2 \eta}{\partial \xi^2}\right)$$

Nondimensional form

$$u_t + 6uu_x + u_{xxx} = 0$$

Solitary wave solution

$$u(x,t) = 2\kappa^2 \operatorname{sech}^2(\kappa(x - 4\kappa^2 t - x_0))$$

Fermi, Pasta and Ulam
Lattice of coupled anharmonic oscillators:

$$m\ddot{x}_n = k(x_{n+1} + x_{n-1} - 2x_n)[1 + \alpha(x_{n+1} - x_{n-1})]$$

No energy equilibration but recurrence

Kruskal and Zabusky: continuous limit is KdV!
Confirmation of recurrence
Discovery of “solitons” (solitary waves interacting elastically)

Properties of KdV equation:
- ∞ number of conservation laws
- Miura transformation to modified KdV:

$$v_t + 6v^2 v_x + v_{xxx} = 0$$

- Arbitrary number of solitons (Hirota, bilinear formalism)
- Linearization

Linear time-independent Schrödinger problem

$$\Phi_{xx} + u\Phi = \lambda\Phi$$

and

$$\Phi_t = u_x\Phi + (4\lambda + 2u)\Phi_x$$

with u, the solution of KdV

Compatibility of the two ($\lambda_t = 0$)
$\rightarrow$ KdV for u

Quantum Mechanical Inverse Scattering (IST)

Lax formulation $L\Phi = \lambda\Phi$, $\Phi_t = M\Phi$
Compatibility

$$L_t + [L, M] = 0$$

Integrable nonlinear lattice
Toda system:

$$\frac{d^2x_n}{dt^2} = e^{x_{n+1}-x_n} + e^{x_n-x_{n-1}}$$

- ∞-number of conservation laws
- Lax pair

More integrable PDE's

Nonlinear Schrödinger equation

$$iu_t + u_{xx} + \kappa|u|^2u = 0$$

Sine-Gordon equation

$$u_{xt} = \sin u$$

More discoveries on integrability

Ablowitz-Segur
Linearization of the Painlevé equations (IST)

Painlevé property and Integrability:
Reductions of integrable PDEs are of Painlevé type

Integrability detector

Singularity analysis ("Painlevé method")

Ablowitz-Ramani-Segur algorithm

Fixed and movable singularities
Linear ODE's: only fixed singularities

Nonlinear eqs.: also movable singularities

$w' + w^2 = 0$ with solution $w = (z - z_0)^{-1}$

$2w' + w^3 = 0$ with solution $w = (z - z_0)^{-1/2}$

$ww'' - w' + 1 = 0$ with solution $w = (z - z_0)\ln(z - z_0) + \alpha(z - z_0)$

$\mu ww'' - (1 - \mu)w'^2 = 0$ with solution $w = \alpha(z - z_0)^{\mu}$

$(ww'' - w'^2)^2 + 4zw'^3 = 0$ with solution $w = \alpha e^{(z-z_0)^{-1}}$

$(1 + w^2)w'' + (1 - 2w)w'^2 = 0$ with solution $w = \tan[\alpha + \ln(z - z_0)]$

Ablowitz-Ramani-Segur algorithm
necessary condition for the absence of movable branch points
(movable essential singularities cannot be detected)

$$w_i' = F_i(w_1, w_2, \ldots, w_n; z) \qquad i = 1, \ldots, n$$

Assumption

$$w_i \sim \alpha_i (z - z_0)^{p_i}, \qquad z \to z_0$$

(Dominant logarithmic branches ?)

In some cases w_i do not diverge and only some higher derivative becomes singular

How to treat w_i^*

Step 1: Dominant Behaviours

$$w_i = a_i(z - z_0)^{p_i}$$

z_0 is arbitrary.
One must find all possible dominant behaviours

Example

$$x' = x(a - x - y) \qquad y' = y(x - 1)$$

We set $p = p_1$ and $q = p_2$ ($\tau = z - z_0$)

$$x = \alpha\tau^p \quad y = \beta\tau^q$$

$\rightarrow p = -1$, $q = \alpha$, with
either $\alpha = +1$ and β free or $\alpha = -1$ and $\beta = 2$.
Two leading behaviours:
(i) $x = \tau^{-1},\ y = \beta\tau$ (leading terms $x' = -x^2$, $y' = xy$)
(ii) $x = \tau^{-1},\ y = 2\tau^{-1}$ (leading terms $x' = -x^2 - xy$, $y' = xy$)

Step 2: Resonances.
Leading terms

$$w_i = a_i \tau^{p_i}(1 + \gamma_i \tau^r), \quad r > 0, \quad i = 1, \ldots, n$$

$$Q(r)\gamma = 0, \qquad \gamma = (\gamma_1 \ldots, \gamma_n)$$

where $Q(r)$ is an $n \times n$ matrix
Resonances from

$$\det Q(r) = (r+1)(r^{n-1} + A_2 r^{n-2} + \ldots + A_n) = 0$$

Example
$\gamma = \gamma_1$, $\delta = \gamma_2$
case (i) Resonances $r = -1$ and $r = 0$
case (ii) Resonances $r = -1$, $r = 2$.

The constants of Integration.
Truncated expansion to substitute in full equation

$$w_i = \alpha_i \tau_i^{p_i} + \sum_{1}^{r_s} a_i^{(m)} \tau^{p_i+m}$$

$\rightarrow$ compatibility condition

$$Q(m)a^{(m)} = R^{(m)}(z_0; a^{(j)}), \qquad j = 1, \ldots, m-1$$

If resonance condition is not satisfied then

$$w_i = \sum_{0}^{r-1} a_i^{(m)} \tau^{p_i+m} + (a_i^{(r)} + b_i^{(r)} \ln\tau)\tau^{p_i+r} + \ldots$$

Example: case (i) is OK, but case (ii) has one resonance at $r = 2$
Expand $x = -\tau^{-1} + a_1 + a_2\tau + \ldots \qquad y = 2\tau^{-1} + b_1 + b_2\tau + \ldots$

Compatibility condition $a = -1$

Painlevé cases are integrable

Example for $a = -1$
New variables

$$X = e^z x, \quad Y = e^z y, \quad Z = e^{-z}$$

lead to

$$X' = X^2 + XY, \qquad Y' = -XY$$

reduction to

$$Y' - \frac{1}{2}Y^2 = 2c_1^2$$

Integrated to $Y(Z) = 2c_1 \tan[c1(Z + c2)]$, c_1, c_2 free constants

ARS approach not failsafe
(movable essential critical singularities?)

2-D Hamiltonian with cubic potential

$$H = \frac{1}{2}\left(p_x^2 + p_y^2\right) + y^3 + ay^2x + byx^2 + cx^3$$

'rotation' to

$$V(x, y) = y^3 + byx^2 + cx^3$$

Painlevé analysis:

$$\ddot{x} = -2bxy - 3cx^2 \qquad \ddot{y} = -3y^2 - bx^2$$

(i) $x \propto \alpha\tau^{-2} \quad y \propto \beta\tau^{-2}$ (ii) $x \propto \tau^s \quad y \propto -2\tau^{-2}$ with $s(s-1) = 4b$
(s must be an integer or for $c = 0$ also half-integer)

At leading order
$6 = -2b\beta - 3c\alpha \qquad 6\beta = -3\beta^2 - b\alpha^2$

Resonances' equation:

$$(N + 2\beta b + 6\alpha c)(N + 6\beta) - 4b^2 a^2 = 0$$

where $N = (r-2)(r-3)$

Resonances

$r = -1$ and $r = 6$ or $N = (2b - 6)\beta$

N_1 and N_2 corresponding to β_1, β_2

$N_1 + N_2 = -2(2b-6) + \frac{1}{9} N_1 N_2 b$

Introducing $N_3 = s(s-1)$, we obtain

$$36(N_1 + N_2 + N_3 - 12) = N_1 N_2 N_3$$

with N_i's consecutive integers

Painlevé cases (all integrable) ($N_1 = N_2 = 6$ leads to logarithms)

a) $N_1 = 0$, $N_2 = 12$, $N_3 = 0$, separable potential

$$V = y^3 + \lambda x^3$$

b) $N_1 = 90$, $N_2 = 90$, $N_3 = 3/4$, with $c = 0$ and $s = -1/2$ yielding

$$V = y^3 + \frac{3}{16} y x^2$$

c) $N_1 = 30$, $N_2 = 30$, $N_3 = 2$ giving

$$V = y^3 + \frac{1}{2} y x^2$$

d) $N_1 = 20$, $N_2 = 90$, $N_3 = 2$, with

$$V = y^3 + \frac{1}{2} y x^2 + \frac{i}{6\sqrt{3}} i x^3$$

Painlevé:

$$F(x', x, t) = 0$$

F polynomial in x' and x, analytic in t
Movable singularities: poles and/or algebraic branch points

Fuchs:

$$x' = f(x, t)$$

f is rational in x and analytic in t
Only Riccati equation has Painlevé property

$$x' = a(t)x^2 + b(t)x + c(t)$$

Integration:
if $a = 0$ linear, otherwise $x = -\frac{u'}{au}$ and linearize

$$au'' - (a' + ab)u' + a^2cu = 0$$

Painlevé:

$$x'' = f(x', x, t)$$

f rational in x', polynomial in x and analytic in t

Rather than the Painlevé α-method present Gambier's method
Start with

$$x'' = x^2 + f(t)$$

and put

$$x \sim a\tau^p$$

where $\tau = t - t_0$
We find $p = -2$ and $a = 1$
Next look for the power of τ at which a second constant appears
(Fuchs: "index", ARS "resonance")

$$x = \tau^{-2} + \gamma\tau^{r-2}$$

Linearizing for γ:

$$(r-2)(r-3)-12=0$$

roots $r=-1$ and $r=6$
Compatibility condition at $r=6$

$$d^2f/dt^2=0$$

Only (nontrivial) Painlevé case:
the $\mathrm{P_I}$ equation

$$x''=6x^2+t$$

Painlevé:
$\mathrm{P_I}$ is free of movable essential singularities

One of Gambier's fundamental remarks:

Je rencontrais des systèmes de conditions différentielles dont l'intégration était, quoiqu'au fond bien simple, assez difficile à apercevoir. Par un mécanisme qui est général, mais qui était difficile à prévoir, la résolution de ce premier problème, intégration des conditions, est intimement liée à l'intégration de l'équation différentielle elle-même.

In other words, the integration of the (integrability) conditions is intimately related to the integration of the nonlinear equation itself.

Properties of Painlevé equations:

- they form coalescence cascades,
- they possess Lax pairs,
- their solutions are related through Bäcklund and Miura transformations,
- they have particular solutions in terms of special functions or rational solutions for special values of their parameters. These solutions can be written in terms of Wronskians,
- they can be cast into bilinear forms,
- they can be written as Hamiltonian systems

Hirota-sensei in the mid 70's

Observation: N-soliton of KdV

$$u = 2\frac{\partial^2}{\partial x^2}\log F \qquad (*)$$

(F determinant of some matrix)

Use (*) in KdV

$$FF_{xt} - F_tF_x + FF_{xxxx} - 4F_xF_{xxx} + 3F_{xx}^2 = 0$$

Bilinear form!

Two soliton solution

$$F = 1 + \exp(\eta_1) + \exp(\eta_2) + A_{12}\exp(\eta_1 + \eta_2)$$

with $\eta_i = k_i x - k_i^3 t + \phi_i$ and $A_{12} = (k_1 - k_2)^2/(k_1 + k_2)^2$

Introducing the bilinear (Hirota) operator

$$D_x F \cdot G = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right) F(x) G(x') |_{x'=x}$$

Rewrite KdV

$$(D_x D_t + D_x^4) F \cdot F = 0$$

For modified-KdV

$$v_t + 6v^2 v_x + v_{xxx} = 0$$

we put

$$v = \frac{G}{F}$$

and find

$$(D_t + D_x^3) F \cdot G = 0$$

$$D_x^2 F \cdot F = 2G^2$$

Moving to discrete

The Hirota-Satsuma nonlinear network

$$\frac{d^2}{dt^2}\log(1+u_n) = u_{n+1} - 2u_n + u_{n-1}$$

Introduce

$$u_n = \frac{d^2}{dt^2}\log F_n$$

Bilinearisation

$$D_t^2 F_n \cdot F_n = 2(F_{n+1}F_{n-1} - F_n^2)$$

Discrete Hirota operator

$$e^{D_n} F_n \cdot G_n = F_{n+1}G_{n-1}$$

and

$$D_t^2 F_n \cdot F_n = 2(\cosh D_n - 1) F_n \cdot F_n$$

Discretising the KdV equation

Semi-discrete form

$$\frac{d}{dt}\frac{w_n}{1+w_n} = w_{n-1/2} - w_{n+1/2}$$

Semi-discretisation

$$D_x(D_t+D_x^3)F\cdot F = 0 \;\rightarrow\; \sinh\left(\frac{D_n}{4}\right)\left(D_t + 2\sinh\left(\frac{D_n}{2}\right)\right) F_n\cdot F_n = 0$$

Full discretisation

$$\sinh\left(\frac{D_n+\delta D_t}{4}\right)\left(\frac{2}{\delta}\sinh\left(\frac{\delta D_t}{2}\right) + 2\sinh\left(\frac{D_n}{2}\right)\right) F_n\cdot F_n = 0$$

Reduction to more familiar form

Introduce

$$u_n = \frac{\cosh(D_n/2)F_n \cdot F_n}{\cosh(\delta D_t/2)F_n \cdot F_n}$$

and obtain

$$\delta(u_{n+1/2}(t) - u_{n-1/2}(t)) = \frac{1}{u_n(t+\delta/2)} - \frac{1}{u_n(t-\delta/2)}$$

Finally

$$U_{n+1}^{m+1} - U_n^m = \frac{1}{U_{n+1}^m} - \frac{1}{U_n^{m+1}}$$

and potential form

$$w_{n+1}^{m+1} - w_n^m = \frac{1}{w_{n+1}^m - w_n^{m+1}}$$

Many more disrete equations were derived by Hirota-sensei

Modified-KdV

$$u_{n+1}^{m+1} = u_n^m \frac{u_n^{m+1} + \mu u_{n+1}^m}{\mu u_n^{m+1} + u_{n+1}^m}$$

sine-Gordon

$$u_{n+1}^{m+1} u_n^m = \frac{1 + \mu u_n^{m+1} u_{n+1}^m}{\mu + u_n^{m+1} u_{n+1}^m}$$

Also linearisable equations

Liouville ($\phi_{xt} = \exp(-2\phi)$)

$$u_{n+1}^m u_{n-1}^m - u_n^{m+1} u_n^{m-1} = 1$$

Burgers

$$u_n^{m+1} = u_n^m \frac{1 + \mu u_{n+1}^m}{1 + \mu u_n^m}$$

Most important discovery:
the discrete form of the 2+1 dim Kadomtsev-Petviashvili equation

The Hirota equation

$$\Big(z_1 \exp(D_1) + z_2 \exp(D_2) + z_3 \exp(D_3)\Big) F \cdot F = 0 \qquad (*)$$

From (*) obtain many integrable lattice equations by reduction

Hirota-sensei used the bilinear formalism to obtain
Soliton solutions
Bäcklund transformations
Lax pairs

A brief introduction to the QRT mapping

TTwo families: symmetric and asymmetric

Start with 3×3 matrices, A_0 and A_1 and vector $\vec{X}$

$$A_i = \begin{pmatrix} \alpha_i & \beta_i & \gamma_i \\ \delta_i & \epsilon_i & \zeta_i \\ \kappa_i & \lambda_i & \mu_i \end{pmatrix} \quad \text{and} \quad \vec{X} = \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix}$$

Construct $\vec{F} \equiv \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$ and $\vec{G} \equiv \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}$

$$\vec{F} = (A_0\vec{X}) \times (A_1\vec{X}) \quad \text{and} \quad \vec{G} = (\widetilde{A}_0\vec{X}) \times (\widetilde{A}_1\vec{X})$$

The f_i, g_i are, in general, quartic polynomials of x

Asymmetric mapping

$$x_{n+1} = \frac{f_1(y_n) - x_n f_2(y_n)}{f_2(y_n) - x_n f_3(y_n)}$$

$$y_{n+1} = \frac{g_1(x_{n+1}) - y_n g_2(x_{n+1})}{g_2(x_{n+1}) - y_n g_3(x_{n+1})}$$

Symmetric

$$x_{m+1} = \frac{f_1(x_m) - x_{m-1} f_2(x_m)}{f_2(x_m) - x_{m-1} f_3(x_m)}$$

with dentification $x_n \to x_{2n}$, $y_n \to x_{2n+1}$

Parameter counting: 8 for the asymmetric and 5 for the symmetric

Invariant relation (biquadratic in x and y)

$\alpha x_n^2 y_n^2 + \beta x_n^2 y_n + \gamma x_n^2 + \delta x_n y_n^2 + \epsilon x_n y_n + \zeta x_n + \kappa y_n^2 + \lambda y_n + \mu = 0$

where $\alpha \equiv \alpha_0 + K\alpha_1$ etc. and K integration constant

Integration of QRT mapping: symmetric case is well-known

$$\alpha x^2y^2 + \beta xy(x+y) + \gamma(x^2+y^2) + \epsilon xy + \zeta(x+y) + \mu = 0$$

Through homographic transformation (common to x and y)

$$X^2Y^2 + \Gamma(X^2+Y^2) + EXY + 1 = 0$$

Elliptic functions: $X = A\,\mathrm{sn}(z)$,$Y = A\,\mathrm{sn}(z+q)$ modulus k ($A^2 = k$)

$$k^2 + \big(\Gamma + \frac{1}{\Gamma} - \frac{E^2}{4\Gamma}\big)k + 1 = 0$$

Step q given by $\Gamma k\,\mathrm{sn}^2(q) + 1 = 0$

Asymmetric case: integrated in a similar way.

A special case, $\alpha = \beta = 0$

Linearisable case (solution in terms of the exponential function)

$$\gamma(x_{n+1}^2 + x_n^2) + \epsilon x_{n+1} x_n + \zeta(x_{n+1} + x_n) + \mu = 0$$

Canonical form

$$x_{n+1}^2 + x_n^2 + \epsilon x_{n+1} x_n + 1 = 0$$

Solution

$$x_n = \frac{\phi_n}{p} + \frac{q}{\phi_n}$$

with

$$\phi_{n+1} = \lambda \phi_n$$

λ is given by

$$\lambda^2 + \epsilon\lambda + 1 = 0 \quad \text{and} \quad \frac{p}{q} = \epsilon^2 - 4$$

Solution of generic QRT mapping: sampling of an elliptic function

Why are QRT mappings pertinent?

Continuous Painlevé equations:
non-autonomous extensions of elliptic functions

This means:
same functional forms as the autonomous equations
with coeffs depending on the independent variable

Strategy for the derivation of discrete analogues:

Start from QRT
allow coeffs to depend on independent variable
select the integrable cases (through integrability detector)

Do integrable mappings have the Painlevé property?

Singularity confinement criterion

Lattice KdV equation

$$x_j^{i+1} = x_{j+1}^{i-1} + \frac{1}{x_j^i} - \frac{1}{x_{j+1}^i}$$

"what if a singularity appears spontaneously?"

$x = 0$ at (i, j)
$x = \infty$ at both $(i+1, j-1)$ and $(i+1, j)$
and $x = 0$ at $(i+2, j-1)$

At $(i+3, j-2)$ and $(i+3, j-1)$ finite values!

The singularity does not propagate beyond a few lattice points: it is confined
Discrete Painlevé property

An example

$$x_{n+1} + x_{n-1} = \frac{a}{x_n} + \frac{1}{x_n^2}$$

Singularity, whenever x_n=0
Iterate → sequence $\{0, \infty, 0\}$
and then *indeterminate form* $\infty - \infty$

Kruskal:
The real problem is the indeterminate form not the simple infinity
Solution
Use continuity with respect to the initial conditions
Introduce a small parameter ϵ
Start from $x_n = \epsilon$, obtain: $x_{n+1} \approx 1/\epsilon^2$, $x_{n+2} \approx -\epsilon$
Compute carefully x_{n+3}
Finite and depends on initial condition x_{n-1}
The singularity has disappeared!

Consider the McMillan mapping:

$$x_{n+1} + x_{n-1} = \frac{2\mu x_n}{1 - x_n^2}$$

Singularity: whenever x passes through ± 1

Assume, x_0 is finite and $x_1 = 1 + \epsilon$

We find:
$x_2 = -\mu/\epsilon - (x_0 + \mu/2) + \mathcal{O}(\epsilon)$,
$x_3 = -1 + \epsilon + \mathcal{O}(\epsilon^2)$
$x_4 = x_0 + \mathcal{O}(\epsilon)$

Singularity confined
and
mapping recovered memory of the initial conditions through x_0

Deautonomise the McMillan mapping

$$x_{n+1} + x_{n-1} = \frac{a(n) + b(n)x_n}{1 - x_n^2}$$

Assume: regular x_n and $x_{n+1} = \sigma + \epsilon$ where $\sigma = \pm 1$
Compute
x_{n+2}(infinite) and x_{n+3}($= -\sigma$at lowest order)
Condition for x_{n+4} to be finite:

$$b_{n+1} - 2b_{n+2} + b_{n+3} + \sigma(a_{n+1} - a_{n+3}) = 0$$

Solution:
$b_n(\equiv z_n) = \alpha n + \beta$ and $a_n = \delta + \gamma(-1)^n$
Ignore even-odd dependence (a=constant)

$$x_{n+1} + x_{n-1} = \frac{a + z_n x_n}{1 - x_n^2}$$

Discrete form of $\mathrm{P_{II}}$!

d-$\mathrm{P_I}$ from singularity confinement (deautonomisation)

$$x_{n+1} + x_n + x_{n-1} = a(n) + \frac{b(n)}{x_n}$$

Assume: x_n regular and x_{n+1} vanishes

$x_{n+1} = \epsilon$

$x_{n+2} = \frac{b_{n+1}}{\epsilon} + a_{n+1} - x_n + \mathcal{O}(\epsilon)$

$x_{n+3} = -\frac{b_{n+1}}{\epsilon} + a_{n+2} - a_{n+1} + x_n + \mathcal{O}(\epsilon)$

x_{n+4} diverges unless $a_{n+3} - a_{n+2}$=0 (for confinement a=constant)

For x_{n+5} finite, second condition: $b_{n+1} - b_{n+2} - b_{n+3} + b_{n+4} = 0$

Solution $b_n = \alpha n + \beta + \gamma(-1)^n$

If we ignore even-odd dependence: $b_n \equiv z_n = \alpha n + \beta$

$$x_{n+1} + x_n + x_{n-1} = a + \frac{z_n}{x_n}$$

Our conjecture (no known counterexample)

All mappings integrable through spectral methods
have confined singulmarities

The Hietarinta-Viallet (H&V) discovery:

Confinement is not sufficient for integrability

Integrability related to low–growth properties (complexity)

Mapping of degree d
$\rightarrow$ n-th iterate: degree d^n, unless there exist simplifications

Integrable mappings: massive simplifications
$\rightarrow$ polynomial degree growth

Algebraic entropy: $\lim_{n\rightarrow\infty} \frac{\log d_n}{n}$

Example

$$x_{n+1} + x_{n-1} = \frac{a}{x_n} + \frac{1}{x_n^2}$$

Introduce homogeneous coordinates

$$x_0 = r,\ x_1 = p/q$$

Assume r to be of degree zero
and compute the degree of homogeneity in p and q at every iteration

Obtain the degrees:

$0, 1, 2, 5, 8, 13, 18, 25, 32, 41, \ldots,$

Degree growth is polynomial: $d_{2m} = 2m^2$ and $d_{2m+1} = 2m^2 + 2m + 1$

The mapping is integrable (QRT)

Nonintegrable mapping, (the H&V) example

$$x_{n+1} + x_{n-1} = x_n + \frac{1}{x_n^2}$$

Singularity pattern is $\{0, \infty, \infty, 0\}$
but
chaotic behaviour

Degree growth: 0, 1, 3, 8, 23, 61, 162, 425, ...,

Exponential!

$d_{n+4} = 3(d_{n+3} - d_{n+1}) + d_n$ with ratio of $(3 + \sqrt{5})/2$

The linearisable case

A Gambier mapping

$$x_{n+1}x_{n-1} - x_{n-1}x_n = \frac{x_n^2}{1 - x_n}$$

Degree growth: 0, 1, 2, 3, 4, 5, 6, 7, . . . ,

A so-called "third-kind" mapping

$$\frac{1}{x_{n+1} + x_n} + \frac{1}{x_n + x_{n-1}} = \frac{1}{x_n} + 1$$

Degree growth: 0, 1, 3, 5, 7, 9,. . . ,

In both cases, linear growth (but different steps)

Algebraic entropy is *not necessary*

Simplest example

$$x_{n+1}x_{n-1} = x_n^3$$

Put $\omega_n = \log x_n$ and find for ω a linear equation

$$\omega_{n+1} + \omega_{n-1} = 3\omega_n$$

Algebraic entropy $\epsilon = \log((3+\sqrt{5})/2)$

Another example

$$x_{n+1} = \frac{3x_n - x_n^3 + x_{n-1}(1-3x_n^2)}{1-3x_n^2 + x_{n-1}(3x_n - x_n^3)}$$

Put $\omega_n = \tan x_n$ and find for ω the same linear equation

Infinitely many such examples exist

Application of singularity confinement to lattice eqs.

The KdV example

$$z_1 f(m+1,n)f(m-1,n-1) + z_2 f(m+1,n-1)f(m-1,n) \\ + z_3 f(m,n)f(m,n-1) = 0$$

Singularity:
when one of the f's becomes 0 or ∞ (0 et previous step)

Singularity confinement:
the vanishing of an f *never* induces a divergence at the next stage

Condition for the vanishing of $f(m,n)=0$:

$$z_1 f(m+1,n)f(m-1,n-1) + z_2 f(m+1,n-1)f(m-1,n) = 0$$

The vanishing of $f(m,n)$ may lead to a diverging $f(m,n+1)$

This does not happen!

Compute $f(m \pm 2, n)$ and $f(m \pm 1, n+1)$

$$z_3 f(m+1, n+1) f(m+1, n) + z_2 f(m+2, n) f(m, n+1) = 0$$

$$z_3 f(m-1, n+1) f(m-1, n) + z_1 f(m-2, n) f(m, n+1) = 0$$

$$z_3 f(m-1, n) f(m-1, n-1) + z_2 f(m-2, n) f(m, n-1) = 0$$

$$z_3 f(m+1, n) f(m+1, n-1) + z_1 f(m+2, n) f(m, n-1) = 0$$

Eliminating $f(m \pm 2, n)$ we find:

$$z_1 f(m+1, n+1) f(m-1, n) + z_2 f(m+1, n) f(m-1, n+1) = 0$$

This guarantees a finite value for $f(m, n+1)$

Deautonomising the Hirota equation

$$z_1(k,m,n)\tau(k-1,m,n)\tau(k+1,m,n) \\ +z_2(k,m,n)\tau(k,m-1,n)\tau(k,m+1,n) \\ +Z_3(k,m,n)\tau(k,m,n-1)\tau(k,m,n+1)=0$$

For singularity confinement:
assume $\tau(k,m,n)=0$

$$z_2(k-1,m,n)\tau(k-1,m-1,n)\tau(k-1,m+1,n) \\ +z_3(k-1,m,n)\tau(k-1,m,n-1)\tau(k-1,m,n+1)=0$$

while $\tau(k-2,m,n)$ finite

$$z_2(k+1,m,n)\tau(k+1,m-1,n)\tau(k+1,m+1,n) \\ +z_3(k+1,m,n)\tau(k+1,m,n-1)\tau(k+1,m,n+1)=0$$

Compute the necessary τ's at $(k, m \pm 1, n)$ and at $(k, m, n \pm 1)$

We find the confinement condition is satisfied provided:

$$
\begin{aligned}
&z_1(k, m-1, n) z_1(k, m+1, n) z_2(k, m, n-1) z_2(k, m, n+1) \\
&\quad \times z_3(k-1, m, n) z_3(k+1, m, n) = z_1(k, m, n-1) z_1(k, m, n+1) \\
&\qquad\qquad \times z_2(k-1, m, n) z_2(k+1, m, n) z_3(k, m-1, n) z_3(k, m+1, n)
\end{aligned}
$$

Automatic for *constant* z's
However, by gauge $z_2 = z_3$ and by division $z_2 = z_3 = 1$
Condition

$$
z_1(k, m-1, n) z_1(k, m+1, n) = z_1(k, m, n-1) z_1(k, m, n+1)
$$

Solution $z_1 = g(k, m+n) h(k, m-n)$ with g, h free functions

Unfortinately, a gauge transforms z_1 to 1 (back to autonomous)

Deautonomisation of potential KdV (find m, n dependence of z_n^m)

$$x_{n+1}^{m+1} = x_n^m + \frac{z_n^m}{x_n^{m+1} - x_{n+1}^m}$$

Degrees of the iterates for constant z:$d_n^m = mn + 1$

	$v_{n-1}ts$	$v_{n-1}ts$	$v_{n-1}ts$	$v_{n-1}ts$	$v_{n-1}ts$	$v_{n-1}ts$	$\iddots$
	1	4	7	10	13	16	$\cdots$
	1	3	5	7	9	11	$\cdots$
	1	2	3	4	5	6	$\cdots$
$m \uparrow$	1	1	1	1	1	1	$\cdots$

$\overrightarrow{}$
n

Deautonomisation:

degrees from autonomous and nonautonomous must be identical

First constraint: degree of x_2^2 must be 5 (and not 6)
Condition: $z_1^1 - z_0^1 - z_1^0 + z_0^0 = 0$

Same as from singularity confinement

Generically

$$z_{n+1}^{m+1} - z_n^{m+1} - z_{n+1}^m + z_n^m = 0$$

suffices

Solution: $z_n^m = f(n) + g(m)$ (f, g arbitrary functions)

Result known in convergence acceleration algorithms

Lattice mKdV

$$x_{n+1}^{m+1} = x_n^m \; \frac{x_n^{m+1} + q_n^m x_{n+1}^m}{q_n^m x_n^{m+1} + x_{n+1}^m}$$

Growth in autonomous case: $d_n^m = mn + 1$
Condition on z

$$q_{n+1}^{m+1} q_n^m - q_n^{m+1} q_{n+1}^m = 0$$

Solution $q_n^m = f(n) g(m)$

Reduction $x_n^{m+1} = x_{n+2}^m$
Introduce $y_n = x_{n+2}/x_{n+1}$

$$y_{n+1} y_{n-1} = \frac{1 + q_n y_n}{y_n (q_n + y_n)}$$

with q_n $q_{n+3} q_n = q_{n+1} q_{n+2}$
Solution: $\log q_n = an + b + c(-1)^n$
Equation is q-$\mathrm{P_{III}}$

Discrete Burgers equation

$$x_n^{m+1} = x_n^m \frac{1 + z_n^m x_{n+1}^m}{1 + z_n^m x_n^m}$$

For z constant: $d_n^m = m + 1$ Condition for same growth

$$z_{n+1}^m - z_n^m = 0$$

i.e. $z_n^m = g(m)$

Nonautonomous extension:
cannot be removed by gauge, is compatible with linearisability
Putting $x_n^m = X_{n+1}^m / X_n^m$ we find (f is arbitrary)

$$X_n^{m+1} = f(m)(X_n^m + g(m) X_{n+1}^m)$$

(Continuous Burgers also possesses nonautonomous extension)

The discrete Painlevé equations

Some historical results

Shohat (1939), orthogonal polynomials (Laguerre?)

$$x_{n+1} + x_{n-1} + x_n = \frac{z_n}{x_n} + 1$$

with $z_n = \alpha n + \beta + \gamma(-1)^n$
(many years later was recognised as d-$\mathrm{P_I}$)

Jimbo & Miwa (1981), contiguity relations of c-Painlevé equations
From $\mathrm{P_{II}}$:

$$x'' = 2x^3 + tx + \alpha$$

contiguity relation ($\alpha_n = n + \alpha_0$):

$$\frac{\alpha_n + {}^1\!/_2}{x_{\alpha_n+1} + x_{\alpha_n}} + \frac{\alpha_n - {}^1\!/_2}{x_{\alpha_n} + x_{\alpha_n-1}} = -(2x_{\alpha_n}^2 + t)$$

No continuous limit was derived!

Brézin & Kazakov (1990) Field-theoretical model of 2-D gravity
Recursion relation of Shohat
Computed the continuous limit!
Obtained $w'' = 6w^2 + t$, i.e. Painlevé I

Periwal & Shevitz (1990)
Obtained

$$x_{n+1} + x_{n-1} = \frac{z_n x_n}{1 - x_n^2}$$

Continuous limit $w'' = 2w^3 + tw$, i.e. Painlevé II

Nijhoff & Papageorgiou (1991)

Since similarity reduction of mKdV $\rightarrow$ P_{II}
Similarity reduction of disrete mKdV should give d-P_{II}
They found the same as Periwal & Shevitz

Derivation of discrete Painlevé equations

Start from QRT mapping:

$$x_{n+1} = \frac{f_1(x_n) - x_{n-1}f_2(x_n)}{f_2(x_n) - x_{n-1}f_3(x_n)}$$

and deautonomize

Rewrite QRT as:

$$f_3(x_n)\Pi - f_2(x_n)\Sigma + f_1(x_n) = 0$$

where $\Sigma = x_{n+1} + x_{n-1}$, $\Pi = x_{n+1}x_{n-1}$

Ask that this equation go over to c-Painlevé equation

Lattice parameter ϵ and obtain:

$$\Sigma = 2x + \epsilon^2 x'' + \mathcal{O}(\epsilon^4), \quad \Pi = x^2 + \epsilon^2(xx'' - x'^2) + \mathcal{O}(\epsilon^4)$$

Derivative part at continuous limit, $\epsilon \to 0$:

$$x'' = \frac{f_3(x)}{xf_3(x) - f_2(x)} x'^2 + g(x)$$

Proper choice of f_2, f_3

For $\mathrm{P_I}$ and $\mathrm{P_{II}}$ we have $f_3 = 0$

$$x_{n+1} + x_{n-1} + x_n = a + \frac{\alpha n + \beta + \gamma(-1)^n}{x_n}$$

and

$$x_{n+1} + x_{n-1} = \frac{x_n(\alpha n + \beta) + \delta + \gamma(-1)^n}{1 - x_n^2}$$

For P_{III} take $f_2 = 0$

$$x_{n+1}x_{n-1} = \frac{\kappa(n)x_n^2 + \zeta(n)x_n + \mu(n)}{x_n^2 + \beta(n)x_n + \gamma(n)}$$

rewrite as

$$x_{n+1}x_{n-1} = \frac{ab(x_n - cq_n)(x_n - dq_n)}{(x_n - a)(x_n - b)}$$

where a, b, c, and d are constants

From singularity confinement

$$q_n = q_0\lambda^n$$

Neglecting even-odd dependence, the continuous limit is P_{III}
(if we do not, we get PVI, as shown by Jimbo and Sakai)

Not a difference equation, but a q- (multiplicative) mapping

For d-P_{IV}

$$(x_{n+1} + x_n)(x_{n-1} + x_n) = \frac{(x_n^2 - a^2)(x_n^2 - b^2)}{(x_n + z_n)^2 - c^2}$$

a, b and c are constants

Algebraic entropy approach

Start from autonomous: if z_n is constant

degrees: d_n=0, 1, 3, 6, 11, 17, 24, …,

quadratic growth

For a generic z_n, sequence d_n=0, 1, 3, 6, 13,…,

Condition for $d_4 = 11$

z linear in n

Lax pairs

Difference equations
linear isospectral deformation problem:

$$\zeta\Phi_{n,\zeta} = L_n(\zeta)\Phi_n$$
$$\Phi_{n+1} = M_n(\zeta)\Phi_n$$

Compatibility condition:

$$\zeta M_{n,\zeta} = L_{n+1}M_n - M_nL_n$$

Multiplicative equations
q-difference linear isospectral problem:

$$\Phi_n(\lambda\zeta) = L_n(\zeta)\Phi_n(\zeta)$$
$$\Phi_{n+1} = M_n(\zeta)\Phi_n(\zeta)$$

Compatibility condition:

$$M_n(\lambda\zeta)L_n(\zeta) = L_{n+1}(\zeta)M_n(\zeta)$$

For d-P_{I} we start from Lax pair:

$$L_n = \begin{pmatrix} 0 & x_n & 1 \\ \zeta & z_n & x_{n+1} + z_n/x_n \\ \zeta x_{n-1} & \zeta & z_{n+1} \end{pmatrix}$$

and

$$M_n = \begin{pmatrix} -z_n/x_n & 1 & 0 \\ 0 & 0 & 1 \\ \zeta & 0 & 0 \end{pmatrix}$$

where $z_n = n/2 + \beta$

Consistency conditions:

$$x_{n+2} + z_{n+1}/x_{n+1} = x_{n-1} + z_n/x_n$$

Integrate (a is the integration constant)

$$x_{n+1} + x_{n-1} + x_n - \frac{z_n}{x_n} = a$$

A multiplicative equation

$$L_n = \begin{pmatrix} 0 & 0 & \frac{k_n}{x_n} & 0 \\ 0 & 0 & x_{n-1} & qx_{n-1} \\ hx_n & 0 & 1 & q \\ 0 & \frac{hk_{n-1}}{x_{n-1}} & 0 & 0 \end{pmatrix}$$

and

$$M_n = \begin{pmatrix} 0 & \frac{x_n}{k_n(x_n+1)} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{x_n} & \frac{q}{x_n} \\ h & 0 & 0 & 0 \end{pmatrix}$$

Compatibility:

$$x_{n+1}x_{n-1} = k_n k_{n+1}(x_n + 1)/x_n^2$$

where $k_{n+1} = qk_{n-1}$

q-Painlevé I

q-P$_{\mathrm{V}}$ $(q_n = q_0\lambda^n)$

$$(x_{n+1}x_n - 1)(x_nx_{n-1} - 1) = \frac{(x_n - a)(x_n - 1/a)(x_n - b)(x_n - 1/b)}{(1 - cx_nq_n)(1 - x_nq_n/c)}$$

δ-P$_{\mathrm{V}}$ $(z_n = \alpha n + \beta)$

$$\frac{(x_n + x_{n+1} - z_n - z_{n+1})(x_n + x_{n-1} - z_n - z_{n-1})}{(x_n + x_{n+1})(x_n + x_{n-1})} = \frac{((x_n - z_n)^2 - a^2)((x_n - z_n)^2 - b^2)}{(x_n^2 - c^2)(x_n^2 - d^2)}$$

q-P$_{\mathrm{VI}}$ $(q_n = q_0\lambda^n)$

$$\frac{(x_nx_{n+1} - q_nq_{n+1})(x_nx_{n-1} - q_nq_{n-1})}{(x_nx_{n+1} - 1)(x_nx_{n-1} - 1)} = \frac{(x_n - aq_n)(x_n - q_n/a)(x_n - bq_n)(x_n - q_n/b)}{(x_n - c)(x_n - 1/c)(x_n - d)(x_n - 1/d)}$$

Canonical forms of discrete Painlevé equations

Are they essentially symmetric? *Not true!*

Form singularity confinement
obtain terms of the form $(-1)^n$, but also j^n where $j^3 = 1$, i^n etc.

They should not be discarded because "they do not possess a continuous limit"

– They indicate that the equation is better written as a system of two, three, etc. equations
– They also introduce one or more extra, parameters
– They lead to richer continuous limits

Profusion of asymmetric forms
terminology with qualifier "asymmetric"

The limit of asymmetric d-$\mathrm{P_{II}}$ is $\mathrm{P_{III}}$
Of asymmetric q-$\mathrm{P_{III}}$ is $\mathrm{P_{VI}}$ (Jimbo & Sakai)
The limits of asym. d-$\mathrm{P_{IV}}$, q-$\mathrm{P_V}$, d-$\mathrm{P_V}$ and q-$\mathrm{P_{VI}}$ are $\mathrm{P_{VI}}$

Higher number of components:

$$x_{n+1}x_{n-1} = a(x_n - 1)$$

From singularity confinement (with $j^3 = 1$):

$$\log a_n = kn + p + rj^n + sj^{2n} + t(-1)^n$$

Can be written as a second-order system of six equations

Properties of discrete Painlevé equations

The discrete Painlevé equations have many special properties

Most are analogues of properties of continuous Painlevé equations

– Lax pairs (already discussed)

– Degeneration through coalescence

$$\begin{array}{ccccccccc}
q\text{-P}_{\text{VI}} & \longrightarrow & q\text{-P}_{\text{V}} & \longrightarrow & q\text{-P}_{\text{III}} & \longrightarrow & q\text{-P}_{\text{III}}^{1} & \longrightarrow & q\text{-P}_{\text{III}}^{0} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
d\text{-P}_{\text{V}} & \longrightarrow & d\text{-P}_{\text{IV}} & \longrightarrow & d\text{-P}_{\text{II}} & \longrightarrow & d\text{-P}_{\text{I}} & &
\end{array}$$

Convention
'higher' equation in capital letters
'lower' equation in lowercase letters
Introduce the coalescence limit: δ

From d-$\mathrm{P_{II}} \to$ d-$\mathrm{P_I}$
Start with the equation:

$$X_{n+1} + X_{n-1} = \frac{Z_n X_n + A}{1 - X_n^2}$$

Put $X = 1 + \delta x$:

$$4 + 2\delta(x_{n+1} + x_{n-1} + x_n) = -\frac{Z_n(1 + \delta x_n) + A}{\delta x_n}$$

$Z = -A - 2\delta^2 z$ to cancel A up to order δ
$\mathcal{O}(\delta^0)$ term in rhs must cancel 4 of lhs
so $A = 4 + 2\delta a$
At $\delta \to 0$:

$$x_{n+1} + x_{n-1} + x_n = \frac{z_n}{x_n} + a$$

precisely d-$\mathrm{P_I}$

Degeneration of d-$\mathrm{P_{III}}$ to d-$\mathrm{P_{II}}$

Start from:

$$X_{n+1}X_{n-1} = \frac{AB(X_n - P_n)(X_n - Q_n)}{(X_n - A)(X_n - B)}$$

Aansatz for X: , $X = 1 + \delta x$. For the remaining quantities we find:

$$A = 1 + \delta, \quad B = 1 - \delta$$

$$P = 1 + \delta + \delta^2(z + a)/2 + \mathcal{O}(\delta^3)$$

$$Q = 1 - \delta + \delta^2(z - a)/2 + \mathcal{O}(\delta^3)$$

At the limit $\delta \to 0$:

$$x_{n+1} + x_{n-1} = \frac{z_n x_n + a}{1 - x_n^2}$$

exactly d-$\mathrm{P_{II}}$

In the case of q-$\mathrm{P_V}$:

$$(X_{n+1}X_n - 1)(X_nX_{n-1} - 1)$$
$$= \frac{(X_n - A)(X_n - 1/A)(X_n - B)(X_n - 1/B)}{(1 - CX_nQ_n)(1 - X_nQ_n/C)}$$

two different limits exist

Limit to disrete $\mathrm{P_{IV}}$
Put $X = 1 + \delta x$, $\lambda = 1 - \alpha\delta$ and take:

$$A = 1 + \delta a, \quad B = 1 - \delta b$$

$$C = 1 + \delta c, \quad Q_n = 1 - \delta z_n$$

At the limit $\delta \to 0$ we find d-$\mathrm{P_{IV}}$

$$(x_{n+1} + x_n)(x_n + x_{n-1}) = \frac{(x_n^2 - a^2)(x_n^2 - b^2)}{(x_n - z_n)^2 - c^2}$$

Limit to discrete $\mathrm{P}_{\mathrm{III}}$

Put $X = x/\delta$ and take:

$$C = c, \quad Q_n = \frac{q_n}{\delta}, \quad A = \frac{a}{\delta}, \quad B = \frac{b}{\delta}$$

We find then at the limit $\delta \to 0$:

$$x_{n+1}x_{n-1} = \frac{(x_n - a)(x_n - b)}{(1 - cx_n q_n)(1 - x_n q_n/c)}$$

precisely q-$\mathrm{P}_{\mathrm{III}}$

– Special solutions

Elementary solutions for specific values of the parameters
Special functions (of hypergeometric type)
or rational
Example q-P_{V}

$$(x_{n+1}x_n - 1)(x_n x_{n-1} - 1 = \frac{pr(x_n - u)(x_n - 1/u)(x_n - v)(x_n - 1/v)}{(x_n - p)(x_n - r)}$$

Factorization

$$xx_{n+1} - 1 = \frac{p(x-u)(x-v)}{uv(xz-p)}$$

$$xx_{n-1} - 1 = \frac{uvr(x-1/u)(x-1/v)}{(xz-r)}$$

Compatibility $\quad uv = p/r\lambda$

$\rightarrow$ discrete Riccati

$$z(xx_{n+1}-1)=px_{n+1}+\lambda r(x-u-v)$$

Linearization

$$x_{n+1}=\frac{\lambda r(x-u-v)+z}{zx-p}$$

Cole-Hopf $x=B/A$

$$A_{n+2}+(p-r)A_{n+1}-(\lambda z^2-zr(u+v)+pr)A_n=0$$

discrete form of confluent hypergeometric

Rational solutions

$x = \pm 1$ when u or v equal ± 1

Nontrivial solutions

$$x = \pm 1 + (p + r)/z$$

for u (or $1/u$) $= \mp 1/\lambda$
and v (or $1/v$) $= \mp p/r$ (or $u \leftrightarrow v$)

Also

$$x = (p + r)/z$$

for $u = \sqrt{\lambda}$, $v = -\sqrt{\lambda}$

Solutions by direct linearisation

Instead of discrete Riccati equation

$$x_{n+1} = -\frac{\alpha x_n + \beta}{\gamma x_n + \delta}$$

when γ or β vanishes
we get linear equation for x_n or $1/x_n$

Some further constraint must be satisfied

integration of the linear equation

$$\delta_n x_{n+1} + \alpha_n x_n + \beta_n = 0$$

First solution, ξ_n of the homogeneous equation

$$\delta_n x_{n+1} + \alpha_n x_n = 0$$

Formally

$$\xi_n = A \prod_{k=0}^{n-1} (-\alpha_k/\delta_k)$$

With "variation of constant"

$$A_{n+1} - A_n = \frac{\beta_n}{\alpha_n \prod_{k=0}^{n-1} (-\alpha_k/\delta_k)}$$

Formally (c is the integration constant)

$$A_n = \sum_n \beta_n / (\alpha_n \prod_{k=0}^{n-1} (-\alpha_k/\delta_k)) + c$$

If $\beta\gamma = 0$ is impossible
we must find one special solution η_n of the Riccati equation
We set $x = \eta + 1/y$
and y satisfies the linear equation,

$$(\gamma_n \eta_{n+1} + \alpha_n) y_{n+1} + (\gamma_n \eta_n + \delta_n) y_n + \gamma_n = 0$$

q-discrete P_{III} $(z_n = \lambda^n)$

$$x_{n+1} x_{n-1} = \frac{(x_n - a)(x_n - b)}{(1 - x_n z_n / c)(1 - x_n z_n / d)}$$

Linearisability condition $ad = bc\lambda$ leads to

$$x_{n+1} = \frac{d}{\lambda} \frac{a - x_n}{c - x_n z_n}$$

Solutions in terms of discrete Bessel functions

This Riccati cannot be reduced to a linear

But can obtain one special solution

We find $x_n = \sqrt{ac/z_n}$ provided $c\sqrt{\lambda} + d = 0$ is satisfied

Putting $x_n = k/\sqrt{z_n} + 1/y_n$ we find

$$y_{n+1}(\sqrt{az_n/c} + 1) + \mu y_n(\sqrt{az_n/c} - 1) + \mu z_n/c = 0$$

Solution from

$$A_{n+1} - A_n = \frac{\sqrt{z}_n}{(\sqrt{acz_n} - c)\prod^{n-1} \tanh \frac{1}{4} \ln\left(\frac{c}{az_k}\right)}$$

Formally discrete quadrature needed

At the continuous limit the special solution goes precisely to the special solution of $\mathrm{P_{III}}$ in the form of a tangent

– Miura/auto-Bäcklund/Schlesinger transformations

The discrete Painlevé equations have many interrelations

– Miura transformations: relate two different equations

– auto-Bäcklund relate solutions of the same equation with different values of the parameter

– Schlesinger transformations are particular auto-Bäcklund transformations

Continuous Schlesingers relate solutions corresponding to the same monodromy data except for *integer* differences in the monodromy exponents

In the discrete case the analogy requires a proper parametrisation (auto-Bäcklund with elementary changes of the parameters can be dubbed Schlesinger)

Miura transformations for d-P_{II}

Introduce the system:

$$y_n = (1 + x_n)(1 - x_{n+1}) - z_{n+1/2}/2$$

$$x_n = \frac{m + y_n - y_{n-1}}{y_n + y_{n-1}}$$

Eliminating y we obtain d-P_{II}:

$$x_{n+1} + x_{n-1} = \frac{m - (z_{n+1} - z_n)/2 + z_n x_n}{1 - x_n^2}$$

Eliminating x we find

$$(y_{n+1} + y_n)(y_n + y_{n-1}) = \frac{4y_n^2 - m^2}{y_n + z_{n+1/2}/2})$$

Discrete form of the equation 34 (d-P_{34})

in the Painlevé/Gambier classification

Miura transformations for d-P_I:

$$x_{n+1} + x_{n-1} = \frac{z_n}{x_n} + \frac{a}{x_n^2}$$

Miura $y_n = x_n x_{n+1}$ leads to:

$$(y_n + y_{n-1} - z_n)(y_n + y_{n+1} - z_{n+1}) = \frac{a^2}{y_n}$$

Another form of d-P_I

Miura $y_n = x_{n+1}/x_n$ on discrete derivative of d-P_I $\to$ 4-point eq.

$$\frac{y_{n+1}y_n + 1 - y_{n+1}^2 y_n(y_{n+2}y_{n+1} + 1)}{y_n y_{n-1} + 1 - y_n^2 y_{n-1}(y_n y_{n+1} + 1)} = \frac{y_{n+1}z_{n+2} - z_{n+1}}{y_n z_{n+1} - z_n}\frac{1}{y_n y_{n-1}}$$

Continuous limit $ww''' = (w'' - 1)w' + 12w^3$ Integrate to

$$(w'' - 1)^2 - 24w^2(w' - t) = 0$$

i.e. Cosgrove's equation SD_V (modified P_I)

How can one find auto-Bäcklund transformations for a given d-P?

General principle

– Obtain a Miura that transforms the equation into a new one

– Use invariance of the latter under some discrete transformation

– Implement these transformations and return to the initial equation

In the process the parameters of initial equation have been modified

The chain of transformations defines an auto-Bäcklund

Clue: all known Miura's are homographic mappings

– Quadratic, "folding", relations

d-$\mathrm{P_I}$ equation

$$x_{n+1} + x_{n-1} + x_n = \frac{z_n}{x_n} + t$$

Take $t = 0$ and multiply by x_n

Introduce $X_n = x_n^2$ and $y_n = x_n x_{n+1}$

Find $y_n + y_{n-1} + X_n = z_n$ and $X_n X_{n+1} = y_n^2$

Eliminating X

$$(y_{n+1} + y_n - z_{n+1})(y_n + y_{n-1} - z_n) = y_n^2$$

Another special form of a d-$\mathrm{P_I}$

The asymmetric d-$\mathrm{P_{II}}$

$$y_n + y_{n-1} = \frac{z_n x_n + a}{x_n^2 - 1}$$

$$x_n + x_{n+1} = \frac{z_{n+1/2} y_n + b}{y_n^2 - 1}$$

Folding when $a = b = 0$

$$v_{n-1} + v_{n+1} = \frac{z_n v_n}{v_n^2 - 1}$$

Multiply by v_n and introduce

$$X_n = v_n^2 \quad \text{and} \quad W_n = v_n v_{n+1}$$

$$W_n + W_{n-1} = \frac{z_n X_n}{X_n - 1}$$

$$X_n X_{n+1} = W_n^2$$

Eliminate X to find an equation for W:

$$\frac{(W_n + W_{n+1} - z_{n+1})(W_n + W_{n-1} - z_n)}{(W_n + W_{n+1})(W_n + W_{n-1})} = \frac{1}{W_n^2}$$

Miura transformed of the “alternate d-$\mathrm{P_{II}}$”

$$\frac{z_{n+1}}{1 + u_n u_{n+1}} + \frac{z_n}{1 + u_n u_{n-1}} = u_n - \frac{1}{u_n} + z_n + \mu$$

Contiguity relations
Start from continuous $\mathrm{P}_{\mathrm{III}}$

$$w'' = \frac{w'^2}{w} - \frac{w'}{t} + \frac{1}{t}(\alpha w^2 + \beta) + w^3 - \frac{1}{w}$$

Relations

$$w(-\alpha, -\beta) = -w(\alpha, \beta)$$

$$w(-\beta, -\alpha) = w^{-1}(\alpha, \beta))$$

$$w(-\beta - 2, -\alpha - 2) = w(\alpha, \beta)\left(1 + \frac{2 + \alpha + \beta}{t(\frac{w'}{w} + w + \frac{1}{w}) - 1 - \beta}\right)$$

Assume further $\alpha \neq \beta$

Start from $w(-\beta, -\alpha)$ find $w(\alpha - 2, \beta - 2)$ and eliminate w'

Obtain a relation between $w(\alpha-2, \beta-2)$, $w(\alpha, \beta)$ and $w(\alpha+2, \beta+2)$

One-dimensional 3-point mapping on the (α, β)-plane

Introduce independent variable $z = (\alpha + \beta + 2)/4$
and parameters $\mu = (\beta - \alpha - 2)/4, \kappa = -it/2$

Choose $x = i/w$ and

$$\frac{z_n}{x_{n+1}x_n + 1} + \frac{z_{n-1}}{x_n x_{n-1} + 1} = \kappa(-x_n + \frac{1}{x_n}) + z_n + \mu$$

Contiguity relation for the solutions of P_{III}
This is the "alternate" discrete Painlevé II

Alt-dP_{II} is a (discrete) Painlevé equation

So it must have Schlesinger transformations and contiguities

What is the evolution along the parameters?

Schlesinger transform of alt-d-P_{II}

$$x_n(\mu-1) = \frac{1}{x_n} + \frac{\mu(1+x_n x_{n-1})}{\kappa(1+x_n x_{n-1}) - z_{n-1}x_n}$$

Similarly

$$x_n(\mu+1) = \left(x_n - \frac{(\mu+1)(1+x_n x_{n-1})}{\kappa(1+x_n x_{n-1}) - z_{n-1}x_{n-1}}\right)^{-1}$$

Eliminate x_{n-1}

μ is now the independent variable (z is now a parameter)

$$\frac{\mu+1}{x_\mu x_{\mu+1}-1} + \frac{\mu}{x_\mu x_{\mu-1}-1} = \kappa(x_\mu + \frac{1}{x_\mu}) - \mu - z$$

Again the alternate d-P_{II} itself (self-duality).

Try to understand the underlying mathematical structures

Hamiltonian approach for Painlevé equations

$$\frac{dx}{dt} = \frac{\partial H}{\partial p}$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial x}$$

Hamiltonians for Painlevé II:

$$H(x,p) = p^2/2 - p(x^2 + t/2) - (\alpha + {}^1\!/_2)x$$

Hamiltonian equations are Miura relations:
Eliminate p find $\mathrm{P_{II}}$ equation for x
Eliminate x and find $\mathrm{P_{34}}$ for p

Geometrical description of Painlevé equations

Consider their birational transformations

Start $H(\alpha)$ for $\mathrm{P_{II}}$

$$\tilde{x} = -x + \frac{\alpha - 1/2}{p}, \qquad \tilde{p} = -p$$

Solution of $\mathrm{P_{II}}$ corresponding to Hamiltonian $H(\alpha - 1)$

Birational transformation is an auto-Bäcklund for $\mathrm{P_{II}}$

Group generated by the transformations $\alpha \to 1 - \alpha$ and $\alpha \to -\alpha$.

Realisation of the affine Weyl group with root system of type $A_1^{(1)}$

Okamoto: result valid for all Painlevé equations
$\mathrm{P_{II}}$ - $A_1^{(1)}$, $\mathrm{P_{III}}$ - $(2A_1^{(1)})$, $\mathrm{P_{IV}}$ - $A_2^{(1)}$, $\mathrm{P_{V}}$ - $A_3^{(1)}$, $\mathrm{P_{VI}}$ - $D_4^{(1)}$

Fundamental notion (Okamoto): τ-function

Relation to the Hamiltonian :

$$\frac{d}{dt}\log\tau = H$$

For Painlevé equations:
τ-function: entire function on the complex plane of the ind. variable

Birational transformations expressed in terms of the τ-function

$$x = \frac{d}{dt}\log\frac{\tau(\alpha-1)}{\tau(\alpha)}$$

Successive application of auto-B̈'s: sequence of τ-functions
$\rightarrow$ translation in the space of parameters

Okamoto: “space of initial conditions”

Continuous Painlevé equations: 2nd differential equations

Space of initial conditions should be $\mathbb{C}^2$

For some t_0 solution specified by data of function and derivative

(precautions for singular coefficients)

But there exist solutions diverging at t_0

Must compactify $\mathbb{C}^2$

It may happen that several solutions pass through the point at ∞

We must then separate themt through a blowing-up of the space

(introducing local coordinates making the divergence disappear)

Sakai: geometrical description of discrete Painlevé equations

Rational surfaces obtained by successive blow-ups of $\mathbb{P}^1 \times \mathbb{P}^1$
studied through connection between Weyl groups
and the gr. of Cremona isometries on the Picard gr. of the surfaces

When 8 points in generic position in projective plane are blown up
group of Cremona isometries isomorphic to Weyl group $E_8^{(1)}$

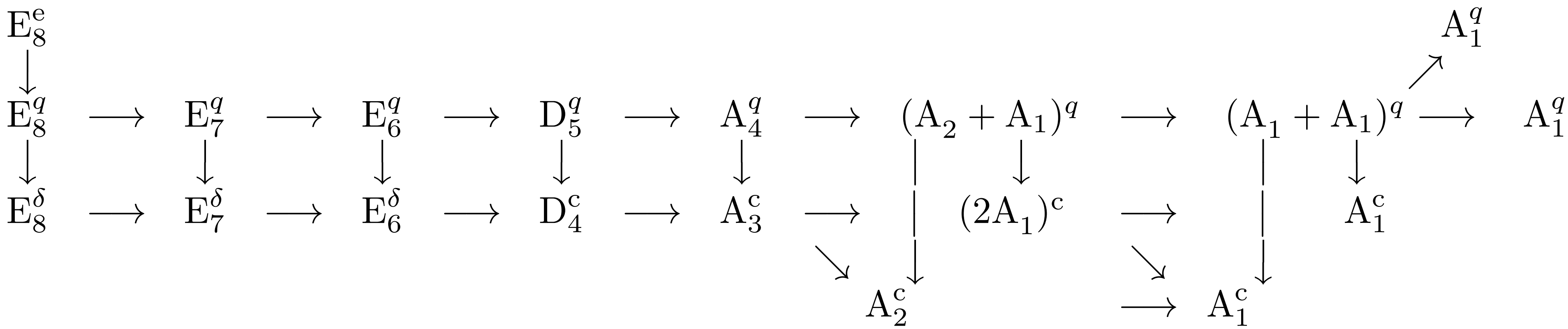
Sakai studied the case where the 8 points are not in generic position

Birational (bi-meromorphic) mappings on $\mathbb{P}^1 \times \mathbb{P}^1$
are obtained by interchanging the procedure of blow-downs

Discrete Painlevé equations:
birational mappings corr. to translations of affine Weyl groups

Classification of discrete Painlevé equations

Degeneration pattern:



Upper index e: a discrete equation involving elliptic functions
Upper index q: equation of q-type
Upper index δ: difference equation not related continuous equation
Upper index c: difference equation, contiguity of continuous Painlevé
P_{VI} for $D_4^{(1)}$, P_V for $A_3^{(1)}$, P_{IV} for $A_2^{(1)}$, P_{III} for $2A_1^{(1)}$, P_{II} for the $A_1^{(1)}$ on the last line and P_{III}^1 for the $A_1^{(1)}$ on the line above last

Sakai's construction is a global one

To construct explicit examples one must specify
a periodically repeated nonclosed pattern in the appropriate space
and obtain the corresponding discrete Painlevé equation

New definition of "discrete Painlevé equation"
A discrete Painlevé equation is the mapping obtained by the periodic repetition of a non-closed pattern on a lattice associated to an affine Weyl groups belonging to the degeneration cascade of $E_8^{(1)}$

Consequence:
the potential number of discrete Painlevé equations is *infinite*

any pattern, compatible with the above definition
in *each* of the affine Weyl groups of the degeneration pattern
would lead to a *different* discrete Painlevé equation

Important finding of Sakai:

elliptic discrete Painlevé equations

Example:

$$
\begin{aligned}
&\mathrm{cn}(\gamma_n)\mathrm{dn}(\gamma_n)(1-k^2\mathrm{sn}^4(z_n))x_n(x_{n+1}+x_{n-1})\\
&\quad -\mathrm{cn}(z_n)\mathrm{dn}(z_n)(1-k^2\mathrm{sn}^2(z_n)\mathrm{sn}^2(\gamma_n))(x_n^2+x_{n+1}x_{n-1})\\
&\quad\quad +\mathrm{cn}(z_n)\mathrm{dn}(z_n)(\mathrm{cn}^2(z_n)-\mathrm{cn}^2(\gamma_n))(1+k^2x_n^2x_{n+1}x_{n-1})=0
\end{aligned}
$$

$z_n=(\gamma_e+\gamma_o)n+z_0$ and $\gamma_n=\gamma_{e,o}$ n-parity dependent
Sakai provided link between singularity confinement
and the construction of the space of initial conditions

All d-Painlevé equations have a max. of 8 confined singularities
They can be described by a maximum of 8 blow-ups

Procedure first advocated by Kruskal