Lie Symmetries of Difference Equations

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1. Lie symmetries; its generalities.
2. Lie point symmetries of Difference Equations; their derivation and their applications.
3. From Point Symmetries to Generalized Symmetries for Difference Equations.
4. Generalized Symmetries from the Integrability of Difference Equations.

1 Lie symmetries of Differential Equations and their extension

Sophus Lie (1842 – 1899) introduced the notion of continuous group of transformations to unify the various techniques for solving ODE's. He was motivated by the work of Evariste Galois (1811– 1832) for solving algebraic equations.

A one–parameter Lie group of transformations $\tilde{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \epsilon)$ is defined by the following axioms:

1. Closure property: $\tilde{\tilde{\mathbf{x}}} = \mathbf{F}(\tilde{\mathbf{x}}, \delta) = \mathbf{F}(\mathbf{x}, \phi(\epsilon, \delta))$
2. Associativity property: $\mathbf{F}(\mathbf{x}, \phi(\mathbf{a}, \phi(\mathbf{b}, \mathbf{c}))) = \mathbf{F}(\mathbf{x}, \phi(\phi(\mathbf{a}, \mathbf{b}), \mathbf{c}))$.
3. Identity element: $\mathbf{x} = \mathbf{F}(\mathbf{x}, \mathbf{e})$.
4. Inverse element: $\mathbf{x} = \mathbf{F}(\tilde{\mathbf{x}}, \delta^{-\mathbf{1}}) = \mathbf{F}(\mathbf{x}, \phi(\delta, \delta^{-\mathbf{1}}))$.
5. $\mathbf{F}$ is differentiable in $\mathbf{x}$ and analytic in ϵ.
6. $\phi(\epsilon, \delta)$ is analytic in ϵ and δ.

ϵ is a continuous parameter and we can always take $\epsilon = 0$ as the identity.

Infinitesimal transformation: $\tilde{\mathbf{x}} = \mathbf{x} + \epsilon \left[\frac{\partial \mathbf{F}}{\partial \epsilon}|_{\epsilon=0} \right] + O(\epsilon^2)$

Infinitesimal coefficient: $\boldsymbol{\xi}(\mathbf{x}) = \left[\frac{\partial \mathbf{F}}{\partial \epsilon}|_{\epsilon=0} \right]$

Infinitesimal generator: $\hat{X} = \boldsymbol{\xi}(\mathbf{x})\partial_{\mathbf{x}} = \sum_{i=0}^{n} \xi_i(\mathbf{x})\partial_{x_i}$

First Lie Theorem *There exists a parametrization $\tau(\epsilon)$ such that the Lie group of transformations $\mathbf{F}$ is equivalent to the solution of an initial value problem for a system of first order ODE's*

$$\frac{d\tilde{\mathbf{x}}}{d\tau} = \boldsymbol{\xi}(\tilde{\mathbf{x}}) \tag{1}$$

with $\tilde{\mathbf{x}} = \mathbf{x}$ when $\tau = 0$.

$\tau(\epsilon)$ is a well defined function of ϵ given by:

$$\begin{aligned} \tau(\epsilon) &= \int_0^{\epsilon} \Gamma(\eta) d\eta, \\ \Gamma(\eta) &= \left. \frac{\partial \phi(\alpha, \beta)}{\partial \beta} \right|_{(\alpha,\beta)=(\eta^{-1},\eta)}. \end{aligned}$$

In terms of the *infinitesimal generators* the **First Lie Theorem** can be written as

Theorem: *The one–parameter Lie group of transformations is equivalent to*

$$\tilde{\mathbf{x}} = e^{\epsilon \hat{X}} \mathbf{x} = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} \hat{X}^k \mathbf{x}. \tag{2}$$

The expression (2) is called the **Lie series**.

Corollary: *If $f(\mathbf{x})$ is an infinitely differentiable function, then its transformed $f(\tilde{\mathbf{x}})$, under a one–parameter Lie group $\mathbf{F}$, is:*

$$f(\tilde{\mathbf{x}}) = f(e^{\epsilon \hat{X}} \mathbf{x}) = e^{\epsilon \hat{X}} f(\mathbf{x}). \tag{3}$$

1.1 Invariant functions

An infinitely differentiable function $f(\mathbf{x})$ is an *invariant function* of the one–parameter Lie group of transformations iff $f(\tilde{\mathbf{x}}) = f(\mathbf{x})$. The invariant function $f(\mathbf{x})$ is called an *invariant* of the one–parameter Lie group of transformations $\mathbf{F}$ and we may also say that $f(\mathbf{x})$ is *invariant under* the one–parameter Lie group of transformations $\mathbf{F}$.

Theorem: $f(\mathbf{x})$ *is invariant under the one–parameter Lie group of transformations* $\mathbf{F}$ *iff*

$$\hat{X} f(\mathbf{x}) = 0 \tag{4}$$

1.2 Invariant surface

A surface $\mathcal{F}(x) = 0$ is invariant under the one parameter Lie group of transformations of infinitesimal generator $\hat{X}$ if

$$\hat{X}\mathcal{F}(x) = 0 \text{ when } \mathcal{F}(x) = 0$$

1.3 Evolutionary form of the infinitesimal generators and higher order transformations

Let us consider a symmetry of a system characterized by a dependent variable $u(x)$ and an independent x (u and x could be vectors too). The infinitesimal generator for the Lie point symmetry for such a system will be given by

$$\hat{X} = \xi(x,u)\partial_x + \eta(x,u)\partial_u$$

An infinitesimal generator of a 1–parameter Lie group of transformations will be in *evolutionary or characteristic form* if it is written as:

$$\hat{X} = [\eta(x,u) - u_1\xi(x,u)]\frac{\partial}{\partial u}$$

By the *First Lie Theorem* the Lie Group will be obtained by solving the following system of coupled first order differential equations

$$\frac{d\tilde{x}}{d\epsilon} = \xi(\tilde{x},\tilde{u}), \quad \frac{d\tilde{u}}{d\epsilon} = \eta(\tilde{x},\tilde{u}). \tag{5}$$

In the evolutionary form we have

$$\frac{\partial \tilde{x}}{\partial \epsilon} = 0, \quad \frac{\partial \tilde{u}}{\partial \epsilon} = \eta(\tilde{x}, \tilde{u}) - \frac{\partial \tilde{u}}{\partial \tilde{x}}\xi(\tilde{x}, \tilde{u}) \tag{6}$$

i.e. a partial differential equation which, on the characteristics, provide the same solutions as the system (5).

The evolutionary form is a natural starting point to **extend the transformations**.

We generalize the infinitesimal coefficient of the Lie point symmetries in the evolutionary form by **introducing a dependence on derivatives of u up to order** k:

$$\hat{X} = \eta(x, u, u_1, \ldots, u_k)\frac{\partial}{\partial u}$$

Then the group transformations are obtained by solving the differential equation $\frac{\partial \tilde{u}}{\partial \epsilon} = \eta(x, \tilde{u}, \tilde{u}_1, \ldots, \tilde{u}_k)$. This is a **PDE of order k, maybe nonlinear,** which, even if it is of first order in the group parameter ϵ, **in general it will not be solvable**. Formally we can solve it by exponentiating and obtain the corresponding one–parameter group of transformations acting on the space of the function $u(x)$,

$\tilde{x} = x$, $\tilde{u} = u + \epsilon\eta + O(\epsilon^2)$. To get the coefficient of ϵ^2, one needs to prolong $\hat{X}$ so as to be able to act on the variable $u_1, u_2, \ldots, u_k$ appearing in η. If we look for the higher order terms in the ϵ expansion we need to prolong still further. So to preserve the contact condition we will need to use

$$\hat{X}^{(\infty)} = \sum_{j=0}^{\infty} D^j \eta \frac{\partial}{\partial u_j}$$

with $u_0 = u$, and thus

$$\tilde{x} = x,\ \tilde{u} = u + \sum_{j=1}^{\infty} \frac{\epsilon^j}{j!} (\hat{X}^{(\infty)})^{j-1} \eta.$$

If $\eta = \eta(x, u, u_1)$ and $\frac{d^2\eta}{du_1^2} \neq 0$ then we say that the one–parameter Lie transformation is a **contact transformation**.

The meaning of contact transformations can be clarified by considering them in the standard formalism. In this case

$$\hat{X} = \xi(x, u, u_1)\partial_x + \phi(x, u, u_1)\partial_u + \phi^{(1)}(x, u, u_1, u_2)\partial_{u_1}.$$

In general, as ϕ and ξ contain u_1 the coefficient of the first prolongation will contain u_2. **A contact transformation is such that $\phi^{(1)}$ depends just on u_1.** This gives a constraint $\phi_{u_1} = u_1 \xi_{u_1}$. A contact transformation exists only if $\xi_{u_1} \neq 0$.

Contact transformations exist only if we have a scalar differential equations.

For systems of differential equations, contact transformations reduce to point transformations (the group of contact transformations reduce to the first prolongation of a group of point transformations).

Bäcklund proved that group of transformations containing derivatives of order higher than one, are infinite [42].

2 Symmetries of equations defined on a lattice

2.1 Lattices and equations defined on them

By a difference equation we mean a functional relation, linear or non-linear, between functions calculated at different points of a lattice [2, 5, 27, 48, 85]. These systems appear in many applications. First of all they can be written down as **discretizations of a differential equation** when one is trying to solve it with a computer. In such a case one reduces the differential equation to a recurrence relation:

$$\frac{du}{dx} = f(x, u) \qquad \Rightarrow \qquad v(n+1) = g(n, v(n)).$$

On the other hand we can consider **dynamical systems defined on a lattice**, i.e. systems where the real independent fields depend on a set of independent variables which vary partly on the integers and partly on the reals. For example we can consider

$$\frac{d^2 u(n,t)}{dt^2} = F(t, u(n,t), u(n-1,t), .., u(n-a,t), u(n+1,t), .., u(n+b,t)).$$

These kind of equations can appear in many different setting. Among them they are associated to the evolution of many body problems, to the study of crystals, to biological and economical systems, etc..

As an example of possible applications we consider the problem of the **transmission of energy in one dimensional molecular system**, problem which is of particular relevance for understanding the functioning of physical systems of biological interest [15]. This is a particularly hot topic as some relevant biological processes require the transport of energy with low dispersion along essentially one dimensional chains, such as the spines in an α helix [25]. A mechanism for the nondispersive transport of vibrational energy along hydrogenon bonded chains was proposed by Davydov and its continuous limit for small lattice spacing gave rise to a Nonlinear Schrödinger equation (NLS) which has soliton solutions [81]. If such **soliton like solutions are valid also at biological temperatures** is an open problem.

In the case of diatomic nonlinear lattices we can describe such systems by the equations

$$M_1\ddot{x}_n - k_1(y_n - x_n) + k_2(x_n - y_{n-1}) - \epsilon\beta_1(y_n - x_n)^2 + \epsilon\beta_2(x_n - y_{n-1})^2 = 0$$
$$M_2\ddot{y}_n + k_1(y_n - x_n) - k_2(x_{n+1} - y_n) + \epsilon\beta_1(y_n - x_n)^2 - \epsilon\beta_2(x_{n+1} - y_n)^2 = 0$$

where M_1 and M_2 are the different values of the two atomic masses, ϵ is a small parameter while k_1, k_2, β_1 and β_2 are four constants of order 1. When $k_2 \ll k_1$ these equation represent a molecular chain with intramolecular interaction stronger than the intermolecular one. This is the case, for example, of an **hydrogen-bonded polypeptide chain**.

Further examples are connected to **Quantum Gravity** (QG), where one of the most fruitful recent approaches to the problem are based on the **discretization of space–time**. In this way one hopes to obtain a non–trivial theory without the use of perturbation expansions. In the approach to QG introduced by Gambini and Pullin [33] one considers a **discrete action** $S = \sum_{n=1}^{N} \mathcal{L}(q_n, q_{n+1})$. The Lagrange equation of motion are given by $\frac{\partial S}{\partial q_n} = \frac{\partial \mathcal{L}(q_{n-1}, q_n)}{\partial q_n} + \frac{\partial \mathcal{L}(q_n, q_{n+1})}{\partial q_n} = 0$ and **a simple example of Lagrangian** is given by $\mathcal{L} = \frac{m}{2\tau}(q_{n+1} - q_n)^2 - \tau V(q_n)$. In this approach one then quantizes the systems by a discrete canonical transformation and then looks for the consequences of the theory by performing a quantum continuum limit.

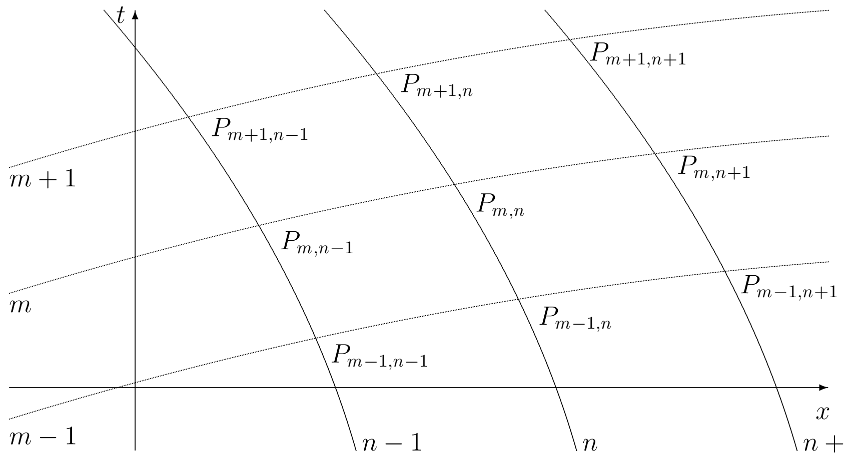
Further examples of discrete equations are provided by **recurrence relations**. For an example one can consider the **logistic map** which describe the growth of a population: $x_{n+1} = r x_n (1 - x_n)$. These maps can appear as approximations of differential equations (see up above) or as **functional equations** in various fields as statistics, engineering, etc. . The problem of **solving functional equations** has been discussed by many mathematicians, see for example, D'Alambert, Cauchy, Abel, Hilbert, etc. [3].

Summing up **difference equations either appear in themselves** and we would like to use **Lie theory to get classes of exact solutions** or we obtain them by **discretizing the continuum equation in such a way to preserve the symmetries** (Dorodnitsyn, and collaborators), i.e. we create sets of discrete equations which provide **numerical schemes approximating the continuum equation**.

For simplicity in the following I will consider just the case of a scalar equation in at most two independent variables but, similar results can be obtained in the case of N independent and M dependent variables.

A discrete equation in R^2 is thus a functional relation for a field u at different points P_i in R^2, i.e. $E = E(x, t, u(P_1), \ldots, u(P_L)) = 0$. A differential difference equation is obtained by considering the points P_i uniformly spaces in one direction, say t, with spacing h_t, in such a way that we are allowed to consider the continuous limit when h_t goes to zero.

The points P_i in R^2 can be labeled by two discrete indexes, **the computational variables**, which characterize the points with respect to two independent directions, $P_{n,m}$ and can be displayed on lines characterized by the constancy of one index.



In cartesian coordinates we have:

(7) $$P_{n,m} = (x_{n,m}, t_{n,m})$$

and the function $u(P)$ reads

(8) $$u_{P_{n,m}} = u(x_{n,m}, t_{n,m}) = u_{n,m}.$$

A **difference scheme** will be a *set of relations* among the values of $\{x, t, u(x,t)\}$ at a finite number, say L, of points in R^2 $\{P_1, \ldots, P_L\}$ around a reference point, say P_1. Some of these relations will define where the points are in R^2 and others how $u(P)$ transforms in R^2. In our case, as we have one only dependent variable and two independent variables we expect to have at most five equations, four which define the two independent variables in the two independent directions in R^2, and one the dependent variable in terms of the lattice points:

$$
\begin{aligned}
(9) \qquad & E_a(\{x_{n+j,m+i}, t_{n+j,m+i}, u_{n+j,m+i}\}) = 0 \\
& 1 \leq a \leq 5; \qquad -i_1 \leq i \leq i_2, \quad -j_1 \leq j \leq j_2 \quad (i_1, i_2, j_1, j_2) \varepsilon Z \\
& i_1 + i_2 = N, \qquad j_1 + j_2 = M.
\end{aligned}
$$

System (9) must be such that, starting from L points we are able to calculate $\{x, t, u\}$ in all points of interest. So if we give four equations for the lattice, two for each independent direction, then these equations must be compatible among themselves. If the lattice is not defined a priory then we can have less equations. Three equations may be sufficient if we solve a Cauchy Problem.

Example: discrete $u_{,xt} = 0$

$$\frac{1}{t_{m+1,n} - t_{m,n}} \left[\frac{u_{m+1,n+1} - u_{m+1,n}}{x_{m+1,n+1} - x_{m+1,n}} - \frac{u_{m,n+1} - u_{m,n}}{x_{m,n+1} - x_{m,n}} \right] = 0, \tag{10}$$

$$t_{m,n+1} - t_{m,n} = 0, \qquad x_{m+1,n} - x_{m,n} = 0, \tag{11}$$

whose solution is given by $u_{m,n} = f(x_{m,n}) + g(t_{m,n})$ with $t_{m,n} = \alpha_m$ and $x_{m,n} = \beta_n$. If we define the functions $t_{m,n}$ and $x_{m,n}$ by two other equations, for example

$$t_{m+1,n} - t_{m,n} = h_n, \qquad x_{m,n+1} - x_{m,n} = k_m, \tag{12}$$

the compatibility of eqs. (11, 12) implies $h_{n+1} = h_n$ and $k_{m+1} = k_m$, i.e. h_n and k_m constants.

If a continuous limit of (9) exists, then one of the equations will go over to a partial differential equation and the others will be identically satisfied (generically $0 = 0$). We can also do partial continuous limits when only one of the independent variables become continuous while the other is still discrete. In this case only part of the lattice equations are identically satisfied and we obtain a differential difference equation for the dependent variable and an equation for the lattice variable.

Lie point symmetries are characterized by transformations of the form:

$$\begin{aligned} \tilde{x} &= F_g(x,t,u) = x + g\ \xi(x,t,u) + \dots \\ \tilde{t} &= G_g(x,t,u) = t + g\ \tau(x,t,u) + \dots \\ \tilde{u} &= H_g(x,t,u) = u + g\ \phi(x,t,u) + \dots \end{aligned} \tag{13}$$

where g, as before, is a group parameter. The transformation (13) is such that if $\{x, t, u\}$ satisfy the difference scheme $E_a = 0$, $\{\tilde{x}, \tilde{t}, \tilde{u}\}$ will be a solution of the same scheme. Such a transformation acts on the whole space of the independent and dependent variables $\{x, t, u\}$, at least in some neighborhood of P_1 including all points up to P_L. This means that **the set of functions F_g, G_g and H_g** must be well behaved in the region where P_L are defined and **will determine the transformation in all points of the scheme**. In the point P_1 we define the infinitesimal generator as:

$$\hat{X}_{P_1} = \xi(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \phi(x,t,u)\partial_u \tag{14}$$

and then we prolong it to all other $L-1$ points of the scheme. Since the transformation is given by the same set of functions $\{F_g, G_g, H_g\}$ at all points, the *prolongation* **of $\hat{X}_{P_j}$ is obtained simply by evaluating $\hat{X}_{P_1}$ at the corresponding**

points involved in the scheme. So

$$pr\hat{X} = \sum_{i=1}^{L} \hat{X}_{P_i}. \tag{15}$$

Consequently the invariance condition for the difference scheme is:

$$pr\hat{X} E_a|_{E_a=0} = 0. \tag{16}$$

Eq.(16) is a set of functional equations whose solution is obtained, following Abel [1], by turning them into differential equations by successive derivation with respect to the independent variables $\{x, t, u\}$ at the different points of the lattice [3].

The solution of (16) provide the function $\xi(x, t, u)$, $\tau(x, t, u)$ and $\phi(x, t, u)$, the infinitesimal coefficients of the local Lie point symmetry group. The transformation is obtained by integrating the vector field, i.e. by solving the following system of differential equations:

$$
\begin{aligned}
\frac{d\tilde{x}}{dg} &= \xi(\tilde{x}, \tilde{t}, \tilde{u}), \qquad && \tilde{x}|_{g=0} = x, \\
\frac{d\tilde{t}}{dg} &= \tau(\tilde{x}, \tilde{t}, \tilde{u}), \qquad && \tilde{t}|_{g=0} = t, \\
\frac{d\tilde{u}}{dg} &= \phi(\tilde{x}, \tilde{t}, \tilde{u}), \qquad && \tilde{u}|_{g=0} = u.
\end{aligned} \tag{17}
$$

In general **we expect the infinitesimal coefficients ξ and τ to be determined by the lattice equations. So according to the form of the lattice, different symmetries can appear.**

Example: the discrete heat equation on a uniform orthogonal lattice:

$$
\frac{u_{n+1,m} - u_{n,m}}{t_{n+1,m} - t_{n,m}} = \frac{u_{n,m+2} - 2u_{n,m+1} + u_{n,m}}{(x_{n,m+1} - x_{n,m})^2} \tag{18}
$$

$$
\begin{aligned}
x_{n,m+1} - x_{n,m} &= h_x; \qquad t_{n,m+1} - t_{n,m} = 0 \\
x_{n+1,m} - x_{n,m} &= 0; \qquad t_{n+1,m} - t_{n,m} = h_t,
\end{aligned} \tag{19}
$$

where h_x, h_t are two a priory fixed constants which define the spacing between two neighboring points in the two directions of the orthogonal lattice. By applying the infinitesimal generator (14) to the lattice equations we get:

$$\xi(x_{n,m+1}, t_{n,m+1}, u_{n,m+1}) = \xi(x_{n,m}, t_{n,m}, u_{n,m});$$
$$\xi(x_{n+1,m}, t_{n+1,m}, u_{n+1,m}) = \xi(x_{n,m}, t_{n,m}, u_{n,m}).$$

From (18) $u_{n,m+1}$, $u_{n+1,m}$ and $u_{n,m}$ can be chosen as independent functions and thus we get $\xi = \xi(x,t)$. As $t_{n,m+1} = t_{n,m}$ and $x_{n,m+1} \neq x_{n,m}$ we get $\xi = \xi(t)$. As $x_{n+1,m} = x_{n,m}$ and $t_{n+1,m} \neq t_{n,m}$ we get that the only possible value is ξ=costant. In a similar fashion we derive that also τ must be a constant and that ϕ=$u + s(x,t)$, where $s(x,t)$ is a solution of the discrete heat equation (18), the linear superposition formula. Summarizing we get that the infinitesimal generators of the symmetries for the discrete heat equation (18) are given by

(20) $$\hat{P}_0 = \partial_t; \quad \hat{P}_1 = \partial_x; \quad \hat{W} = u\partial_u; \quad \hat{S} = s(x,t)\partial_u.$$

Let us prove, in the case of ordinary difference equations when we have just one discrete independent variable x_n and one dependent variable $u_n(x_n)$ that **the prolongation formula given above (15) has the proper continuous limit.**

To do so we show that it can be recast in a form which corresponds to the well know to you continuous formula. We consider a prolonged vector field

$$
\begin{aligned}
(21) \qquad \mathrm{pr}^1 \hat{X}_n &= \xi(x_n, u_n)\partial_{x_n} + \phi(x_n, u_n)\partial_{u_n} \\
&+ \xi(x_{n+1}, u_{n+1})\partial_{x_{n+1}} + \phi(x_{n+1}, u_{n+1})\partial_{u_{n+1}},
\end{aligned}
$$

depending on two neighboring points x_n, u_n, x_{n+1} and u_{n+1} as we want to approximate a first derivative. We can define the *new variables* $\tilde{x}_n$, $\tilde{u}_n$, h_{n+1} and $u_{x,n+1}$, given by the incremental ratio

$$
(22) \qquad \tilde{x}_n = x_n, \quad \tilde{u}_n = u_n, \quad h_{n+1} = x_{n+1} - x_n, \quad u_{x,n+1} = \frac{u_{n+1} - u_n}{x_{n+1} - x_n}.
$$

Rewriting the prolonged vector field (21) in the new variables, we get:

$$
\begin{aligned}
(23) \qquad \mathrm{pr}^1 \hat{X}_n &= \xi(\tilde{x}_n, \tilde{u}_n)\partial_{\tilde{x}_n} + \phi(\tilde{x}_n, \tilde{u}_n)\partial_{\tilde{u}_n} \\
&+ [\xi(\tilde{x}_n + h_{n+1}, \tilde{u}_n + h_{n+1}u_{x,n+1}) - \xi(\tilde{x}_n, \tilde{u}_n)]\partial_{h_{n+1}} \\
&+ \Big[\frac{\phi(\tilde{x}_n + h_{n+1}, \tilde{u}_n + h_{n+1}u_{x,n+1}) - \phi(\tilde{x}_n, \tilde{u}_n)}{h_{n+1}} \\
&- u_{x,n+1}\frac{\xi(\tilde{x}_n + h_{n+1}, \tilde{u}_n + h_{n+1}u_{x,n+1}) - \xi(\tilde{x}_n, \tilde{u}_n)}{h_{n+1}}\Big]\partial_{u_{x,n+1}}.
\end{aligned}
$$

When $h_{n+1} \to 0$ eq. (23) gives the first continuous prolongation in term of derivatives:

$$
\lim_{h_{n+1}\to 0} \mathrm{pr}^1 \hat{X}_n = \xi(x,u)\partial_x + \phi(x,u)\partial_u + [D_x\phi - u_x D_x\xi]\partial_{u_x}.
$$

Equation (23) gives a formula for the discrete prolongation

$$
\begin{aligned}
(24) \qquad \phi^{(1)} &= \Big[\frac{\phi(\tilde{x}_n + h_{n+1}, \tilde{u}_n + h_{n+1}u_{x,n+1}) - \phi(\tilde{x}_n, \tilde{u}_n)}{h_{n+1}} \\
&- u_{x,n+1}\frac{\xi(\tilde{x}_n + h_{n+1}, \tilde{u}_n + h_{n+1}u_{x,n+1}) - \xi(\tilde{x}_n, \tilde{u}_n)}{h_{n+1}}\Big].
\end{aligned}
$$

In the following we will show that

1. **Given a discrete equation we can apply to it the same procedures as for differential equations.**
2. **Given a differential equation with symmetries we can construct a difference scheme which approximate it while preserving the symmetries.**

2.1.1 Symmetries of the discrete–time Toda lattice

Let us consider the discrete–time Toda lattice (the Hirota equation)

$$\begin{aligned} \mathcal{F} &= e^{u_{n,m}-u_{n,m+1}} - e^{u_{n,m+1}-u_{n,m+2}} \\ &- \alpha^2 \left[e^{u_{n-1,m+2}-u_{n,m+1}} - e^{u_{n,m+1}-u_{n+1,m}} \right] = 0, \end{aligned} \tag{25}$$

and the associated non–transformable orthogonal lattice

$$\mathcal{L} = [x_{n,m} - n \cdot \sigma_x, t_{n,m} - m \cdot \sigma_t] = [0,0] = 0. \tag{26}$$

where σ_x and σ_t are given real constants.

Let us define $t = t_{n,m}$, $v_n(t) = u_{n,m}$ and $\alpha = \sigma_t^2$. Then the limit of (25) when $\sigma_t \to 0$, $m \to \infty$ such that t is finite gives the Toda lattice equation

$$\ddot{v}_n = e^{v_{n-1}-v_n} - e^{v_n - v_{n+1}}. \tag{27}$$

The infinitesimal generator for (25, 26) is

$$\hat{X}_{n,m} = \xi_{n,m}\partial_{x_{n,m}} + \tau_{n,m}\partial_{t_{n,m}} + \phi_{n,m}\partial_{u_{n,m}},$$

and the invariance condition

$$\mathrm{pr}\hat{X}_{n,m}\mathcal{F}\Big|_{\mathcal{F}=0,\mathcal{L}=0} = 0, \quad \mathrm{pr}\hat{X}_{n,m}\mathcal{L}\Big|_{\mathcal{F}=0,\mathcal{L}=0} = 0.$$

As the lattice is non–transformable with a given origin $\xi_{n,m} = 0$ and $\tau_{n,m} = 0$. In (25) one can take $u_{n,m}$, $u_{n,m+1}$, $u_{n,m+2}$, $u_{n+1,m}$ as independent and can express $u_{n-1,m+2}$ in term of them. Then the determining equation reads

$$\begin{aligned}
& e^{u_{n,m}-u_{n,m+1}}[\phi_{n,m} - \phi_{n-1,m+2}] \\
+ \quad & e^{u_{n,m+1}-u_{n,m+2}}[\phi_{n-1,m+2} + \phi_{n,m+2} - 2\phi_{n,m+1}] \\
- \quad & \alpha^2 e^{u_{n,m+1}-u_{n+1,m}}[\phi_{n-1,m+2} + \phi_{n+1,m} - 2\phi_{n,m+1}] = 0.
\end{aligned} \tag{28}$$

Differentiating (28) with respect to $u_{n,m+2}$ and eliminating a non–zero multiplicative factors we get

$$\frac{d\phi_{n,m+2}}{du_{n,m+2}} = \frac{d\phi_{n-1,m+2}}{du_{n-1,m+2}} + \phi_{n,m+2} + \phi_{n-1,m+2} - 2\phi_{n,m}. \tag{29}$$

Differentiating (29) with respect to $u_{n,m}$ we get $\frac{d\phi_{n,m+2}}{du_{n,m+2}} + \phi_{n,m+2} = c_1$, i.e. $\phi_{n,m} = c_0 + c_1 e^{-u_{n,m}}$ where c_0 and c_1 are integration constants and thus are arbitrary functions of n and m. Introducing this result in (28) we get $c_1(n,m) = 0$ and $c_0(n,m) = c_0$.

So the only point symmetry of the equation $\mathcal{F} = 0$ on a nontransformable uniform lattice is a translation in $u_{n,m}$ as the equation depends on differences.

The discrete time Toda lattice admits generalized symmetries. One of these symmetries is

$$\begin{aligned} \hat{X}_{n,m} &= \mathcal{Q}(u_{n,m}, u_{n-1,m+1}, u_{n,m+1})\partial_{u_{n,m}}, \\ \mathcal{Q} &= \alpha e^{u_{n-1,m+1}-u_{n,m}} + \frac{1}{\alpha} e^{u_{n,m}-u_{n,m+1}}. \end{aligned} \tag{30}$$

A symmetry reduction with respect to the generalized symmetry (30) is obtained by requiring the simultaneous solution of $\mathcal{F}=0$ and $\mathcal{Q}=0$. Taking into account $\mathcal{Q}=0$ the nonlinear equation $\mathcal{F}=0$ reduces to $\Delta_n\Delta_m u_{n,m}=0$ where $\Delta_n u_{n,m}=u_{n+1,m}-u_{n,m}$ and similarly for Δ_m. So $u_{n,m}=g_m+f_n$ and introducing this result into the equation $\mathcal{Q}=0$ we get as a solution $u_{n,m}=u_0+n\log(k_0)+m\log(k_1)$ with $\alpha^2k_0+k_1=0$.

2.1.2 From symmetries of discrete equations to symmetries of differential difference equations

We consider here scalar difference equations where one of the independent variable is continuous, i.e. $v_n(t)$.

$$\mathcal{E}(\ddot{v}_n,\dot{v}_n,\{v_{n+i}\}_{i=i_a}^{i_b})=0,\quad i_b\geq i_a. \tag{31}$$

The symmetry generator will have the form:

$$\hat{X}=\tau_n(t,v_n)\partial_t+\phi_n(t,v_n)\partial_{v_n(t)}$$

Any symmetry generator as $\hat{X}$ can be rewritten in an equivalent evolutionary form as

$$\hat{X}_e = Q_n(t, v_n, \dot{v}_n)\partial_{v_n}, \qquad Q_n = \phi_n(t, v_n) - \tau_n(t, v_n)\dot{v}_n.$$

From Lie theorem we deduce that the existence of a symmetry for the equation $\mathcal{E} = 0$ is equivalent to *commuting flows*

$$\mathcal{E} = 0, \qquad \frac{dv_n(t,\epsilon)}{d\epsilon} = Q_n(t, v_n(t,\epsilon), \dot{v}_n(t,\epsilon)).$$

The compatibility of the two equations implies a determining equation

$$\mathcal{E}_{,\ddot{v}_n}\ddot{Q}_n + \mathcal{E}_{,\dot{v}_n}\dot{Q}_n + \sum_{i=i_a}^{i_b} \mathcal{E}_{,v_{n+i}} Q_{n+i}\Big|_{\mathcal{E}=0} = 0 \tag{32}$$

Taking into account the explicit form of Q_n in term of τ_n we get from the last term of (32) that the following equation must be satisfied

$$\tau_n(t, v_n) = \tau_{n+i_a}(t, v_{n+i_a}) \tag{33}$$

By proper differentiation this last equation implies that $\tau_n(t, v_n) = \tau(t)$.

So, the symmetries of any differential difference equation of the form (31) will be given by a symmetry generator

$$\hat{X}_n = \tau(t)\partial_t + \phi_n(t, v_n(t))\partial_{v_n}. \tag{34}$$

As an example we construct the symmetries of the Toda lattice equation (27). The determining equation is

$$\begin{aligned} \phi_n^{(2)} \quad &- \quad \Big[e^{v_{n-1}-v_n}\big(\phi_{n-1} - \phi_n\big) \\ &- \quad e^{v_n - v_{n+1}}\big(\phi_n - \phi_{n+1}\big)\Big]\Big|_{\ddot{v}_n = e^{v_{n-1}-v_n} - e^{v_n - v_{n+1}}}. \end{aligned} \tag{35}$$

where $\phi_n^{(2)} = \ddot{\phi}_n + [2\dot{\phi}_{n,v_n} - \ddot{\tau}]\dot{v}_n + [\phi_{n,v_n} - 2\dot{\tau}]\ddot{v}_n$. As $v_{n\pm1}$ are independent fields we get by differentiating with respect to them the following four dimensional symmetry algebra

$$\begin{aligned} \hat{D} &= t\partial_t + 2v_n\partial_{v_n}, & \hat{T} &= \partial_t, \\ \hat{W} &= t\partial_{v_n}, & \hat{U} &= \partial_{v_n}. \end{aligned} \tag{36}$$

Non–trivial solutions for the Toda lattice (27), different from the soliton solutions, can be obtained by symmetry reduction with respect respectively to the symmetries

$\hat{T} + c\hat{W}$ and $\hat{D} + d\hat{U}$:

$$v_n = p - \frac{1}{2}ct^2 - \sum_{j=1}^{n} \log(q - cj), \tag{37}$$

$$v_n = p + 2(n+d)\log t - \sum_{j=0}^{n} \log[q + (2d-1)j + j^2],$$

where p and q are some integration constants. Conditional symmetries will provide in this case new special solutions [58].

2.1.3 Symmetry preserving discretization of differential equations

In this Section we consider a given differential equation, for example an ODE $\mathcal{E}(x, u, \dot{u}, \ddot{u}, \ldots) = 0$ and its point symmetries, $\hat{X} = \xi(x,u)\partial_x + \phi(x,u)\partial_u$. We want to **construct a discrete equation whose continuous limit gives the differential equation $\mathcal{E} = 0$ at a certain order in the lattice parameter approximation and which has the same symmetry group $\hat{X}$ or one of its nontrivial subgroups**.

For simplicity we limit our discussion to the case of ODEs but similar results can be presented also in the case of PDEs.

- Discretizations which preserve the symmetry group, also preserve the set of exact solutions.
- The discrete scheme can be implemented on a computer and it represents a **symmetry preserving numerical discretization scheme**.

If the continuous equation has $\xi \neq 0$ then **the lattice spacing cannot be constant** and the differential equation will be represented by a difference scheme, where **the lattice is described by a difference equation**, possibly giving an orthogonal coordinate system.

We describe here in general **the procedure necessary to discretize an ODE and write down the discrete scheme** and then we will present with all details an example.

If $\mathcal{E} = 0$ is an ODE of order N then the simplest scheme $\Delta = 0$ we can construct which describe it must involve essentially $N + 1$ lattice points for the independent and dependent variables $\{x_j, u_j; j = 1, 2, \cdot, N + 1\}$ to be able to reconstruct the first approximation to the N–derivative.

i We consider the M infinitesimal generators $\hat{X}_j$, $j = 1, \cdots, M$ of the symmetry algebra $\mathcal{L}$ of the Lie point symmetries of the equation $\mathcal{E} = 0$ and we prolong them to the $N + 1$ points of the lattice.

ii We construct a basis given by all the invariant obtainable from the prolonged generators. This basis will include K functionally independent invariants

$$\mathcal{I}_a = \mathcal{I}_a(x_1, x_2, \ldots, x_{N+1}, u_1, \ldots, u_{N+1}), \quad 1 \leq a \leq K, \tag{38}$$

obtained as constant of integration when solving the invariance equations

$$\mathrm{pr}\hat{X}_j \mathcal{I}_a = 0, \qquad j = 1, \ldots, M. \tag{39}$$

We need at least two independent invariants to be able to construct an invariant scheme $\Delta = 0$ which in the continuous limit gives the ODE $\mathcal{E} = 0$. If the number of invariants is not sufficient we can consider also the invariant varieties, i.e. those invariant equations which are identically satisfied when the difference scheme is satisfied.

iii We perform the continuous limit of the constructed invariants. Taking the continuous limits into account we construct a difference scheme whose continuous limit gives the continuous ODE we want to discretize. If we want a better approximation we have to increase the number of points involved.

Example [12]

Let us consider the second order nonlinear ordinary differential equation

$$x^2 u_{,xx} + 4xu_{,x} + 2u = (2xu + x^2 u_{,x})^{\frac{k-2}{k-1}}, \quad k \neq 0, \frac{1}{2}, 1, 2. \tag{40}$$

The choice of the parameter k is such that the equation is non–singular, non–linear and not linearizable. For these values of k the equation has a three dimensional symmetry algebra given by

$$\begin{aligned} \hat{X}^{(1)} &= \partial_x - \frac{2u}{x}\partial_u, \quad \hat{X}^{(2)} = \frac{1}{x^2}\partial_u, \\ \hat{X}^{(3)} &= x\partial_x + (k-2)u\partial_u, \\ [\hat{X}_1, \hat{X}_2] = 0, \quad [\hat{X}_1, \hat{X}_3] &= \hat{X}_1, \quad [\hat{X}_2, \hat{X}_3] = k\hat{X}_2. \end{aligned} \tag{41}$$

As the equation is an ODE of second order and has a three dimensional symmetry group which has an Abelian subalgebra, it will be solvable and its general solution is $u = \left(\frac{1}{k-1}\right)^{k-1} \frac{1}{k} \frac{x-x_0}{x^2} + \frac{u_0}{x}$.

As the equation is of second order the minimum number of point necessary to describe it is three: (x, x_+, x_-), (u, u_+, u_-), where $x = x_n$, $x_+ = x_{n+1}$ and $x_- = x_{n-1}$. The invariance condition reads:

$$\mathrm{pr}\hat{X} F(x_-, x, x_+, u_-, u, u_+) = 0, \tag{42}$$

where F, an apriori arbitrary function of its arguments, is an invariant and eq. (42) must be satisfied by F for the prolongation of all generators $\hat{X}$ given by eq. (41)

$$\mathrm{pr}\hat{X}^{(1)} = \hat{X}^{(1)} + \partial_{x_-} - \frac{2u_-}{x_-}\partial_{u_-} + \partial_{x_+} - \frac{2u_+}{x_+}\partial_{u_+}, \text{ etc..} \tag{43}$$

An invariant of $\hat{X}_1$, $\hat{X}_2$ and $\hat{X}_3$ depending only on the lattice is given by $\xi_1 = \frac{x_+ - x}{x - x_-}$.

Other invariants are obtained by solving among the equations $\mathrm{pr}\hat{X}^{(1)} F = 0$ the characteristic differential equation $\frac{du}{dx} = -\frac{u}{x}$ and its shifted ones. We get $\xi_2 = \frac{x^2 u - (x_+)^2 u_+}{(x_+ - x)^k}$ and $\xi_3 = \frac{(x_-)^2 u_- - x^2 u}{(x - x_-)^k}$ as an invariant.

When we perform the continuous limit, $h_{n+1} = h_+$ and $h_n = h$ go to zero while

$$
\begin{aligned}
(44) \qquad u_+ &= u(x_+) = u(x) + h_+\dot{u} + \frac{(h_+)^2}{2!}\ddot{u} + \mathcal{O}((h_+)^3), \\
u_- &= u(x_-) = u(x) - h\dot{u} + \frac{h^2}{2!}\ddot{u} + \mathcal{O}(h^3)
\end{aligned}
$$

Combining ξ_1, ξ_2 and ξ_3 we get in the continuous limit

$$
\begin{aligned}
(45) \qquad \frac{2\xi_1}{\xi_1+1}\big(\xi_2 - \frac{\xi_2}{\xi_1^{k-1}}\big) &= (h_+)^{2-k}\Big[(x^2\ddot{u} + 4x\dot{u} + 2u) + \\
&\quad + \frac{1}{3}(h_+ - h)(x^2\dddot{u} + 6x\ddot{u} + 6\dot{u}) + \mathcal{O}(h^2)\Big], \\
\frac{1}{2}\big(\xi_2 + \frac{\xi_3}{\xi_1^{k-1}}\big)^{(k-2)/(k-1)} &= (h_+)^{2-k}(x^2\dot{u} + 2xu)^{(k-2)/(k-1)} \\
&\quad \Big[1 + (h_+ - h)\frac{k-2}{k-1}\,\frac{x^2\ddot{u} + 4x\dot{u} + 2u}{x^2\dot{u} + 2xu} + \mathcal{O}(h^2)\Big].
\end{aligned}
$$

We can thus write down in terms of the invariants ξ_1, ξ_2 and ξ_3 the difference equation

$$\frac{2\xi_1}{\xi_1+1}\big(\xi_2-\frac{\xi_2}{\xi_1^{k-1}}\big)=\frac{1}{2}\big(\xi_2+\frac{\xi_3}{\xi_1^{k-1}}\big)^{(k-2)/(k-1)}, \tag{46}$$

which, taking into account (45), will approximate up to order h^2 the differential equation (40). The only invariant dependent just on the lattice variable is ξ_1 and thus an admissible lattice is given by

$$\xi_1 = K. \tag{47}$$

When $K \neq 1$ (47) will give a lattice up to order h. When $K = 1$ the lattice equation represent a uniform lattice and will approximate the continuous case up to order h^2.

Let us compare the discrete scheme provided by eqs. (46, 47), for $k = 3$ and $K = 1$

$$x_+^2 u_+ - 2x^2 u + x_-^2 u_- = \frac{h^{\frac{3}{2}}}{\sqrt{2}}(x_+^2 u_+ - x_-^2 u_-)^{\frac{1}{2}}$$

$$x_+ - 2x + x_- = 0$$

with a Runge–Kutta defined on the same number of points

$$(\tilde{u}_+ - 2\tilde{u} + \tilde{u}_-)\tilde{x}^2 + 2\tilde{x}\tilde{h}(\tilde{u}_+ - \tilde{u}_-) + 2\tilde{h}^2\tilde{u} = \tilde{h}^2(2\tilde{x}\tilde{u} + \tilde{x}^2\frac{\tilde{u}_+ - \tilde{u}_-}{2\tilde{h}})^{\frac{1}{2}}.$$

In both schemes the problem of obtaining u_+ from u and u_- is nonlinear and to solve it we need to apply a fixed point iteration up to convergence.

If we choose $x \in [1,3]$ with $u(1) = \frac{13}{12}$ and $\dot{u}(1) = -1$, the exact solution of (40) is $u(x) = \frac{x}{12} + \frac{1}{x^2}$. In the discrete scheme we consider the initial condition $u_0 = u(x=1) = \frac{13}{12}$ and $u_1 = u(x = 1+h) = \frac{1+h}{12} + \frac{1}{(1+h)^2}$. In the following figure we present the differences of the discretization errors of the two methods with respect to the exact result. Both schemes have the same accuracy but the best result is obtained in the symmetry preserving scheme.

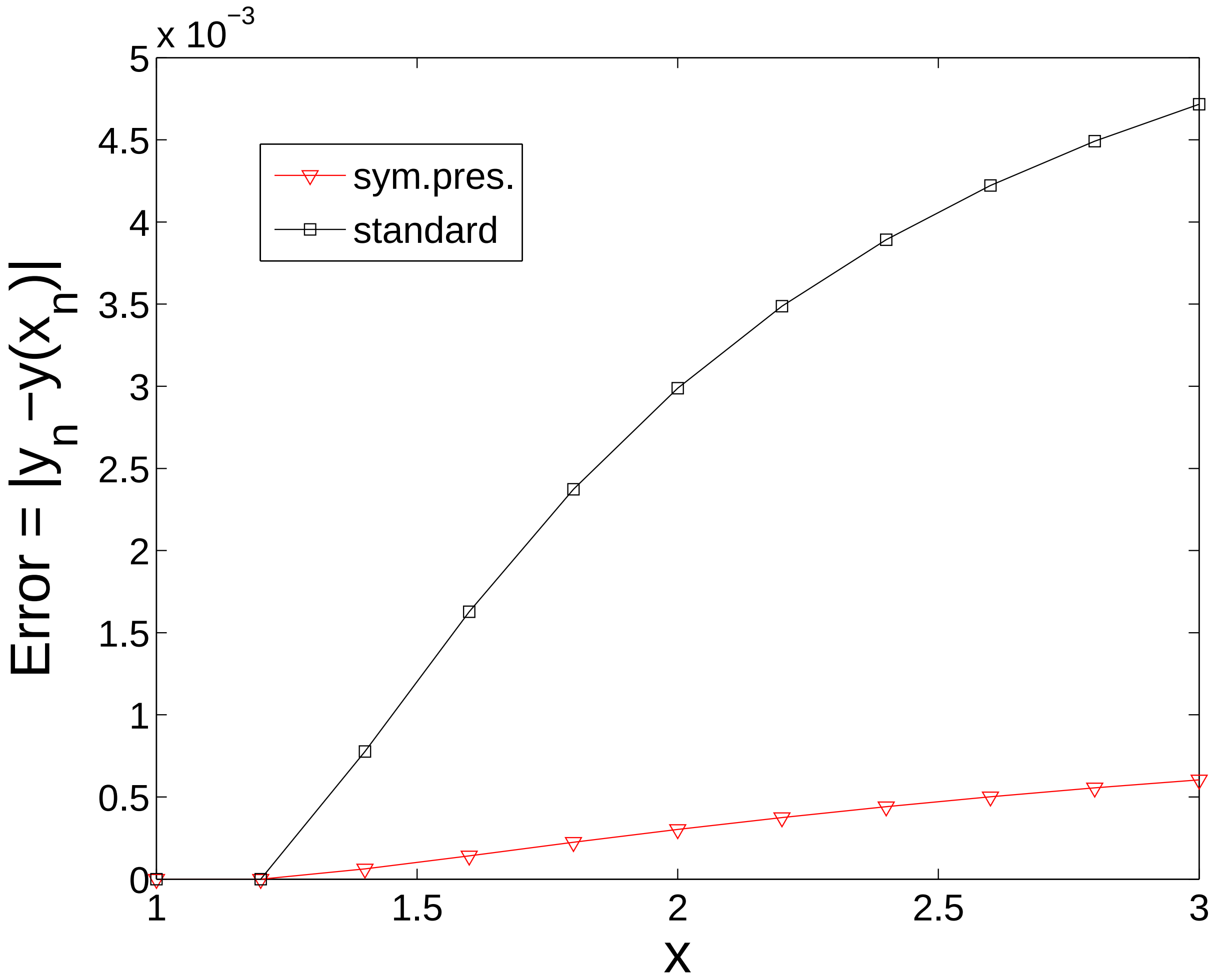


Figure 1. Discretization errors for the symmetry preserving scheme and the standard scheme, Example 1.

3 From point symmetries to generalized symmetries for discrete equations

We have shown up to now that for a given discrete equation very few symmetries are present.

To overcome this problem in the previous Sections we considered the point symmetries of a difference scheme where we allow for a variable lattice. In the following we will analyze the structure of the generalized symmetries for a difference equation. We limit ourselves to consider just **partial difference equations** (with two independent variables) **where the lattice is fixed and non–transformable**.

I will just present an example of **construction of generalized symmetries for completely discrete equations**. The derivation is more complicate as the points on the lattice are no more all independent but are related by the discrete equation. For more results the diligent reader can see the relevant references [34, 35, 60, 61, 65, 66].

3.1 Direct construction of generalized symmetries for partial difference equations on the square lattice: an example

Let us consider the class of nonlinear partial difference equations defined on a square:

$$\mathcal{E}_1(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}) = 0, \tag{48}$$

where the function $\mathcal{E}_1$ is solvable with respect to all its variables.

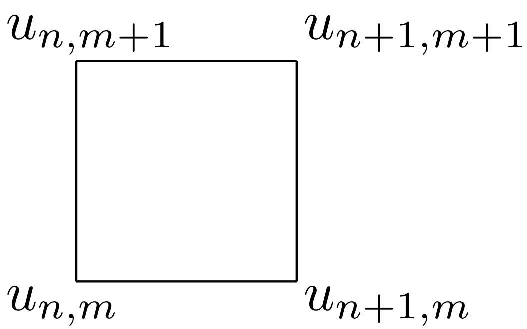


Figure 1: A square lattice

Let us consider a generalized symmetry generator depending on 9 points defined on a square of vertices $u_{n-1,m-1}$, $u_{n-1,m+1}$, $u_{n+1,m+1}$ and $u_{n+1,m-1}$. By taking into account the difference equation (48), we can express the extremal points $u_{n-1,m-1}$, $u_{n-1,m+1}$,

$u_{n+1,m+1}$ and $u_{n+1,m-1}$ in terms of the remaining five points $u_{n-1,m}$, $u_{n+1,m}$, $u_{n,m}$, $u_{n,m-1}$ and $u_{n,m+1}$. In this way the most general n,m–independent 9 points generator is represented by the infinitesimal symmetry generator

$$\hat{X} = Q(u_{n-1,m}, u_{n+1,m}, u_{n,m}, u_{n,m-1}, u_{n,m+1})\partial_{u_{n,m}},$$

and the prolongation necessary to construct the determining equation is given by

$$\begin{aligned}\mathrm{pr}\hat{X} = \sum_{i=0,1}\sum_{j=0,1} Q(&u_{n-1+i,m+j}, u_{n+1+i,m+j}, u_{n+i,m+j}, u_{n+i,m-1+j},\\ &u_{n+i,m+1+j})\partial_{u_{n+i,m+j}}.\end{aligned}$$

Applying the prolonged vector field onto eq. (48), we get:

$$Q\frac{\partial \mathcal{E}_1}{\partial u_{n,m}} + [T_1 Q]\frac{\partial \mathcal{E}_1}{\partial u_{n+1,m}} + [T_2 Q]\frac{\partial \mathcal{E}_1}{\partial u_{n,m+1}} + [T_1 T_2 Q]\frac{\partial \mathcal{E}_1}{\partial u_{n+1,m+1}} = 0, \tag{49}$$

where $T_1 f_{n,m} = f_{n+1,m}$ and $T_2 f_{n,m} = f_{n,m+1}$. Eq. (49) contains $u_{n+i,m+j}$ with $i = -1, 0, 1, 2,\ j = -1, 0, 1, 2$.

The invariance condition requires that (49) be satisfied on the solutions of (48). To be able to do so, as (48) is a relation between points on a square, we need to choose a set of **independent variables** (not on a square) for which (49) must be identically satisfied. A natural choice is to take as independent variables the values on the two axis, i.e. $u_{n,m+2}$, $u_{n,m+1}$, $u_{n,m-1}$, $u_{n,m}$, $u_{n+2,m}$, $u_{n+1,m}$ and $u_{n-1,m}$. An'other choice is given by putting the independent variables on an infinite staircase.

By substituting the equation (48) and its shifted consequences

- $u_{n+2,m+1} \to (u_{n+2,m}, u_{n+1,m+1}, u_{n+1,m})$,
- $u_{n+1,m+1} \to (u_{n+1,m}, u_{n,m+1}, u_{n,m})$,
- $u_{n-1,m+1} \to (u_{n-1,m}, u_{n,m+1}, u_{n,m})$,

etc., we reduce the determining equation to an equation, written just in terms of independent variables, which thus must be identically satisfied. Differentiating (49) with respect to $u_{n,m+2}$ and to $u_{n+2,m}$, we get

(50) $$\frac{\partial^2 T^1 T^2 Q}{\partial u_{n+2,m+1} \partial u_{n+1,m+2}} = T^1 T^2 \frac{\partial^2 Q}{\partial u_{n+1,m} \partial u_{n,m+1}} = 0.$$

Consequently the symmetry generator coefficient Q is the sum of two simpler functions,

$$Q = Q_0(u_{n-1,m}, u_{n+1,m}, u_{n,m}, u_{n,m-1}) + Q_1(u_{n-1,m}, u_{n,m}, u_{n,m-1}, u_{n,m+1}).$$

Introducing this result into the determining equation (49) and differentiating it with respect to $u_{n,m+2}$ and to $u_{n-1,m}$, we have that Q_1 reduces to $Q_1 = Q_{10}(u_{n-1,m}, u_{n,m}, u_{n,m-1}) + Q_{11}(u_{n,m}, u_{n,m-1}, u_{n,m+1})$. In a similar way, if we differentiate the resulting determining equation with respect to $u_{n+2,m}$ and to $u_{n,m-1}$, we have that Q_0 reduces to $Q_0 = Q_{00}(u_{n-1,m}, u_{n,m}, u_{n,m-1}) + Q_{01}(u_{n-1,m}, u_{n+1,m}, u_{n,m})$. Combining these results and taking into account that $\frac{\partial^2 Q}{\partial u_{n-1,m} \partial u_{n,m-1}} = 0$ we obtain the following form for Q:

$$Q = Q_0(u_{n,m-1}, u_{n,m}, u_{n,m+1}) + Q_1(u_{n-1,m}, u_{n,m}, u_{n+1,m}).$$

So, *the infinitesimal symmetry coefficient is the sum of functions that either involve shifts only in n with m fixed or only in m with n fixed* [60, 76].

Let us consider the case when the symmetry generator is given by

$$\frac{du_{n,m}}{d\epsilon} = Q_1(u_{n-1,m}, u_{n,m}, u_{n+1,m}). \tag{51}$$

This is a differential difference equation depending parametrically on m. Setting $u_{n,m} = u_n$ and $u_{n,m+1} = \tilde{u}_n$, a different solution, the compatible partial difference equation (48) turns out to be an ordinary difference equation for a new solution $\tilde{u}_n$ of eq. (51),

$$\mathcal{E}_1(u_n, u_{n+1}, \tilde{u}_n, \tilde{u}_{n+1}) = 0,$$

i.e. a Bäcklund transformation [53] for (51). A similar result is obtained in case of Q_0. The same splitting will also appear for higher order symmetries of this class of equations.

To find the specific form of Q_1 we have to differentiate the determining equation (49) with respect to the independent variables and get some **further necessary conditions** on its shape. If, for example, we differentiate with respect to $u_{n+2,m}$,

$$\frac{\partial \mathcal{E}_1}{\partial u_{n+1,m}} \frac{\partial T_1 Q_1}{\partial u_{n+2,m}} + \frac{\partial \mathcal{E}_1}{\partial u_{n+1,m+1}} \frac{\partial T_1 T_2 Q_1}{\partial u_{n+2,m}} = 0.$$

Dividing by $\frac{\partial \mathcal{E}_1}{\partial u_{n+1,m}}$ and differentiating the resulting expression with respect to $u_{n,m+1}$, we get

$$\frac{\partial}{\partial u_{n,m+1}} \left[\frac{\frac{\partial \mathcal{E}_1}{\partial u_{n+1,m+1}}}{\frac{\partial \mathcal{E}_1}{\partial u_{n+1,m}}} \frac{\partial T_1 T_2 Q_1}{\partial u_{n+2,m}} \right] = 0.$$

This is a partial differential equation for Q_1 which constrains its shape, and by solving it it will give an expression in terms of functions of lower number of independent variables and possibly some integration constants. Proceeding further in the general study of the class of equations possessing generalized symmetries is extremely hard. So we go over to the **construction of generalized symmetries for given equations**. In the specific case when

$$\begin{aligned} \mathcal{E}_1 &= \alpha(u_{n,m}u_{n+1,m} + u_{n+1,m+1}u_{n,m+1}) \\ &- \beta(u_{n,m}u_{n,m+1} + u_{n+1,m+1}u_{n+1,m}) + \delta(\alpha^2 - \beta^2), \end{aligned}$$

where α, β and δ are constants, we get

$$Q_1 = \frac{u_{n,m}(u_{n+1,m} + u_{n-1,m}) + 2\delta\alpha}{u_{n+1,m} - u_{n-1,m}}.$$

This generalized symmetry was complicate to derive. It would be extremely complicate to derive in this way generalized symmetries involving higher number of points. So we need a different procedure to get them in compact form. They can be derived, using the *Recursion Operator*, which can be derived from the linear problem associated to the nonlinear discrete equation.

4 Generalized symmetries from the integrability properties

Equations with generalized symmetries are rare. Here we present results on the discrete–time Toda Lattice or Hirota equation, as for this equation the integrability is well known and well studied.

4.1 Construction of the discrete-time Toda lattice hierarchy

We start from the discrete Schrödinger Spectral Problem

$$\psi_{n-1,m} + a_{n,m}\psi_{n+1,m} + b_{n,m}\psi_{n,m} \equiv L_{n,m}\psi_{n,m} = \lambda\psi_{n,m}, \tag{52}$$

where $a_{n,m}$ and $b_{n,m}$, for any m reduce to 1 and 0 respectively , as n goes to ∞. In eq.(52) λ is an m-independent spectral parameter. An integrable nonlinear difference-difference equation can be written in operator form as

$$L_{n,m+1} - L_{n,m} = L_{n,m+1}M_{n,m} - M_{n,m}L_{n,m} \tag{53}$$

in terms of the operator $M_{n,m}$ which governs the discrete "time" evolution of the wave function $\psi_{n,m}$ of eq.(52)

$$\psi_{n,m+1} = \psi_{n,m} - M_{n,m}\psi_{n,m}. \tag{54}$$

Let us notice that for the operator $L_{n,m}$ given by eq.(52) we can write:

$$L_{n,m+1} - L_{n,m} = (a_{n,m+1} - a_{n,m})E_n^+ + b_{n,m+1} - b_{n,m}, \tag{55}$$

where E_n^+ is the shift operator in the n-variable such that $E_n^+ f_{n,m} = f_{n+1,m}$ for any function $f_{n,m}$.

We use the by now standard Lax technique [13]. We construct a hierarchy of nonlinear partial discrete equations by requiring that an operator $M_{n,m}$ and two scalar functions $U_{n,m}$ and $V_{n,m}$ satisfy

$$L_{n,m+1}M_{n,m} - M_{n,m}L_{n,m} = U_{n,m}E_n^+ + V_{n,m}. \tag{56}$$

We then construct new functions $\tilde{U}_{n,m}$ and $\tilde{V}_{n,m}$ and a new operator $\tilde{M}_{n,m}$, using the following formulas:

$$L_{n,m+1}\tilde{M}_{n,m} - \tilde{M}_{n,m}L_{n,m} = \tilde{U}_{n,m}E_n^+ + \tilde{V}_{n,m} \tag{57}$$

$$\tilde{M}_{n,m} = L_{n,m+1} M_{n,m} + F_{n,m} E_n^+ + G_{n,m}, \tag{58}$$

where $F_{n,m}$ and $G_{n,m}$ are two scalar functions. Imposing the compatibility condition of eqs.(52, 56 - 58) we get the following hierarchy of equations

$$\begin{pmatrix} a_{n,m+1} - a_{n,m} \\ b_{n,m+1} - b_{n,m} \end{pmatrix} = f_m^1(\mathcal{L}_{n,m}) \begin{pmatrix} (b_{n,m+1} - b_{n+1,m}) \frac{\pi_{n,m+1}}{\pi_{n+1,m}} \\ \frac{\pi_{n-1,m+1}}{\pi_{n,m}} - \frac{\pi_{n,m+1}}{\pi_{n+1,m}} \end{pmatrix} + f_m^2(\mathcal{L}_{n,m}) \begin{pmatrix} a_{n,m+1} - a_{n,m} \\ b_{n,m+1} - b_{n,m} \end{pmatrix}. \tag{59}$$

Here f_m^1 and f_m^2 are entire functions of their argument and $\mathcal{L}$ is the recursion operator of the hierarchy, obtained from eqs.(52,56-58) and given by:

$$\mathcal{L}_{n,m}\begin{pmatrix} p_{n,m} \\ q_{n,m} \end{pmatrix} = \begin{pmatrix} a_{n,m+1}S_{n+2,m} - a_{n,m}S_{n,m} \\ p_{n-1,m} + \Sigma_{n-1,m}\frac{\pi_{n-1,m+1}}{\pi_{n,m}} - \Sigma_{n,m}\frac{\pi_{n,m+1}}{\pi_{n+1,m}} \end{pmatrix} + \begin{pmatrix} b_{n,m+1}p_{n,m} + (b_{n,m+1} - b_{n+1,m})\Sigma_{n,m}\frac{\pi_{n,m+1}}{\pi_{n+1,m}} \\ +b_{n,m+1}q_{n,m} + (b_{n,m+1} - b_{n,m})S_{n,m} \end{pmatrix}. \tag{60}$$

The starting points $\begin{pmatrix} (b_{n,m+1} - b_{n+1,m})\frac{\pi_{n,m+1}}{\pi_{n+1,m}} \\ \frac{\pi_{n-1,m+1}}{\pi_{n,m}} - \frac{\pi_{n,m+1}}{\pi_{n+1,m}} \end{pmatrix}$ and $\begin{pmatrix} a_{n,m+1} - a_{n,m} \\ b_{n,m+1} - b_{n,m} \end{pmatrix}$ are obtained as coefficients of the integration constants for the functions $F_{n,m}$ and $G_{n,m}$. The function $\pi_{n,m}$ is given by

$$\pi_{n,m} = \Pi_{j=n}^{\infty} a_{j,m}, \tag{61}$$

while $S_{n,m}$ and $\Sigma_{n,m}$ are defined as the **bounded solutions** of the equations

$$S_{n+1,m} - S_{n,m} = q_{n,m} \tag{62}$$
$$\Sigma_{n+1,m} - \Sigma_{n,m} = -p_{n+1,m}\frac{\pi_{n+2,m}}{\pi_{n+1,m+1}}$$

The boundedness of the solutions of eqs.(62) it is necessary to get a hierarchy of nonlinear difference-difference equations with well defined evolution of the spectra.

Let us define the reflection and transmission coefficients $R_m(z)$ and $T_m(z)$ in terms of the asymptotic behavior of the function $\psi_{n,m}$

$$\begin{aligned} \lim_{n->\infty} \psi_{n,m}(z) = &\quad \phi_m(z^{-n} + R_m z^m), \\ \lim_{n->-\infty} \psi_{n,m}(z) = &\quad \phi_m T_m z^{-n}, \end{aligned} \tag{63}$$

where ϕ_m is an appropriate normalization function depending just on m. In the case of a generic equation of the discrete Toda lattice hierarchy (59) the discrete evolution of the reflection coefficient turns out to be

$$R_{m+1} = \frac{1 - f_m^2(\lambda) - z f_m^1(\lambda)}{1 - f_m^2(\lambda) - \frac{f_m^1(\lambda)}{z}} R_m. \tag{64}$$

The transmission coefficient T_m does not evolve in m.

Let us notice that, as opposed to the usual case of hierarchies of partial differential or differential-difference equations, the recursion operator (60) depends on both the functions $(a_{n,m}, b_{n,m})$ and $(a_{n,m+1}, b_{n,m+1})$. Thus, in order to write the nonlinear partial difference equation as an evolution equation in which we explicitate the fields at the time $m+1$ in terms of those at the time m, we must write down explicitly the complete system of equations and then solve for the fields at the time $m+1$. It is not guaranteed that this can always be carried out since the equation can represent an implicit evolution in the discrete time.

4.1.1 The discrete Toda lattice

Choosing $f_m^2 = 0$ and $f_m^1 = \alpha$ in eq.(59) we get:

$$a_{n,m+1} - a_{n,m} = \alpha(b_{n,m+1} - b_{n+1,m})\frac{\pi_{n,m+1}}{\pi_{n+1,m}}, \tag{65}$$

$$b_{n,m+1} - b_{n,m} = \alpha(\frac{\pi_{n-1,m+1}}{\pi_{n,m}} - \frac{\pi_{n,m+1}}{\pi_{n+1,m}}). \tag{66}$$

Solving eqs. (65, 66) for $b_{n+1,m} - b_{n,m}$ and taking into account the boundary conditions for the fields $a_{n,m}$ and $b_{n,m}$, we get:

$$b_{n,m} = \alpha + \frac{1}{\alpha} - \alpha \frac{\pi_{n-1,m+1}}{\pi_{n,m}} - \frac{\pi_{n,m}}{\alpha \pi_{n,m+1}}. \tag{67}$$

Substituting eq.(67) into eq.(65) we obtain a single equation of higher order for the field $\pi_{n,m}$:

$$\Delta_{Toda} = \pi_{n-1,m+2} - \frac{1}{\alpha^2}\pi_{n,m} - \pi^2_{n,m+1}\left(\frac{1}{\pi_{n+1,m}} - \frac{1}{\alpha^2 \pi_{n,m+2}}\right) = 0, \tag{68}$$

which, for $\pi_{n,m} = e^{u_{n,m}}$ reads

$$e^{u_{n,m} - u_{n,m+1}} - e^{u_{n,m+1} - u_{n,m+2}} = \alpha^2 \left(e^{u_{n-1,m+2} - u_{n,m+1}} - e^{u_{n,m+1} - u_{n+1,m}}\right), \tag{69}$$

i. e. the well known discrete-time Toda lattice equation [39]. On the left hand side of eq.(69) we can easily obtain the second difference of the function $u_{n,m}$ with respect to the discrete-time m. Thus, defining

$$t = m\sigma; \qquad v_n(t) = u_{n,m}; \qquad \alpha = \sigma^2 \tag{70}$$

we find that eq.(69) reduces to the continuous-time Toda lattice equation:

$$\ddot{v}_n = e^{v_{n-1}-v_n} - e^{v_n - v_{n+1}} + O(\sigma). \tag{71}$$

Eq.(69) has the following Lax pair:

$$\psi_{n-1,m} + (\alpha + \frac{1}{\alpha} - \alpha e^{u_{n-1,m+1}-u_{n,m}} - \frac{e^{u_{n,m}-u_{n,n+1}}}{\alpha})\psi_{n,m} + \tag{72}$$
$$+e^{u_{n,m}-u_{n+1,m}}\psi_{n+1,m} = \lambda\psi_{n,m}$$

$$\psi_{n,m+1} = \psi_{n,m} - \alpha e^{u_{n,m+1}-u_{n+1,m}}\psi_{n+1,m} \tag{73}$$

From eq.(64) we get the evolution of the reflection coeffcient R_m and T_m:

$$R_{m+1} = \frac{1-\alpha z}{1-\frac{\alpha}{z}} R_m. \tag{74}$$

4.2 Isospectral and non-isospectral generalized symmetries for the discrete-time Toda lattice

Infinitesimal symmetries for the discrete-time Toda lattice can be obtained as **commuting flows**, i.e. an infinitesimal symmetry is obtained when its flow in the group parameter and the discrete time evolution commute. These are represented by the **hierarchy of nonlinear differential-difference equations associated to the Schrödinger Spectral Problem (52)**. We can also consider **the nonlinear discrete-time difference equations commuting with the discrete-time Toda lattice**, but these turn out not to form a group of symmetry transformations associated to (69). As we shall see later, they provide us with its **Bäcklund transformations**.

To discuss these issues, it is easier **to work in the space of the spectral parameter** where the nonlinear evolution of the fields is substituted by the **linear evolution of the reflection coefficient**. The two spaces are in one to one correspondence for fields which are asymptotically bounded. In such a situation the discrete-time Toda lattice equation is represented in the spectral space by the following evolution of the reflection coefficient $R_m(z, \epsilon)$ (the transmission coefficient $T_m(z, \epsilon)$ is invariant under the m

evolution):

$$R_{m+1}(z,\epsilon) = \frac{1 - z\alpha}{1 - \frac{\alpha}{z}} R_m(z,\epsilon). \tag{75}$$

where ϵ is the infinitesimal group parameter (see eq.(74)).

Any isospectral deformation ($\frac{dz}{d\epsilon_k} = 0$) of the discrete Schrödinger Spectral Problem (52) is given by

$$\begin{pmatrix} a_{n,m} \\ b_{n,m} \end{pmatrix}_{,\epsilon_k} = (\tilde{\mathcal{L}})^k \begin{pmatrix} a_{n,m}(b_{n,m} - b_{n+1,m}) \\ a_{n-1,m} - a_{n,m} \end{pmatrix}. \tag{76}$$

The recursion operator $\tilde{\mathcal{L}}$ is

$$\tilde{\mathcal{L}} \begin{pmatrix} p_{n,m} \\ q_{n,m} \end{pmatrix} = \begin{pmatrix} p_{n,m}b_{n+1,m} + a_{n,m}(q_{n,m} + q_{n+1,m}) + (b_{n,m} - b_{n+1,m})s_{n,m} \\ b_{n,m}q_{n,m} + p_{n,m} + s_{n-1,m} - s_{n,m} \end{pmatrix} \tag{77}$$

with $s_{n,m}$ given by an asymptotically bounded solution of the inhomogeneous first order equation

$$s_{n+1,m} = \frac{a_{n+1,m}}{a_{n,m}}(s_{n,m} - p_{n,m}). \tag{78}$$

The index k of ϵ_k denotes the fact that this group parameter is associated to the k^{th} equation of the Toda lattice hierarchy (76). In correspondence with eq. (76) we have an evolution (in ϵ_k) of the reflection coefficient associated to the discrete Schrödinger Spectral Problem (52), i.e.

$$\frac{dR_m(z, \epsilon_k)}{d\epsilon_k} = \mu\lambda^k R_m(z, \epsilon_k) \tag{79}$$

with

$$\lambda = z + \frac{1}{z}; \qquad \mu = \frac{1}{z} - z \tag{80}$$

It is easy to prove that the flows (65) and (76) commute by checking that the corresponding flows of the reflection coefficients, given by eq. (75) and (79), commute.

A less obvious calculation has to be done to get the nonisospectral symmetries of the discrete-time Toda lattice equation. In this case we have:

$$\begin{pmatrix} a_{n,m} \\ b_{n,m} \end{pmatrix}_{,\epsilon_k} = f_m^k(\tilde{\mathcal{L}}) \begin{pmatrix} a_{n,m}(b_{n,m} - b_{n+1,m}) \\ a_{n-1,m} - a_{n,m} \end{pmatrix}$$

$$+\tilde{\mathcal{L}}^k \begin{pmatrix} a_{n,m}[(2n+3)b_{n+1,m} - (2n-1)b_{n,m}] \\ b_{n,m}^2 - 4 + 2[(n+1)a_{n,m} - (n-1)a_{n-1,m}] \end{pmatrix}. \tag{81}$$

The function $f_m^k(\lambda)$ depends on the equation under consideration and, for the discrete-time Toda lattice, is obtained as a solution of the difference equation:

$$f_{m+1}^k(\lambda) - f_m^k(\lambda) = -2\lambda^k \frac{2\alpha^2 - \alpha\lambda}{1 + \alpha^2 - \alpha\lambda}. \tag{82}$$

Up to an arbitrary inessential constant the function $f_m^k(\lambda)$ is given by:

$$f_m^k(\lambda) = -2m\lambda^k \frac{2\alpha^2 - \alpha\lambda}{1 + \alpha^2 - \alpha\lambda}. \tag{83}$$

The proof that the flow (81) with f_m^k given by (83) commutes with that of eq.(65) is easily obtained in the space of the spectrum, where the reflection coefficient associated to eq. (81) satisfies the equation

$$\frac{dR_m(z,\epsilon_k)}{d\epsilon_k} = \mu f_m^k(\lambda) R_m(z,\epsilon_k), \qquad \lambda_{\epsilon_k} = \mu^2 \lambda^k. \tag{84}$$

On the l.h.s. we have the total derivative of $R_m(z,\epsilon_k)$ with respect to ϵ_k.

Both the isospectral (for $k \neq 0$) and nonisospectral symmetries involve the dependent variable in different points of the lattice and they are effectively generalized symmetries. As such they are not integrable. They can be used to provide solutions via symmetry reduction. As an example of these symmetries we write down the simplest nonisospectral symmetry obtained for $k = 0$ and $\alpha = 1$ and given by:

$$\begin{aligned} \begin{pmatrix} a_{n,m} \\ b_{n,m} \end{pmatrix}_{,\epsilon_0} = & \quad -2m \begin{pmatrix} a_{n,m}(b_{n,m} - b_{n+1,m}) \\ a_{n-1,m} - a_{n,m} \end{pmatrix} \\ & + \begin{pmatrix} a_{n,m}[(2n+3)b_{n+1,m} - (2n-1)b_{n,m}] \\ b_{n,m}^2 - 4 + 2[(n+1)a_{n,m} - (n-1)a_{n-1,m}] \end{pmatrix}. \end{aligned} \tag{85}$$

Taking into account eq. (61), we can rewrite eq. (85) as:

$$(\pi_{n,m})_{,\epsilon_0} = \pi_{n,m}\{-(2m+2n+1)b_{n,m} + 2\sum_{j=n}^{\infty} b_{j,m}\} \tag{86}$$

$$(b_{n,m})_{,\epsilon_0} = b_{n,m}^2 - 4 + 2[(n+m+1)a_{n,m} - (n+m-1)a_{n-1,m}].$$

In view of eq.(67), $b_{n,m}$ can be rewritten in terms of $\pi_{n,m}$ and its shifted values.

A symmetry reduction with respect to the symmetry given by eq.(86) is obtained by solving the discrete time Toda lattice (68) together with the equation we get by equating to zero the r.h.s. of eq.(86), i.e.

$$\begin{aligned} &(2m+2n-1)b_{n,m} - (2m+2n+3)b_{n+1,m} = 0, \\ &a_{n,m}[2(n+1)+2m] - a_{n-1,m}[2(n-1)+2m] = 4 - b_{n,m}^2. \end{aligned} \tag{87}$$

The general solution is given by

$$
\begin{aligned}
(88)\quad b_{n,m} &= \frac{b^0_m}{(2m+2n-1)(2m+2n+1)}, \\
a_{n,m} &= \frac{1}{(2n+2m+2)(2n+2m)}[a^0_m + 4n(2m+1+n) + \frac{(b^0_m)^2}{4(2m+2n+1)^2}].
\end{aligned}
$$

Using

$$
a_{n,m+1} - a_{n,m} = \alpha(b_{n,m+1} - b_{n+1,m})\frac{\pi_{n,m+1}}{\pi_{n+1,m}}, \tag{89}
$$

$$
b_{n,m+1} - b_{n,m} = \alpha(\frac{\pi_{n-1,m+1}}{\pi_{n,m}} - \frac{\pi_{n,m+1}}{\pi_{n+1,m}}). \tag{90}
$$

with $\alpha = 1$, we get two equations for b^0_m and a^0_m, the **reduced equations**.

4.3 Bäcklund Transformations and Symmetries

Bäcklund transformations are obtained by the same kind of formulas as those used to get the difference-difference equations when the new functions $(\tilde{a}_{n,m}, \tilde{b}_{n,m})$ are defined as

$$\tilde{a}_{n,m} = a_{n,m+1}; \qquad \tilde{b}_{n,m} = b_{n,m+1}. \tag{91}$$

With this identification the class of Bäcklund transformations associated to the discrete-time Toda lattice hierarchy reads:

$$\delta(\Lambda) \begin{pmatrix} (\tilde{b}_{n,m} - b_{n+1,m}) \frac{\tilde{\pi}_{n,m}}{\pi_{n+1,m}} \\ \frac{\tilde{\pi}_{n-1,m}}{\pi_{n,m}} - \frac{\tilde{\pi}_{n,m}}{\pi_{n+1,m}} \end{pmatrix} = \gamma(\Lambda) \begin{pmatrix} \tilde{a}_{n,m} - a_{n,m} \\ \tilde{b}_{n,m} - b_{n,m} \end{pmatrix}, \tag{92}$$

where Λ is the Bäcklund recursion operator, obtained in the same way as $\mathcal{L}$, and given by:

$$\Lambda \begin{pmatrix} p_{n,m} \\ q_{n,m} \end{pmatrix} = \begin{pmatrix} \tilde{a}_{n,m}(q_{n,m} + q_{n+1,m}) + (a_{n,m} - \tilde{a}_{n,m})\tilde{P}_{n,m} \\ p_{n,m} + \tilde{\Sigma}_{n-1,m} - \tilde{\Sigma}_{n,m} + \tilde{b}_{n,m} q_{n,m} \end{pmatrix} + \begin{pmatrix} b_{n+1,m} p_{n,m} + (\tilde{b}_{n,m} - b_{n+1,m})\tilde{\Sigma}_{n,m} \\ (b_{n,m} - \tilde{b}_{n,m})\tilde{P}_{n,m} \end{pmatrix}. \tag{93}$$

Above, $\tilde{\Sigma}_{n,m}$ and $\tilde{P}_{n,m}$ are now defined as the bounded solutions to the following difference equations:

$$\begin{aligned} \tilde{P}_{n,m} - \tilde{P}_{n+1,m} &= q_{n,m} \\ \tilde{\Sigma}_{n,m}\frac{\pi_{n+1,m}}{\tilde{\pi}_{n,m}} - \tilde{\Sigma}_{n+1,m}\frac{\pi_{n+2,m}}{\tilde{\pi}_{n+1,m}} &= p_{n,m}\frac{\pi_{n+1,m}}{\tilde{\pi}_{n,m}}. \end{aligned} \tag{94}$$

γ and δ are entire functions of their arguments. Eq. (93) corresponds asymptotically to

$$\tilde{R}_m = \frac{\gamma(\lambda) - z\delta(\lambda)}{\gamma(\lambda) - \frac{\delta(\lambda)}{z}} R_m. \tag{95}$$

The simplest Bäcklund transformation is obtained by choosing $\gamma = 1$ and δ constant and reads:

$$\begin{aligned} \tilde{a}_{n,m} - a_{n,m} &= \delta(\tilde{b}_{n,m} - b_{n+1,m})\frac{\tilde{\pi}_{n,m}}{\pi_{n+1,m}}, \\ \tilde{b}_{n,m} - b_{n,m} &= \delta\left[\frac{\tilde{\pi}_{n-1,m}}{\pi_{n,m}} - \frac{\tilde{\pi}_{n,m}}{\pi_{n+1,m}}\right]. \end{aligned} \tag{96}$$

It is worthwhile to recall that while **the composition of two Bäcklund transformations is still a Bäcklund transformation**, however of higher order, **the Bäcklund transformations do not form a Lie group as the product of two Bäcklund transformations does not give a Bäcklund transformation of the same form as the original ones**.

Any Bäcklund transformation can be written as a superposition of an infinite number of symmetries and a symmetry as a superposition of an infinite number of Bäcklund.

Similar results on the generalized symmetries of the lattice potential KdV

$$(p - q + u_{n,m+1} - u_{n+1,m})(p + q - u_{n+1,m+1} + u_{n,m}) - (p^2 - q^2) = 0.$$

have been published in *J. Phys. A* (2007), but more on this is still to be done.

References

[1] N.H. Abel, Méthode générale pour trouver des fonctions d'une seule quantité variable, lorsqu'une propriété de cettes fonctions est esprimée par une équation entre deux variables, *Magazin for Naturvidenskaberne* Aargang 1, Bind 1, Christiania 1823.

[2] M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform*, (SIAM, Providence, R.I., 1981).

[3] see J. Aczel, *Lectures on Functional Equations and their Applications*, (AP, New York 1966) expecially chapter IV and references therein included.

[4] V. E. Adler, A.B. Shabat and R. Yamilov, 2000 Symmetry Approach to the Integrability Problem *Theor. Math. Phys.* **125** 1603-1661.

[5] R. P. Agarwal, *Difference equations and inequalities: theory, methods and applications*, (Dekker, New York 1992).

[6] R.L. Anderson and N.H. Ibragimov, 1979 *Lie-Bäcklund Transformations in Applications* (SIAM, Philadelphia).

[7] V. A. Baikov, R. K. Gazizov, N. H. Ibragimov, Approximate symmetries of equations with a small parameter. (Russian) *Akad. Nauk SSSR Inst. Prikl. Mat.* Preprint 1987, no. 150, 27 pp.

[8] V. A. Baikov, R. K. Gazizov, N. H. Ibragimov,(2-AOS-M) Approximate symmetries. (Russian)*Mat. Sb.* (N.S.) 136(178) (1988), 435–450, 590; translation in Math. USSR-Sb. 64 (1989), no. 2, 427–441

[9] G. W. Bluman and S.Anco, 2002 *Symmetry and Integration Methods for Differential Equations* (Springer & Verlag, New York).

[10] G. W. Bluman and J. D. Cole, 1969 The general similarity solutions of the heat equation *J. Math. Mech.* **18** 1025–1042.

[11] G. W. Bluman and S. Kumei, 1989 *Symmetries of Differential Equations* (Springer-Verlag, New York).

[12] A. Bourlioux, C. Cyr-Gagnon and P. Winternitz,2006 Difference schemes with point symmetries and their numerical tests *J. Phys. A: Math. Gen.* **39** 6877–6896.

[13] M. Bruschi and O. Ragnisco, Nonlinear differential-difference equations,

associated Bäcklund transformations and Lax technique, *J. Phys. A: Math. Gen.* **14** (1981) 1075-1081

[14] F. Calogero, Why are certain nonlinear PDEs both widely applicable and integrable? *What is integrability?*, 1–62, (Springer, Berlin, 1991).

[15] A. Campa, A. Giansanti, A. Tenenbaum, D. Levi, and O. Ragnisco, 1993 Quasi-solitons on a diatomic chain at room temperature, *Phys. Rev.* **48**, 10168.

[16] B.J. Cantwell, *Introduction to Symmetry Analysis* (CUP, Cambridge, 2002).

[17] D Catalano Ferraioli, Nonlocal aspects of λ-symmetries and ODEs reduction, *Journal of Physics A: Mathematical and Theoretical* **40**:21 5479-5489 2007.

[18] D Catalano Ferraioli and P. Morando, Local and nonlocal solvable structures in the reduction of ODEs, *J. Phys. A: Math. Theor.* **42** 035210 (15pp) 2009.

[19] G. Cicogna, 2004 Weak symmetries and adapted variables for differential equations *Int. J. Geom. Meth. Mod. Phys.* **1** 23–31.

[20] G. Cicogna and G. Gaeta, *Symmetry and perturbation theory in nonlinear dynamics* (Springer, Berlin, 1999).

[21] G. Cicogna and G. Gaeta, 2001 Partial Lie-point symmetries of differential equations *J. Phys. A-Math. Gen.* **34** 491?-512.

[22] G. Cicogna, G. Gaeta and P. Morando, 2004 On the relation between standard and μ-symmetries for PDEs, *J. Phys. A: Math. Gen.* **37** 9467?-9486.

[23] P.A. Clarkson, Painlevé equations – Nonlinear Special Functions, in *Orthogonal Polynomials and Special Functions - Computation and Applications* Lecture Notes in Mathematics 1883 F. Marcellan W.Van Assche (Eds.) (Springer, Berlin 2006).

[24] P. A. Clarkson and M. D. Kruskal, 1989 New similarity reductions of the Boussinesq equation *J. Math. Phys.* **30** 2201–2213.

[25] A. S. Davydov, 1979 Solitons in molecular systems, *Phys. Scr.* **20**, 387–394; *Biology and Quantum Mechanics*, (Pergamon, New York 1982).

[26] Dorodnitsyn V.A.,A finite-difference analogue of Noether's theorem,*Phys. Dokl.* **38**/2 66–68 (1993).

[27] S. N. Elaydi, *An Introduction to Difference Equations*, (Springer, New York, 1999)

[28] A. S. Fokas, 1980 A symmetry approach to exactly solvable evolution equations *J.*

Math. Phys. **21** 1318–1325.

[29] V. I. Fushchich, N. I. Serov and V. I. Chopik, 1988 Conditional invariance and nonlinear heat equations. (Russian. English summary) *Dokl. Akad. Nauk Ukrain. SSR Ser. A* **86** 17–21.

[30] W. I. Fushchich and W. M. Shtelen, 1989 On approximate symmetry and approximate solutions of the non-linear wave equation with a small parameter *J. Phys. A: Math. Gen.* **22** L887-L890.

[31] G. Gaeta 1994 Asymptotic symmetries and asymptotically symmetric solutions of partial differential equations *J. Phys. A: Math. Gen.* **27** 437–451.

[32] G. Gaeta and M. A. Rodriguez, 1996 Discrete symmetries of differential equations. *J. Phys. A* **29**, 859–880.

[33] R. Gambini and J. Pullin, Consistent discretizations as a road to quantum gravity, arXiv:gr-qc/0512065v1.

[34] R. N. Garifullin, E. V. Gudkova and I. T. Habibullin, Method for searching higher symmetries for quad-graph equations, *J. Phys. A: Math. Theo* **44** (2011) 325202.

[35] R. N. Garifullin and R. I. Yamilov, Generalized symmetry classification of discrete equations of a class depending on twelve parameters, *J. Phys. A: Math. Theo* **45** (2012) 345205.

[36] W. Hereman, 1994 Review of symbolic software for the computation of Lie symmetries of differential equations, *Euromath Bulletin*, **1** pp. 45–82.

[37] R. Hernández Heredero, D. Levi, M.A. Rodriguez and P. Winternitz, Lie algebra contractions and symmetries of the Toda hierarchy, *J. Phys. A: Math.Gen.* **33** (2000) 5025–5040

[38] R. Hernández–Heredero and E.G. Reyes, Nonlocal symmetries and a Darboux transformation for the Camassa-Holm equation, *J. Phys. A: Math. Theo* **42** 182002 (9pp).

[39] R. Hirota, Nonlinear partial difference equations. II. discrete-time Toda equation. *J. Phys. Soc. Japan* **43** (1977) 2074–2078

[40] P. E. Hydon, *Symmetry Methods for Differential Equations* (CUP, Cambridge, 2000).

[41] Hydon P.E. and Mansfield E.L.,On the variational complex for difference equations, *Found. Comput. Math.* **4** 187–217 (2004).

[42] N. H. Ibragimov, *Transformation Groups Applied to Mathematical Physics* (Reidel, Boston, 1985) .

[43] N. H. Ibragimov ed., *CRC Handbook of Lie Group Analysis of Differential Equations. Vol. I: Symmetries, exact solutions and conservation laws* (CRC Press, Baton Rouge, 1993).

[44] N. H. Ibragimov ed., *CRC Handbook of Lie Group Analysis of Differential Equations. Vol. II: Applications in Engineering and Physicas Sciences* (CRC Press, Baton Rouge, 1994).

[45] N. H. Ibragimov ed., *CRC Handbook of Lie Group Analysis of Differential Equations. Vol. III* (CRC Press, Baton Rouge, 1995).

[46] Inönüe and Wigner E P, 1953 On the contraction of groups and their representations *Proc. Natl. Acad. Sci. USA* **39** 510-?24

[47] N. Jacobson, *Lie Algebras* (Interscience, New York, 1962).

[48] W. G. Kelley and A. C. Peterson, *Difference equations: an introduction with applications*, (AP, New York 1991).

[49] P. H. M. Kersten and I. S. Krasil'shchik, *Symmetries and Recursion Operators for Classical and Supersymmetric Differential Equations* (Kluwer Academic Publisher, Dordrecht, 2000).

[50] I. S. Krasil'shchik and A.M. Vinogradov, Nonlocal trends in the geometry of differential equations: symmetries, conservation laws, and B?cklund transformations. Symmetries of partial differential equations, Part I. *Acta Appl. Math.* **15**:1-2 (1989) 161–209.

[51] D. Levi, Nonlinear differential-difference equations as B?cklund transformations, *J. Phys. A* **14** 1083–1098 (1981).

[52] D. Levi and R. Benguria, Bäcklund transformations and nonlinear differential difference equations. *Proc. Nat. Acad. Sci. U.S.A.* **77** 5025–5027 (1980).

[53] Levi D., Petrera M., Scimiterna C. and Yamilov R., On Miura transformations and Volterra-type equations associated with the Adler-Bobenko-Suris equations, *SIGMA* **4** (2008) 077, 14 pp.

[54] D. Levi and M.A. Rodriguez, 2004 Lie discrete symmetries of lattice equations, *J. Phys. A* **37** 1711–1725.

[55] D. Levi, Z. Thomova and P. Winternitz, Contact transformations for discrete equations, *J. Phys. A* submitted.

[56] D. Levi, S. Tremblay and P. Winternitz, 2001 Lie point symmetries of difference equations and lattices, *J. Phys. A: Math. Gen.* **33**, 8507–8523 (2000); Lie symmetries of multidimensional difference equations, *J. Phys. A: Math. Gen.* **34**, 9507–9524.

[57] D. Levi D and P. Winternitz, 1989 Nonclassical symmetry reduction: Example of the Boussinesq equation *J. Phys. A: Math. Gen.* **22** 2915–2924.

[58] D. Levi and P. Winternitz, 2006 Continuous symmetries of difference equations *J. Phys. A: Math. Gen.* **39** R1-R63

[59] D. Levi and R. Yamilov, Conditions for the existence of higher symmetries of evolutionary equations on the lattice,*J. Math. Phys.* **38**:12 (1997) 6648–6674.

[60] D. Levi and R.I. Yamilov, The generalized symmetry method for discrete

equations, `arXiv:0902.4421v1`, *J. Phys. A: Math. Theor.* **42** (2009) 454012.

[61] D. Levi and R.I. Yamilov, Generalized symmetry integrability test for discrete equations on the square lattice *J. Phys. A: Math. Theor.* **44** (2011) 145207.

[62] R. Mancinelli, D. Vergni and A. Vulpiani, 2003 Front propagation in reactive systems with anomalous diffusion, *Physica* **D 185**, 175–195.

[63] A.V. Mikhailov, A.B. Shabat and V.V. Sokolov, The symmetry approach to classification of integrable equations, In: *What is Integrability?*, edited by V.E. Zakharov, Springer, Berlin, 1991, pp. 115–184.

[64] A.V. Mikhailov, A.B. Shabat and R.I. Yamilov, The symmetry approach to the classification of nonlinear equations. Complete lists of integrable systems, *Russian Math. Surveys* **42**:4 (1987) 1–63.

[65] A. V. Mikhailov, J. P. Wang and P. Xenitidis, Recursion operators, conservation laws, and integrability conditions for difference equations, *Theo. Math. Phys.* **167** (2011) 421–443.

[66] A. V. Mikhailov and P. Xenitidis, Second Order Integrability Conditions for

Difference Equations: An Integrable Equation, *Letts. Math. Phys.* (2013) 1–20.

[67] Muriel, C., Romero, J.L, New methods of reduction for ordinary differential equations, *IMA Journal of Applied Mathematics* **66**:2 (2001) 111-125.

[68] Noether E 1918 Invariante variationsprobleme *Nachr. Ges. Wiss. Goettingen, Math.-Phys.* Kl.**2** 235?57 See Noether E 1971 *Transp. Theory Stat. Phys.* **1** 186 (Engl. Transl.)

[69] C. Nucci and P. A. Clarkson, 1992 The nonclassical method is more general than the direct method for symmetry reductions. An example of the FitzHugh-Nagumo equation. *Phys. Lett.* **A 164** 49–56.

[70] P. J. Olver *Applications of Lie Groups to Differential Equations* (Springer-Verlag, New York, 1993).

[71] P. J. Olver and Ph. Rosenau, 1987 Group invariant solutions of differential equations *SIAM J. Appl. Math.* **47** 263–278.

[72] P. J. Olver and Ph. Rosenau, 1987 The construction of special solutions to partial differential equations, *Phys. Lett.* **114 A** 107–112.

[73] L. V. Ovsiannikov, *Group Analysis of Differential Equations* (AP, New York, 1982).

[74] J. Patera, P. Winternitz and H. Zassenhausm, 1975 Continuous subgroups of the fundamental groups of physics. I. General method and the Poincar? group. *J. Math. Phys.* **16** 1597–1614.

[75] Pucci, 1992 Similarity reductions of partial differential equations, *J. Phys. A* **25** 2631-2640.

[76] O. G. Rasin, P. E. Hydon, 2007, Symmetries of integrable difference equations on the quad-graph, *Stud. Appl. Math.* **119** 253–269.

[77] G. J. Reid, D. T.Weih, and A. D. Wittkopf, A point symmetry group of a differential equation which cannot be found using infinitesimal methods, *Modern Group Analysis: Advances Analytical and Computational Methods in Mathematical Physics* N. H. Ibraghimov, M. Torrisi, and A. Valenti, eds.,(Kluwer, Dordrecht, 1993), pp. 93–99.

[78] E.G. Reyes, Nonlocal symmetries and the Kaup-Kupershmidt equation. *J. Math. Phys.* **46**:7 (2005), 073507, 19 pp. .

[79] E.G. Reyes, On nonlocal symmetries of some shallow water equations. *J. Phys. A* **40**:17 (2007) 4467–4476.

[80] A.H. Sakka, Linear Problems and Hierarchies of Painlevé Equations, *J. Phys. A: Math. Theor.* **42** 025210 (19 pp) (2009).

[81] A. C. Scott, Davydov's soliton, *Phys. Rep.* **217**, 1–67 (1992).

[82] W. H. Steeb, *Continuous Symmetries, Lie Algebras, Differential Equations and Computer Algebra* (World Scientific, Singapore, 1996).

[83] H. Stephani, *Differential Equations: Their Solution Using Symmetries* ed. MacCallum M (CUP, Cambridge, 1990).

[84] O. Stormark, *Lie's Structural Approach to PDE Systems* (CUP, Cambridge, 2000).

[85] M. Toda, *Theory of Nonlinear Lattice* (Springer, New York 1988).

[86] A.M. Vinogradov and I.S. Krasil?shchik, A method of calculating higher symmetries of nonlinear evolutionary equations, and nonlocal symmetries. (Russian) *Dokl. Akad. Nauk SSSR* **253**:6 (1980), 1289–1293

[87] P. Winternitz, 1993 Lie Groups and solutions of nonlinear partial differential equations. In: *Integrable Systems, Quantum Groups and Quantum Fields Theories*, eds. A. Ibort and M.A. Rodriguez (Kluwer, Dordrecht) 429–495.

[88] Yamilov R.I.,On conservation laws for the discrete Korteweg-de Vries equation, *Dinamika Sploshnoi Sredy* **44** 164–173 (1980) (Soviet Acad. Sci., Sibirian Branch, Institute of Hydrodynamics, Novosibirsk) (in Russian).

[89] R.I. Yamilov, Classification of discrete evolution equations, *Uspekhi Mat. Nauk* **38**:6 (1983) 155–156 [in Russian].

[90] R. Yamilov, 2006 Symmetries as integrability criteria for differential difference equations *J. Phys. A: Math. Gen.* **39** R541–R623.

[91] R.I. Yamilov, Integrability conditions for an analogue of the relativistic Toda chain, *Theor. Math. Phys.* **151**:1 (2007) 492–504.

[92] R. Yamilov and D. Levi, Integrability conditions for n and t dependent dynamical lattice equations,*J. Nonl. Math. Phys.* **11**:1 (2004) 75–101.