Revenue Maximisation with Tatkal Seva

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Background

- Server is available for fixed durations or serves a fixed number of customers, e.g., concerts and plays, and some restaurants.
- Another view: congestion events last for finite durations; transient analysis are of interest.
- Cost to customer has many components; waiting time, time at which service is completed, price paid for higher service grade.

Background

- Heterogeneous customers

- Different customers weight different costs differently and hence make their choices strategically.
- Server can exploit customer heterogeneity and offer different service grades and enhance revenue.

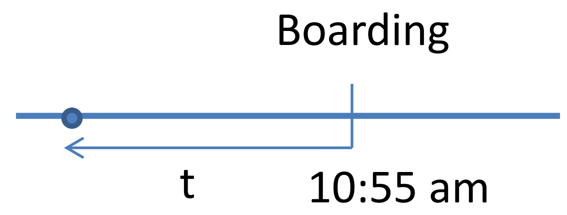
Resource Allocation

- Simple Priority Server
  - Server floats different service grades
  - Server prices the service grades
  - Customer chooses a service grade

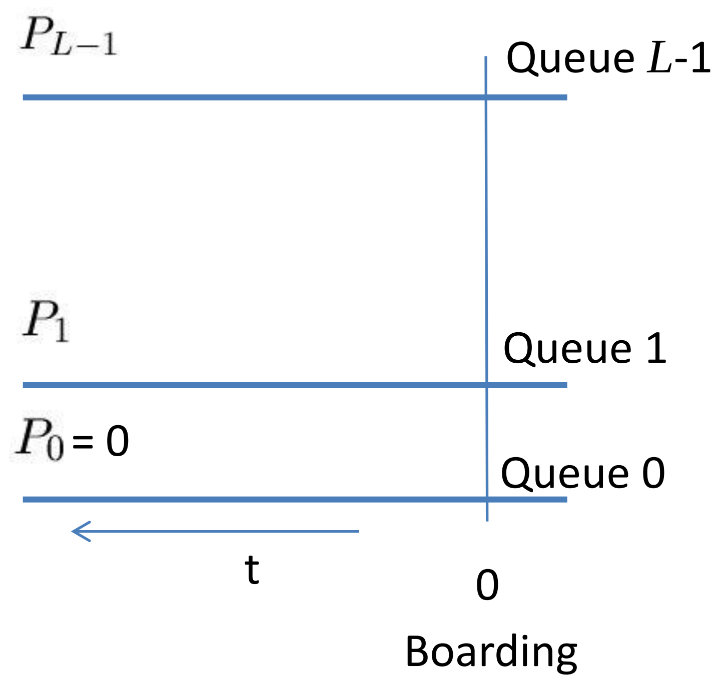
- Complex Priority Server
  - Customers choose two priority parameters
  - They jointly determine the service grade
  - Server prices the priority parameters

Anchor Problem: Airline Boarding

- Boarding begins at time 10:55 am



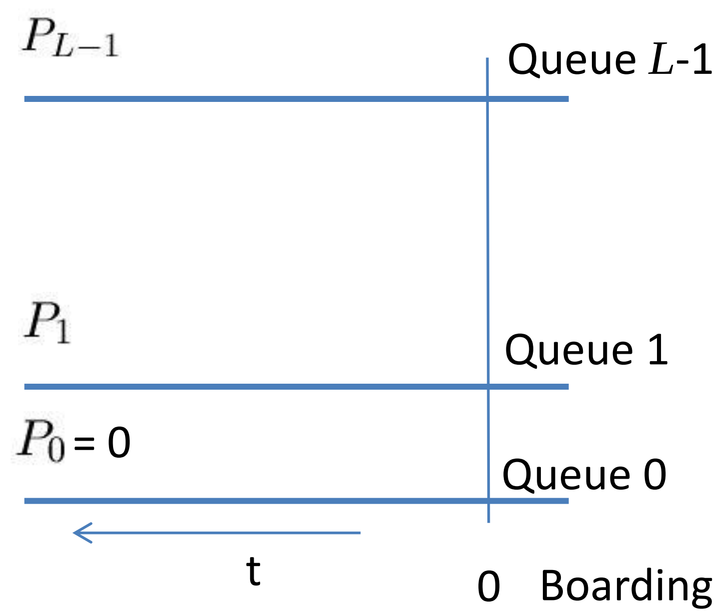
- Passenger: when to arrive?
  - Too late: bad seat
  - Too early: long waiting time
- Airline: Can we earn revenue?
  - Put a `high priority' (tatkal seva) queue and charge for it
  - Customer parameters: arrival time and the queue to join



Airline Boarding Problem

- Complex Priority System
  - Customers choose two priority parameters
  - They jointly determine service grade
  - Server prices the priority parameters

Airline Boarding Game



Simple Priority Server

Simple Priority Server

- Heterogeneous population
  - Customer type $v \in \mathcal{V} = [A, B]$ where $0 \leq A < B \leq \infty$
  - Continuous distribution $C(v)$
- Single priority parameter: $0 \leq w \leq 1$
- Price function: $P(w)$ is ↑ in w, $P(0) = 0$
- Cost for type v customer to pay price P is

$$m_v P$$

  - m_v is a decreasing function

Customer's Cost Function

- Customer v chooses priority $w(v)$
- Total cost for a type v customer

$$c_v(w(v)) = \underbrace{m_v P(w(v))}_{\text{cost of priority}} + h(\underbrace{F(w(v))}_{\text{fraction with better priority}})$$

where $$F(w(v)) = \int \mathbb{1}_{w(x)>w(v)} dC(x)$$

– h is any increasing function

– this makes the second term general!

Nash Equilibrium

- Total cost for a type v customer is

$$c_v(w(v)) = m_v P(w(v)) + h(F(w(v)))$$

where $w(v)$ is his priority choice

Definition 1. *A $w^{NE}(v)$ is a stable, or a Nash Equilibrium, policy if for all $v \in [A, B]$,*

$$w^{NE}(v) = \operatorname*{argmin}_{0 \le w \le 1} c_v(w).$$

Structure of Nash Equilibrium

$$c_v(w(v)) = m_v P(w(v)) + h(F(w(v)))$$

Assuming existence of NE,

- $w^{NE}(v)$ is a non-decreasing function in v
  - Intuition: since value for paying price P ↓ in v

Proof sketch:

- $$\begin{aligned} c_v(w^{\mathrm{NE}}(v)) &\leq c_v\left(w^{\mathrm{NE}}(v+h)\right) \\ &= m_v P\left(w^{\mathrm{NE}}(v+h)\right) + NF\left(w^{\mathrm{NE}}(v+h)\right) \\ &= m_v P\left(w^{\mathrm{NE}}(v+h)\right) + c_{v+h}\left(w^{\mathrm{NE}}(v+h)\right) \\ &\quad - m_{v+h} P\left(w^{\mathrm{NE}}(v+h)\right). \end{aligned}$$

- $$\begin{aligned} c_{v+h}(w^{\mathrm{NE}}(v+h)) \leq m_{v+h} P\left(w^{\mathrm{NE}}(v)\right) + c_v\left(w^{\mathrm{NE}}(v)\right) \\ - m_v P\left(w^{\mathrm{NE}}(v)\right). \end{aligned}$$

Proof sketch:

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- $$\begin{aligned} c_{v+h}(w^{\mathrm{NE}}(v+h)) \leq m_{v+h} P\left(w^{\mathrm{NE}}(v)\right) + c_v\left(w^{\mathrm{NE}}(v)\right) \\ - m_v P\left(w^{\mathrm{NE}}(v)\right). \end{aligned}$$

- Adding the two:

$$0 \leq (m_v - m_{v+h})\left(P\left(w^{\mathrm{NE}}(v+h)\right) - P\left(w^{\mathrm{NE}}(v)\right)\right)$$

- This implies $w^{\mathrm{NE}}(v+h) \geq w^{\mathrm{NE}}(v)$

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Structure of Nash Equilibrium

$$c_v(w(v)) = m_v P(w(v)) + h(F(w(v)))$$

Assuming existence of NE,

- $w^{NE}(v)$ is a non-decreasing function in v
  - Intuition: since value for paying price P $\downarrow$ in v
- $w^{NE}(v)$ is given by

$$w^{NE}(v) = P^{-1}\left(\int_A^v \frac{h'(1-C(v))\,dC(v)}{m_v}\right)$$

Proof sketch:

- $$F(w^{\mathrm{NE}}(v)) = \int \mathbb{1}_{w^{\mathrm{NE}}(x) > w^{\mathrm{NE}}(v)} dC(x)$$
  $$= \int \mathbb{1}_{x>v} dC(x) = 1 - C(v).$$

Since, $w^{NE}(v)$ is non-dec. in v

Proof sketch:

$$c_v(w(v)) = m_v P(w(v)) + h(F(w(v)))$$

- $$F(w^{\text{NE}}(v)) = \int \mathbb{1}_{w^{\text{NE}}(x) > w^{\text{NE}}(v)} dC(x) = \int \mathbb{1}_{x>v} dC(x) = 1 - C(v).$$

Since, $w^{NE}(v)$ is non-dec. in v

- $$0 = \left.\frac{dc_v(w)}{dw}\right|_{w=w^{\text{NE}}(v)} = Nh'\left(F\left(w^{\text{NE}}(v)\right)\right) \left.\frac{dF(w)}{dw}\right|_{w=w^{\text{NE}}(v)} + m_v P'\left(w^{\text{NE}}(v)\right),$$

Optimality of $w^{NE}(v)$

- $$-\frac{dC(v)}{dv} = \frac{dF(w^{\text{NE}}(v))}{dv} = \left.\frac{dF(w)}{dw}\right|_{w=w^{\text{NE}}(v)} \times \frac{dw^{\text{NE}}(v)}{dv}.$$

- $$N\frac{h'\left(F\left(w^{\text{NE}}(v)\right)\right)}{m_v}\frac{dC(v)}{dv} = P'\left(w^{\text{NE}}(v)\right)\frac{dw^{\text{NE}}(v)}{dv}$$

Proof sketch:

$$c_v(w(v)) = m_v P(w(v)) + h(F(w(v)))$$

- $$0 = \left.\frac{dc_v(w)}{dw}\right|_{w=w^{\mathrm{NE}}(v)}$$
  $$= Nh'\left(F\left(w^{\mathrm{NE}}(v)\right)\right)\left.\frac{dF(w)}{dw}\right|_{w=w^{\mathrm{NE}}(v)} + m_v P'\left(w^{\mathrm{NE}}(v)\right),$$

- $$-\frac{dC(v)}{dv} = \frac{dF(w^{\mathrm{NE}}(v))}{dv} = \left.\frac{dF(w)}{dw}\right|_{w=w^{\mathrm{NE}}(v)} \times \frac{dw^{\mathrm{NE}}(v)}{dv}.$$

- $$N\frac{h'\left(F\left(w^{\mathrm{NE}}(v)\right)\right)}{m_v}\frac{dC(v)}{dv} = P'\left(w^{\mathrm{NE}}(v)\right)\frac{dw^{\mathrm{NE}}(v)}{dv}$$

- Integrating on both sides

$$P(w^{\mathrm{NE}}(v)) = N\int_A^v \frac{h'\left(F\left(w^{\mathrm{NE}}(x)\right)\right)}{m_x} dC(x) + \alpha$$
$$= N\int_A^x \frac{h'\left(1 - C(x)\right)}{m_x} dC(x) + \alpha,$$

- $\alpha = 0$ because $w^{\mathrm{NE}}(A) = 0$

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Structure of Nash Equilibrium

$$c_v(w(v)) = m_v P(w(v)) + h(F(w(v)))$$

Assuming existence of NE,

- $w^{NE}(v)$ is a non-decreasing function in v
  - Intuition: since value for paying price P ↓ in v
- $w^{NE}(v)$ is given by

$$w^{NE}(v) = P^{-1}\left(\int_A^v \frac{h'(1-C(v))\,dC(v)}{m_v}\right)$$

- Revenue earned

$$R(P) = \int_A^B P(w^{\mathrm{NE}}(v))dC(v).$$

is independent of pricing function P!!

Example

$$c_v(w(v)) = m_v P(w(v)) + h(F(w(v)))$$

- Let $v \sim \mathcal{U}[0,1]$, $h(x) = x$, and $m_v = \frac{1}{v^{l-1}}$
- Then for $P(w) = w^\beta$, for any $\beta > 0$

  the NE exists and is

$$w^{\text{NE}}(v) = \left(\frac{v^l}{l}\right)^{1/\beta} = P^{-1}\left(\frac{v^l}{l}\right)$$

$$F(w) = 1 - l^{\frac{1}{l}} w^{\frac{n}{l}}$$

- Revenue remains the same

Complex Priority Server

Airline Boarding Problem

- Heterogeneous Population $v \in \mathcal{V} = [A, B]$
- Cost

$$c_v(t,l) = \left[N \sum_{j=l+1}^{L-1} F_j(0) + N F_l(t) \right] + m_v P_l + n_v Q(t)$$

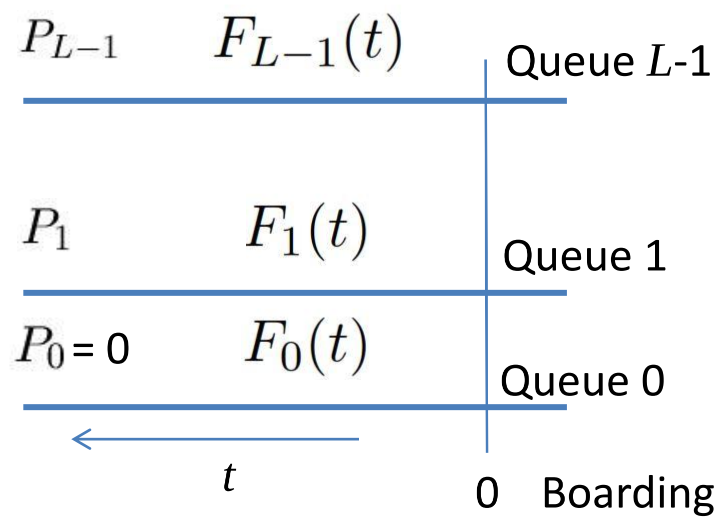
l : queue joined
t : arrival time

fraction that gets served earlier

Cost of paying

Cost of waiting

- Queue l+1 is served before Queue l
- P_l , $Q(t)$ increasing $Q(0) = 0$
- Complementary weights
  - n_v increasing
  - m_v decreasing

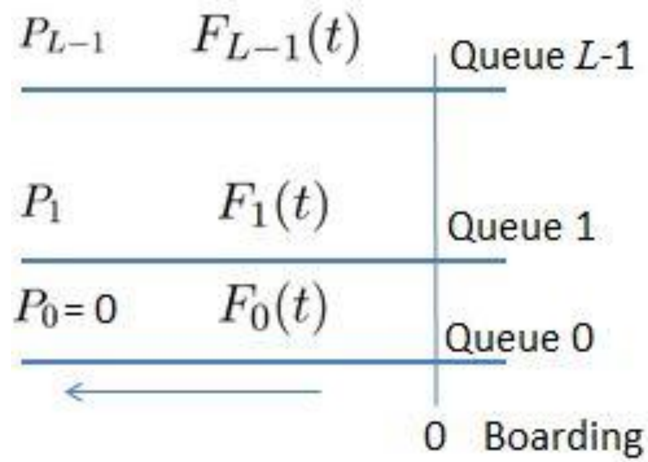


Strategic Customer

- Cost

$$c_v(t,l) = \left[N \sum_{j=l+1}^{L-1} F_j(0) + N F_l(t) \right] + m_v P_l + n_v Q(t)$$

l : queue joined
t : arrival time



- Optimal time to join each queue

$$T_l(v) = \operatorname*{argmin}_{t \geq 0} c_v(l,t)$$

- Decision on the queue to join

$$q(v) = \operatorname*{argmin}_{q_j} \sum_{l \in \mathcal{L}} q_l c_v(l, T_l(v))$$

Nash equilibrium if

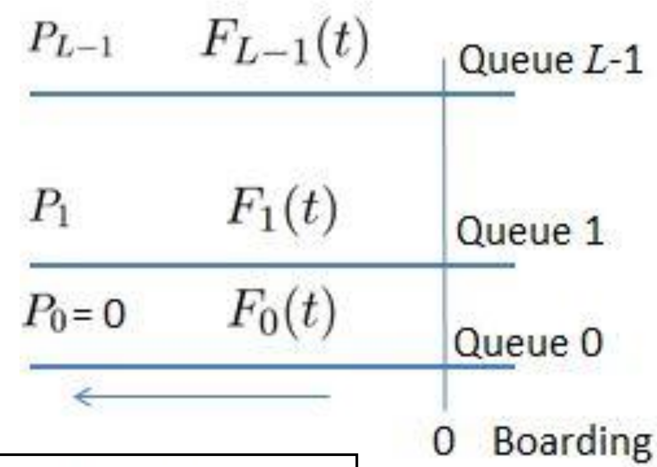
$$F_l(t) = \int_{v \in \mathcal{V}} \mathbb{1}_{T_l(v) \geq t} \; q_l(v) \; dC(v)$$

$$\left(T_l^{\mathrm{NE}}(v), q_l^{\mathrm{NE}}(v)\right)_{l=0}^{L-1}$$

Results and Discussion

$$c_v(t,l) = \left[N \sum_{j=l+1}^{L-1} F_j(0) + N F_l(t) \right] + m_v P_l + n_v Q(t)$$

- $T_l^{NE}(v)$ is non-increasing in v
- $T_l^{NE}(v) \leq$ the NE arrival time if there is only a single queue
- Under certain regularity



Theorem *The NE strategy is unique and is characterized by*

$$q_l^{NE}(v) = \mathbb{1}_{v_l < v \leq v_{l+1}}.$$

Here $A = v_0 < v_1 < v_2 < \cdots < v_{L-1} < v_L = B$ *are given by*

$$c_{l-1}(v_l) = c_l(v_l),$$

for all $l = 1$ *to* $L-1$*, each of which has a unique solution.*

optimal joining costs

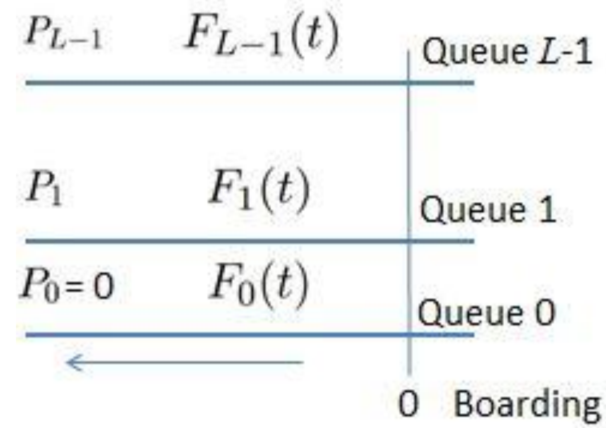
Regularity Conditions

Cost: $c_v(t,l) = \left[N \sum_{j=l+1}^{L-1} F_j(0) + N F_l(t) \right] + m_v P_l + n_v Q(t)$

- $y(v) \triangleq \dfrac{n'_v}{(-m'_v)} \displaystyle\int_v^B \frac{dC(x)}{n_x}$ is bounded
- $\int_A^B \dfrac{dC(x)}{n_x} < \infty$
- $P_{l+1} - P_l > N \max \left\{ \sup_{v \in \mathcal{V}} y(v), \dfrac{2}{m_A} \right\}$

Example

$$c_v(t,l) = \left[N \sum_{j=l+1}^{L-1} F_j(0) + N F_l(t) \right] + m_v P_l + n_v Q(t)$$



- $m_v = \frac{N}{\epsilon(B-A)} \left(B \log\left(\frac{B}{v}\right) - (B-v) \right) + \frac{N\delta}{\epsilon}$
- $n_v = v$
- $v \sim \mathcal{U}[0, 20]$ and L = 3
- $N = 10,\ P_1 = 8.75,\ P_2 = 11.45, \text{ and } \delta = 0.05$

Optimal Arrival Times

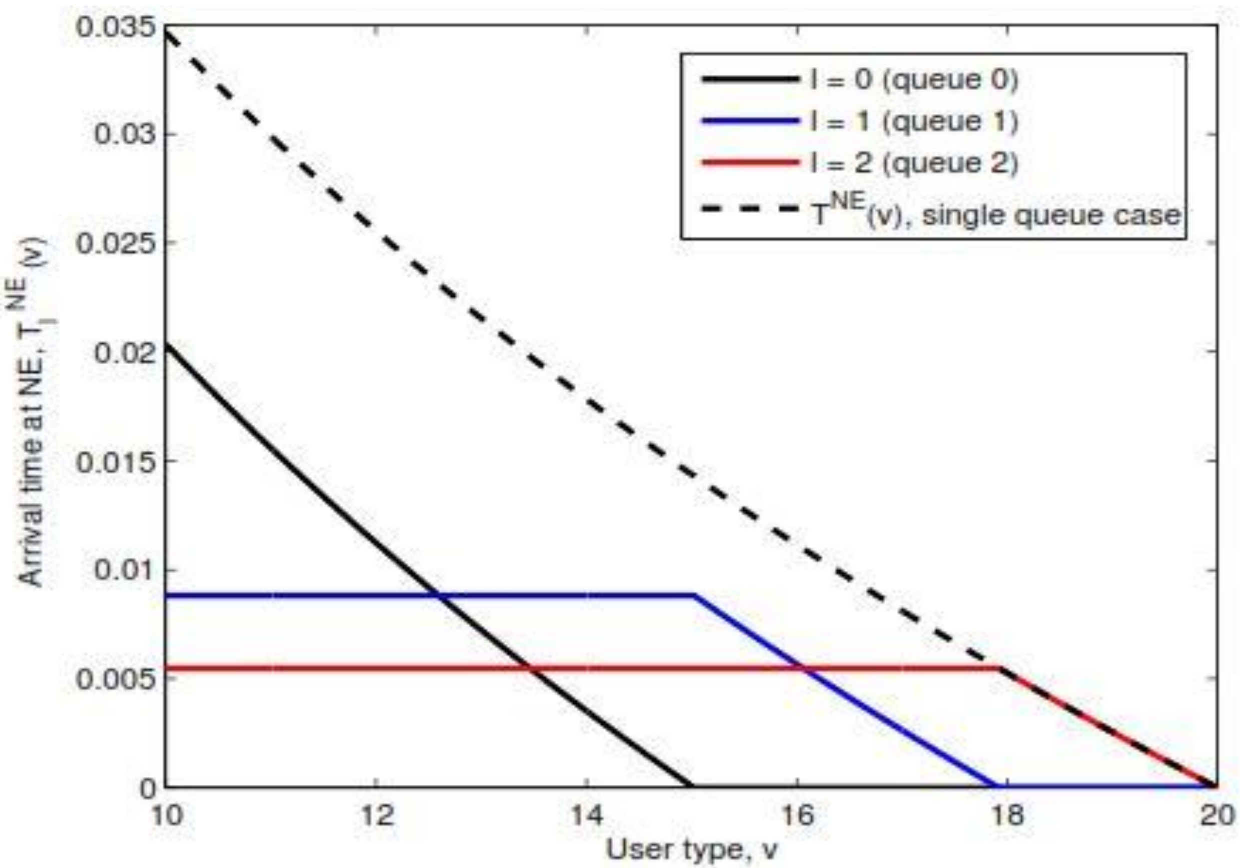
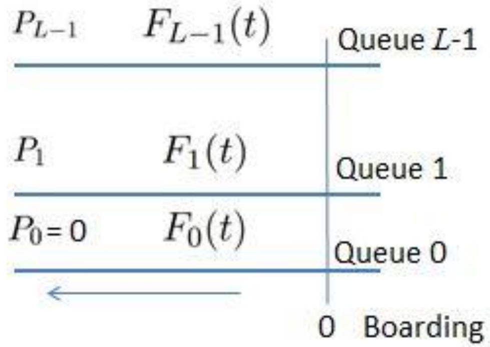


Figure 1: Comparison of NE arrival times at different queues for a system with three queues ($v \sim \mathcal{U}[0, 20]$, $N = 10$, $P_1 = 8.75$, $P_2 = 11.45$ and $\delta = 0.05$).



Optimal Queue Joining Costs

Optimal cost to join Queue l

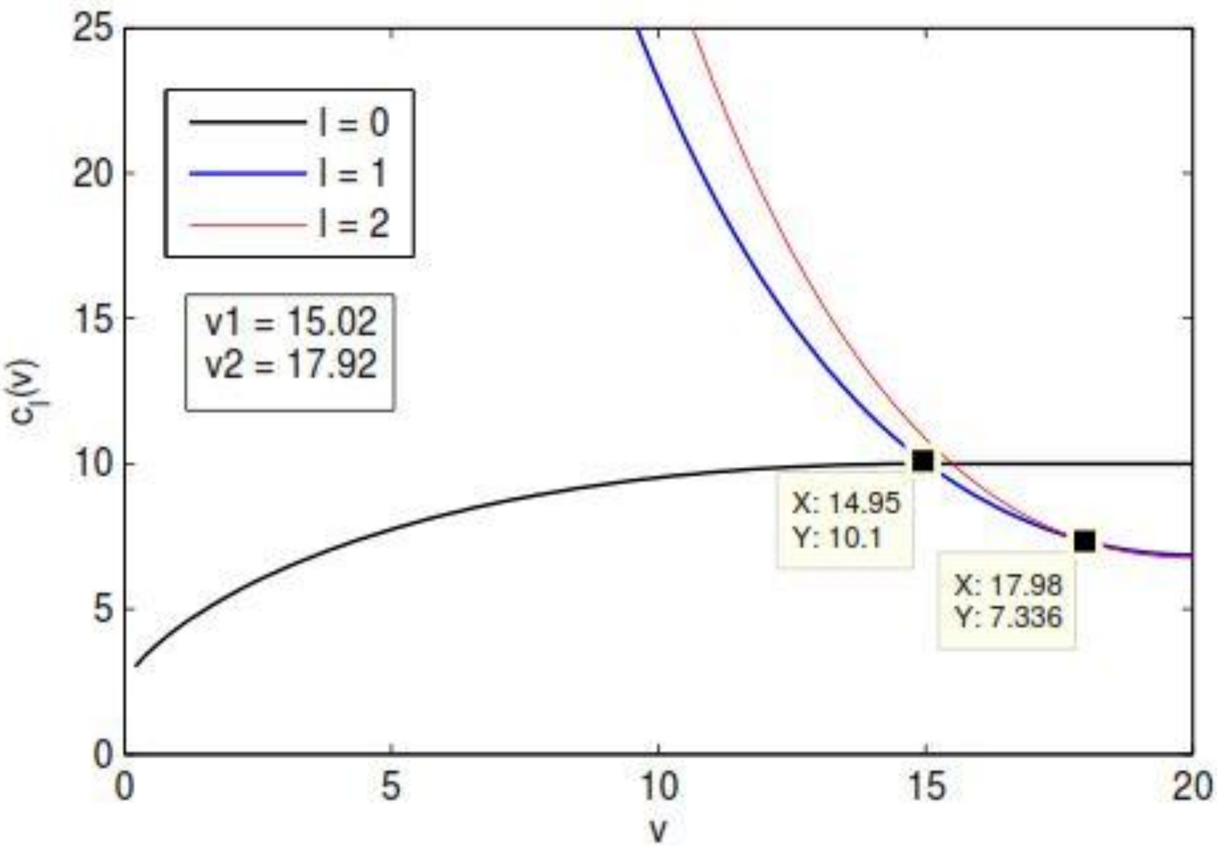
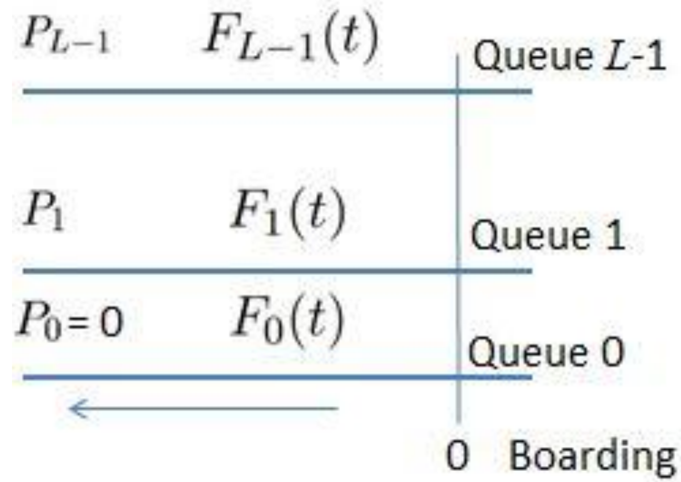


Figure 2: Comparison of optimal costs to join Queue l and illustration of the thresholds ($v \sim \mathcal{U}[0, 20]$, $N = 10$, $P_1 = 8.75$, $P_2 = 11.45$ and $\delta = 0.05$).

Revenue Maximization



- Revenue Earned

$$R(P_1, P_2, \ldots, P_{L-1}) = \sum_{l=1}^{L-1} N F_l(0) P_l$$

- Solved as a Dynamic Program

$$R(P_1, P_2, \ldots, P_{L-1}) = \sum_{j=1}^{L-1} u\left(v_{L-j}, v_{L-j+1}\right)$$

$$A = v_0 < v_1 < v_2 < \cdots < v_{L-1} < v_L = B$$

  – Stage 1: choose v_{L-1}
  – Stage *j*: choose v_{L-j}

Revenue Maximization

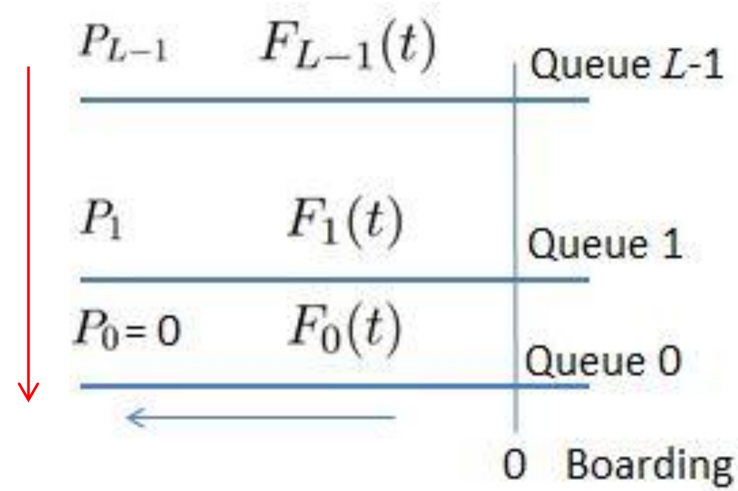
- How many queues?
  - Numerically: four queues gets us near the maximum revenue

Population distribution	Revenue		
	$L=2$	$L=3$	$L=4$
$\mathcal{U}[0,20]$	2.26	2.46	2.50
$\mathcal{U}[0,150]$	7.53	7.83	7.87
$\mathcal{U}[20,150]$	7.41	7.65	7.68

Future Work

- Generalized Prioritization
  - To maximize revenue

??



- Repeated game
  - Learn from outcomes to reach max. revenue

Thank You

Thank you

General Framework and Open Problems

- Cost for a customer v

$$c_v := D(w_1, w_2) + m_v P(w_1) + n_v Q(w_2)$$

  - $w_1 \in \mathcal{W}_1$ and $w_2 \in \mathcal{W}_2$ are priority parameters
  - $D(w_1, w_2)$ determines QoS, depends on others choices

- Airline boarding problem

$$c_v(t, l) = \left[N \sum_{j=l+1}^{L-1} F_j(0) + N F_l(t) \right] + m_v P_l + n_v Q(t)$$