# A dynamic model of capital inflow in an infinite horizon framework

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08 September 2014

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- Extension of Basak et al (2012).
- Discounted utilities over infinite time horizon.
- Multi-country framework.

# Introduction

- Why capital inflow?

1. Capital scarce countries need capital to support production sector.
2. Capital rich countries lend at higher interest rates.

- Construction of an optimal portfolio of foreign and domestic funds.
- Assumption: Banks are symmetric.

# Developed country

- $r^d$ Rate of domestic deposits, $K_t$ amount of deposits at time $t$
- $R^d$ Rate of domestic loans.
- $R_t^*$ Rate of foreign loans at time $t$.
- $F_t$ Funds of bank at time $t$.
- $(1-\mu_t, \mu_t)$ portfolio of lending.

$$\pi_t = F_t((1-\mu_t)(1+R_t^D) + \mu_t(1+R_t^*)\epsilon_t) - K_t^D(1+r_t^D)$$

$$\Omega_t^D = E_t(\pi_t) - \frac{\gamma}{2} V_t(\pi_t^D)$$

# Developed country

- $\max_{\mu_t}[\Omega_t^D + B\Omega_{t+1}^D + B^2\Omega_{t+2}^D + \ldots]$

$$E_t(F_{t+1}) = F_t[(1-\mu_t)(1+R_t^D) + \mu_t(1+R_t^*)E_t(\epsilon_t)] - (1+r_t^D)K_t^D + K_{t+1}^D$$

- Bellman Optimization set-up.

$$V^D(t, F_t) = max_{\mu_t}\{\Omega_t^D + BE_t(V^D(t+1, F_{t+1}))\},$$
$$V^D(F_t) = max_{\mu_t}\{\Omega_t^D + BE_t(V^D(F_{t+1}))\},$$

# Developed country

- Quadratic value function in $F_t$.

$$a + bF_t + cF_t^2 = max_{\mu_t}\{\Omega_t^D + BE_t(V^D(F_{t+1})\},$$

- First order condition gives $\mu_t^*$

$$\mu_t^* = \frac{Z_t(1+Bb) + 2BcZ_tF_t(1+R_t^D) - 2BcZ_tr_{tk}^D}{\gamma F_t^2(1+R_t^*)^2V_t(\epsilon_t) - 2Bc(Z_t^2 + F_t^2(1+R_t^*)^2V_t(\epsilon_t))}$$

$$Z_t = (1+R_t^*)E_t(\epsilon_t) - (1+R^d)$$

- $b$ and $c$ are to be determined.

# $b$ and $c$

- $\mu_t^*$- Optimal portfolio.

$$a + bF_t + cF_t^2 = \Omega_t^D(F_t, \mu_t^*) + BE_t(V(F_{t+1}, \mu_t^*))$$

- Comparing coefficients of $F_t$ and $F_t^2$.
- $b$=$f_1(R_t^*, \mu_t^*)$ and $c = f_2(R_t^*, \mu_t^*)$.
- $\mu_t^* = h(R_t^*, F_t)$
- Key distinction with Basak et al (2012).

# Expression of $mu_t^*$

$$\mu_t^* = \frac{Z_t(1+Bb) + 2BcZ_tF_t(1+R_t^D) - 2BcZ_t r_{tk}^D}{\gamma F_t^2(1+R_t^*)^2 V_t(\epsilon_t) - 2Bc(Z_t^2 + F_t^2(1+R_t^*)^2 V_t(\epsilon_t))}$$

$$c = -\frac{\frac{\gamma}{2}(\mu_t^*)^2(1+R_t^*)^2 V_t(\epsilon_t)}{1 - BE_t(X_t^2)}$$

$$b = \frac{E_t(X_t) - 2Bcr_{tk}^D}{1 - BE_t(X_t)}.$$

$$X_t = (1-\mu_t^*)(1+R_t^D) + \mu_t^*(1+R_t^*)\epsilon_t$$

# Underdeveloped country

- $r^u$ Rate of domestic borrowing(deposits), raises $K_t^U$ amount at time $t$
- $R_t^*$ Rate of foreign borrowing $t$.
- $G_t$ Funds of bank at time $t$.
- $(1-\lambda_t, \lambda_t)$ portfolio of lending.
- $R^U$ Rate of domestic loans, with risk $\eta_t$.

$$\Omega_t^U = E_t(\pi_t^U) - \frac{\beta}{2} V_t(\pi_t) \tag{1}$$

# Bellman Set-up

$$V^U(G_t) = max_{\lambda_t}\{\Omega_t^D + BE_t(V^U(G_{t+1})\}.$$

- Quadratic value function in $G_t$

$$x + yG_t + zG_t^2 = max_{\lambda_t}\{\Omega_t^D + BE_t(V^U(G_{t+1})\}.$$

- First order condition gives $\lambda$.

$$\lambda_t^* = \frac{K_t^U A_t(1 + By) + 2Bz(1 + R_t^U)E_t(\eta_t)G_t K_t^U A_t + 2BzK_t^U A_t r_{tk}^U}{K_t^{U2}[(1 + R_t^*)^2 V_t(\frac{e_{t+1}}{e_t})(\beta - 2Bz) - 2BzA_t^2]}$$

$$A_t = (1 + r_t^U) - (1 + R_t^*)E_t(\frac{e_t + 1}{e_t})$$

# Expressions of $y$, $z$

$$z = -\frac{\frac{\beta}{2}(1 + R_t^U)^2 V_t(\eta_t)}{1 - BE_t(\eta_t^2)(1 + R_t^U)^2}, \qquad (2)$$

- $z$ is a constant, independent of $\lambda_t^*$

$$y = \frac{E_t(\eta_t)(1 + R_t^U) - 2BzE_t(\eta_t)(1 + R_t^U)[K_t^U[(1 - \lambda_t)(1 + r_t^U) + \lambda_t(1 + R_t^*)E_t(\frac{e_t+1}{e_t})] - K_{t+1}^U}{1 - BE_t(\eta_t)(1 + R_t^U)}$$

- Substituting back, a closed form solution of $\lambda_t^*$ as a function of $R_t^*, G_t$.

# Equilibrium in loan market

- $m^D$-banks in developed country.
- $m^U$-banks in underdeveloped country.
- $e_t$-exchange rate at time $t$.

$$m^D \mu_t F_t = \frac{m^U \lambda_t K_t^U}{e_t}$$

# Equilibrium in forex market

$$N_t = -N_0 + N_1 e_t$$

- Current year's supply equals $-N_0 + N_1 e_t + \frac{m^U \lambda_t^* K_t^U}{e_t}$
- Demand for the foreign currency equals $\frac{m^U (1+R_{t-1}^*) \lambda_{t-1}^* K_{t-1}^U}{e_{t-1}^*}$

$$-N_0 + N_1 e_t + \frac{m^U \lambda_t^* K_t^U}{e_t} = \frac{m^U (1 + R_{t-1}^*) \lambda_{t-1}^* K_{t-1}^U}{e_{t-1}^*}$$

# Simulations

- $$\frac{(1+R^D)}{E_t(\epsilon_t)} < (1+R_t^*) < \frac{(1+r^U)}{E_t(\frac{e_{t+1}}{e_t})}$$
- Arbitrary parameters are not feasible for $\lambda_t^*$, $\mu_t^*$ in $\in (0,1)$.
- $\lambda_t^* \Downarrow R_t^*$.
- $\mu_t^* \Uparrow R_t^*$.

# Simulations

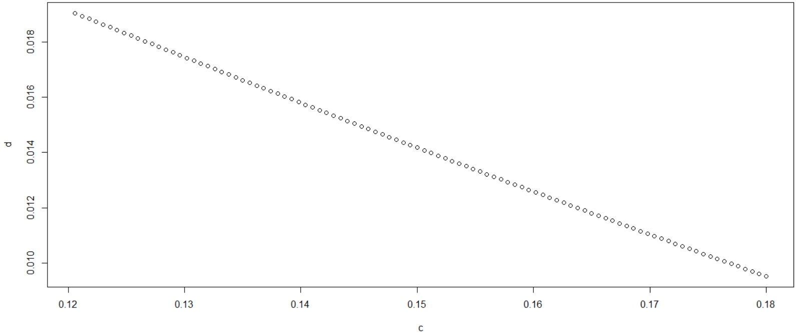


Figure: Proportion of foreign borrowing vs Interest rate

# Simulations

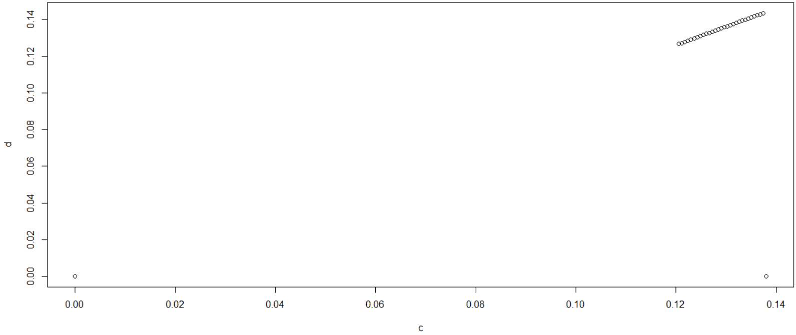


Figure: Proportion of foreign lending vs interest rate

# Simulations

- Take $R_0^*$, $e_0^*$ and other parametric values.
- Calculate $\mu_0^*$ and $\lambda_0^*$ from $R_0^*$ using equations respectively.
- Fixed point iteration to calculate $e_t$, $R_t^*$, $\lambda_t$ and $\mu_t$ at each t

1. Start with an initial value $\lambda_{tj}$, calculate $e_{tj}$ given $\lambda_{tj}$.
2. Calculate $mu_{tj}$ from loan market equilibrium.
3. Calculate $R_{tj}^*$ from $\mu_{tj}$ polynomial.
4. Update $\lambda_{t(j+1)}$ using $R_{tj}^*$, go back to step 1 till all variables converge.
5. Give variables at time $t$, calculate at time $t + 1$.

# Initial values

Table: Initial values of parameters

| $E(\frac{e_t+1}{e_t})$ | $E(\epsilon_t)$ | $V(\epsilon_t)$ | $V(\frac{e_t+1}{e_t})$ | $E(\eta_t)$ | $V(\eta_t)$ | $\gamma$ | $\beta$ | $N_{01}$ | $N_{02}$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.955 | 0.92 | 0.1 | 0.25 | 0.955 | 0.0025 | 50 | 2 | 800 | 900 |

Table: Initial values of variables

| $m^D$ | $m^U$ | $K^D$ | $K^U$ | $R^D$ | $r^D$ | $R^U$ | $r^U$ | $R_0^*$ | $e_0$ | $G_0$ | $F_0$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 200 | 150 | 20 | 10 | 0.05 | 0.04 | 0.2 | 0.15 | 0.14 | 75 | 10 | 20 |

# Figure (1-lam)co30

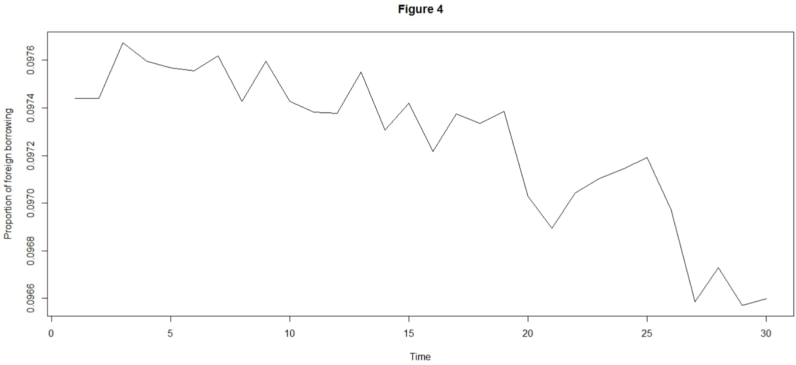


Figure: Proportion of foreign borrowing

# Simulation

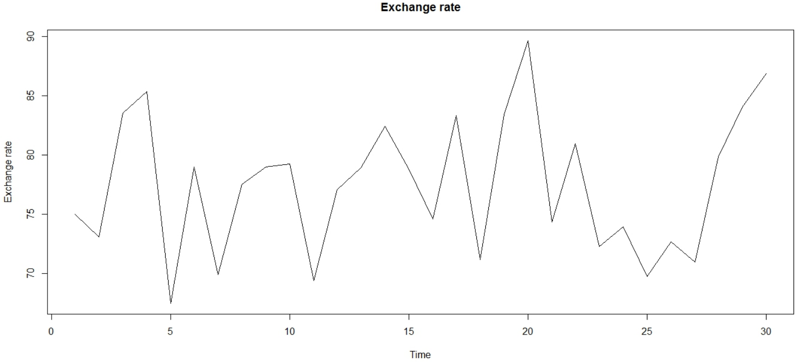


Figure: Exchange rate

- International rate

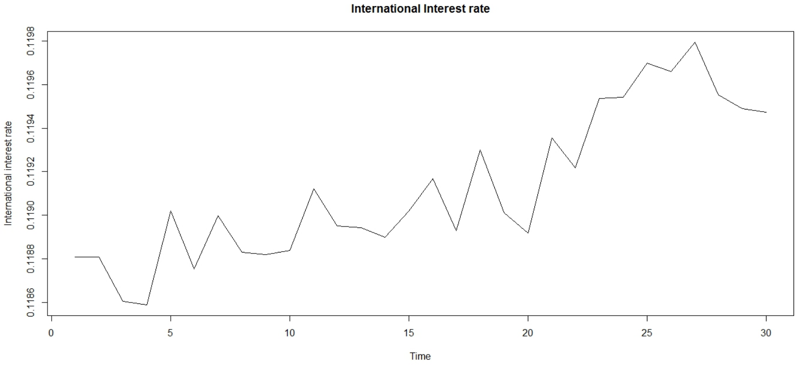


Figure: International interest rate

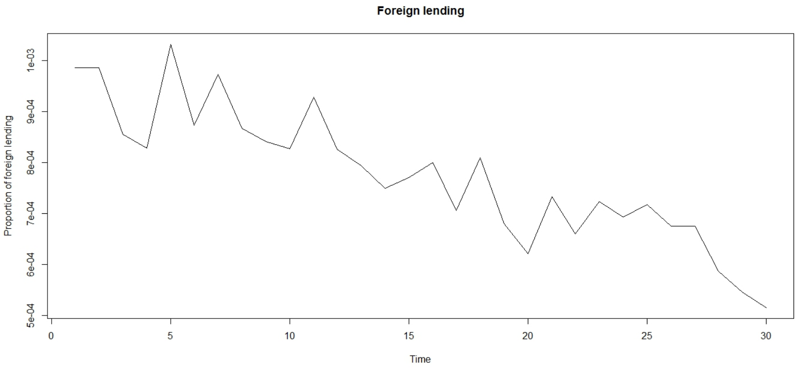


Figure: Proportion of foreign lending

# Mean and Variance of endogenous variables

Table: Mean and variance of the endogenous variables

| Variable | Mean | SD | Coefficient of Variation |
|---|---|---|---|
| $R_t^*$ | 0.1190 | 0.00035 | 0.0029 |
| $e_t$ | 78.4629 | 5.0705 | 0.0646 |
| $\lambda_t^*$ | 0.0939 | 0.0003 | 0.0033 |

# Change in $\gamma$

- More risk averse
- What do we expect?

1. Higher interest rates.
2. Lower proportion of borrowing.

# Change in $\gamma$

- Table justifies our expectation!

Table: Mean and variance of variables-increased $\gamma$

| Variable | Mean | SD | Coefficient of Variation |
|---|---|---|---|
| $R_t^*$ | 0.1197 | 0.0003 | 0.0027 |
| $e_t$ | 77.37 | 4.27 | 0.055 |
| $\lambda_t^*$ | 0.0906 | 0.0038 | 0.042 |

# Change in expected exchange rate

- A decrease in expected exchange rate.
- What do we expect?

1. Increase in proportion of borrowing.
2. Increase in demand of foreign loan, increasing $R_t^*$

# Change in expected exchange rate

- Table justifies it again!

Table: Mean and variance of variables-decreased $E(\frac{e_t+1}{e_t}) = 0.92$

| Variable | Mean | SD | Coefficient of Variation |
|---|---|---|---|
| $R_t^*$ | 0.1197 | 0.0006 | 0.0051 |
| $e_t$ | 76.30 | 5.031 | 0.0659 |
| $\lambda_t^*$ | 0.1238 | 0.0039 | 0.031 |

# $\gamma$ shock

- Sudden increase in $\gamma$ at time 10 and decrease to its original level at $t = 20$.

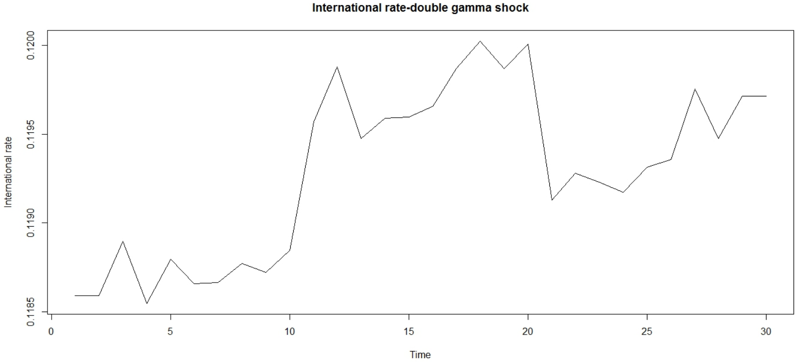


Figure: *gamma* shock on International rate

# $\gamma$ shock

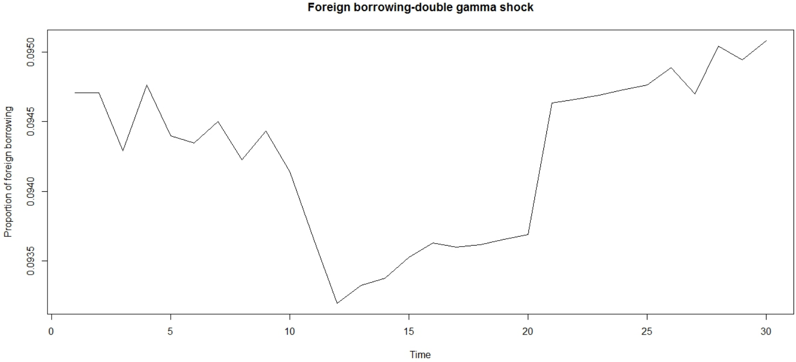


Figure: *gamma* shock on proportion of borrowing

# $E_(\eta_t)$ shock

- Sudden decrease in $E_t(\eta_t)$

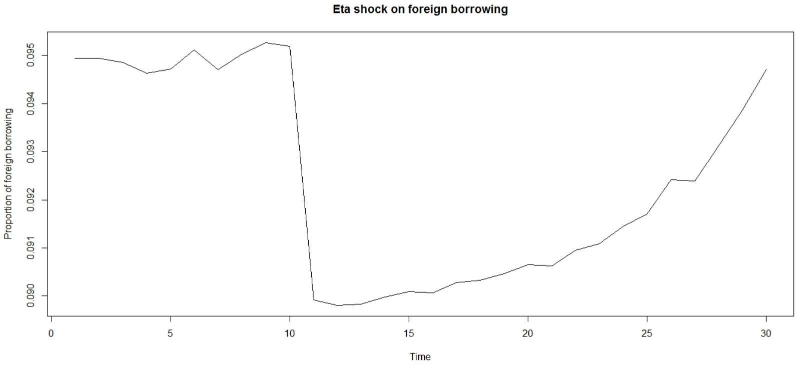


Figure: $\eta$ shock on proportion of borrowing

# $E_(\eta_t)$ shock

- Sudden decrease in $E_t(\eta_t)$

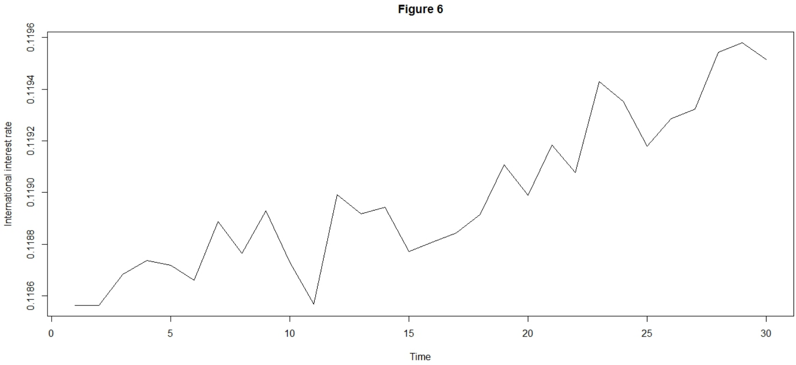


Figure: $\eta$ shock on International interest rate

# Multi country framework

- 1 developed country and 3 underdeveloped countries.
- $\mu_{1t}$-proportion of foreign lending to country 1
- $\mu_{2t}$-proportion of foreign lending to country 2
- $\mu_{3t}$-proportion of foreign lending to country 3
- Quadratic value function.

# Multi country framework

- First order condition imply beautiful equations.

$$\mu_{1t} V_t(\epsilon_{1t}) Z_{2t} = \mu_{2t} V_t(\epsilon_{2t}) Z_{1t},$$

$$\mu_{2t} V_t(\epsilon_{2t}) Z_{3t} = \mu_{3t} V_t(\epsilon_{3t}) Z_{2t},$$

$$\mu_{3t} V_t(\epsilon_{3t}) Z_{1t} = \mu_{1t} V_t(\epsilon_{1t}) Z_{3t}.$$

$$Z_{it} = (1 + R_t^*) E_t(\epsilon_{it}) - (1 + R^D)$$

# Multi country framework

- $\mu_{jt}$ is obtained as function of $R_t^*$, $F_t$.
- $\mu_{jt}$ depends parameters of other countries too!

$$\mu_{1t} = \frac{N}{D}$$

$$N = [Z_{1t} + BbZ_{1t} + 2BcZ_{1t}F_t(1 + R_t^D) - 2Bcr_{kt}^D Z_{1t}]V_t(\epsilon_{2t})V_t(\epsilon_{3t})$$

$$D = (\gamma - 2Bc)F_t^2(1 + R_t^*)^2 V_t(\epsilon_{1t})V_t(\epsilon_{2t})V_t(\epsilon_{3t}) -$$

$$2Bc(Z_{1t}^2 V_t(\epsilon_{2t})V_t(\epsilon_{3t}) + Z_{2t}^2 V_t(\epsilon_{3t})V_t(\epsilon_{1t}) + Z_{3t}^2 V_t(\epsilon_{1t})V_t(\epsilon_{2t}))$$

# Underdeveloped countries

- Same optimization problem as in single country framework.
- Closed form expression of $\lambda_{jt}^*$ as a function of $R_t^*$ , $G_{jt}^*$.
- $\lambda_{jt}$ does not directly depend on parameters of other countries!

# Equilibrium in loan and forex market

- $$m^D[\mu_{1t} + \mu_{2t} + \mu_{3t}]F_t = \sum_{j=1}^{3} \frac{m^j(\lambda_{jt})K_{jt}}{e_{jt}}$$

- $$-N_{0,j} + N_{1,j}e_{t,j} + \frac{m^j \lambda^*_{j,t} K_{j,t}}{e_{j,t}} = \frac{m^j(1 + R^*_{t-1})\lambda^*_{j,t-1}K_{j,t-1}}{e^*_{j,t-1}}.$$

# Simulations

- $G_{jt}$ and $F_t$ play dominant role in determining equilibria.
- Carefully chosen initial values.

Table: Initial values of parameters Country 1

| $E(\frac{e_t+1}{e_t})$ | $E(\epsilon_t)$ | $V(\epsilon_t)$ | $V(\frac{e_t+1}{e_t})$ | $E(\eta_t)$ | $V(\eta_t)$ | $\gamma$ | $\beta$ | $N_{01}$ | $N_{02}$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.955 | 0.95 | 0.1 | 0.2 | 0.955 | 0.00025 | 50 | 2 | 1000 | 1200 |

Table: Initial values of variables-Country 1

| $m^D$ | $m^U$ | $K^D$ | $K^U$ | $R^D$ | $r^D$ | $R^U$ | $r^U$ | $R_0^*$ | $e_0$ | $G_0$ | $F_0$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1500 | 150 | 20 | 10 | 0.05 | 0.04 | 0.2 | 0.15 | 0.14 | 100 | 10 | 20 |

Table: Initial values of parameters Country 2

| $E(\frac{e_t+1}{e_t})$ | $E(\epsilon_t)$ | $V(\epsilon_t)$ | $V(\frac{e_t+1}{e_t})$ | $E(\eta_t)$ | $V(\eta_t)$ | $\gamma$ | $\beta$ | $N_{01}$ | $N_{02}$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.89 | 0.96 | 0.1 | 0.25 | 0.93 | 0.00025 | 50 | 3 | 630 | 850 |

# Simulations

Table: Initial values of parameters Country 3

| $E(\frac{e_t+1}{e_t})$ | $E(\epsilon_t)$ | $V(\epsilon_t)$ | $V(\frac{e_t+1}{e_t})$ | $E(\eta_t)$ | $V(\eta_t)$ | $\gamma$ | $\beta$ | $N_{01}$ | $N_{02}$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.91 | 0.97 | 0.1 | 0.25 | 0.92 | 0.00025 | 50 | 3 | 650 | 850 |

Table: Initial values of variables-Country 2

| $m^D$ | $m^U$ | $K^D$ | $K^U$ | $R^D$ | $r^D$ | $R^U$ | $r^U$ | $R_0^*$ | $e_0$ | $G_0$ | $F_0$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1500 | 150 | 20 | 10 | 0.05 | 0.04 | 0.2 | 0.15 | 0.14 | 80 | 10 | 20 |

# Simulations

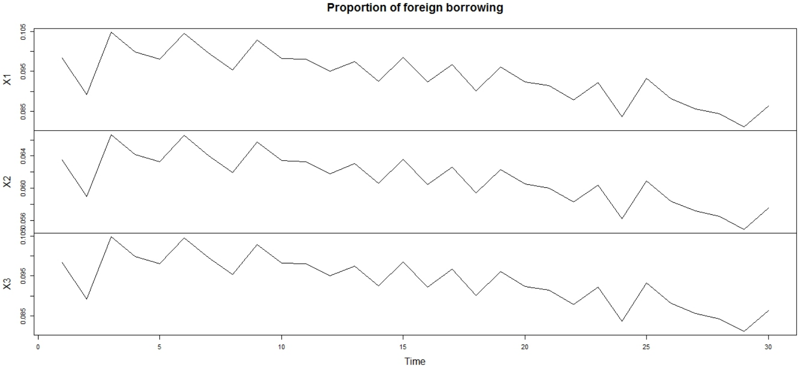


Figure: Proportion of foreign borrowing rate

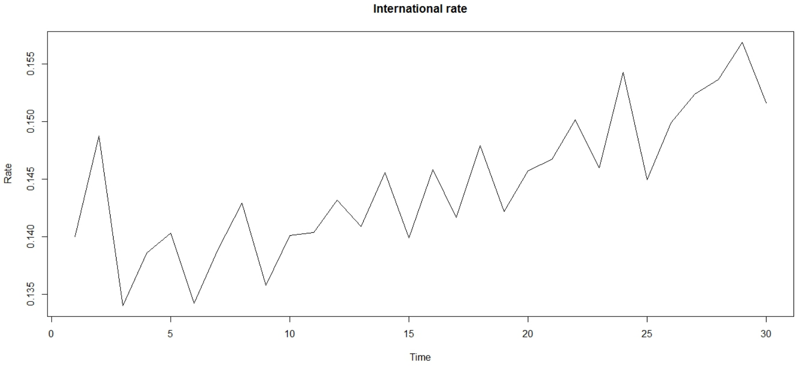


Figure: International interest rate

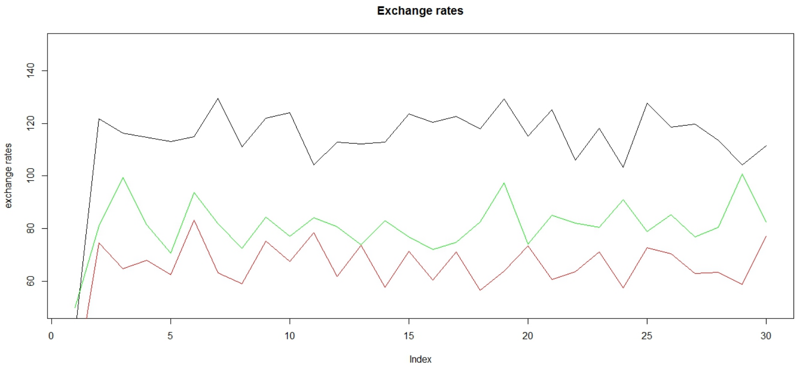


Figure: Exchange rate

# Simulations

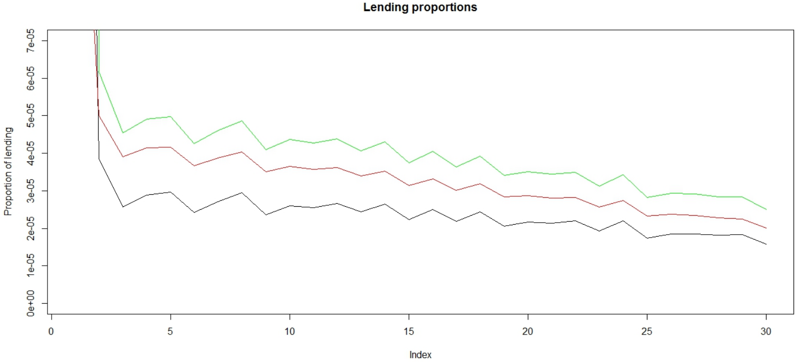


Figure: Lending rate

# Possible reason for the trend

- $G_{jt}$ decreases over time.
- Increase in $R_t^*$.
- Decrease in $(\lambda_{jt})$.

# Comparison of mean values

Table: Mean and variance of the endogenous variables

| Variable | Country 1 | Country 2 | Country 3 |
|---|---|---|---|
| $R_t^*$ | 0.1439 | 0.1439 | 0.1439 |
| $\mu_{jt}$ | $3.10^{-5}$ | $6.10^{-5}$ | $3.6(10^{-5})$ |
| $(\lambda_{jt}^*)$ | 0.0942 | 0.0614 | 0.033 |

Table: Mean and variance of the endogenous variables-increased $\gamma$

| Variable | Country 1 | Country 2 | Country 3 |
|---|---|---|---|
| $R_t^*$ | 0.1445 | 0.1445 | 0.1445 |
| $\mu_{jt}$ | $3.10^{-5}$ | $6.10^{-5}$ | $3.6(10^{-5})$ |
| $(\lambda_{jt}^*)$ | 0.0937 | 0.0611 | 0.031 |

Table: Mean and variance of the endogenous variables-decreased $E_t(\frac{e_{t+1}}{e_t})$

| Variable | Country 1 | Country 2 | Country 3 |
|---|---|---|---|
| $R_t^*$ | 0.1448 | 0.1448 | 0.1448 |
| $\mu_{jt}$ | $3.71^{-5}$ | $6.47^{-5}$ | $3.6(10^{-5})$ |
| $(\lambda_{jt}^*)$ | 0.093 | 0.0609 | 0.0297 |

# Shock in $\gamma$

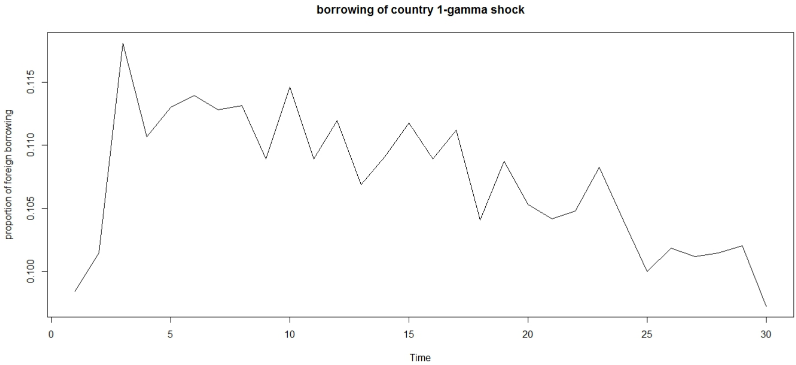


Figure: Proportion of foreign borrowing

# Shock in $\gamma$

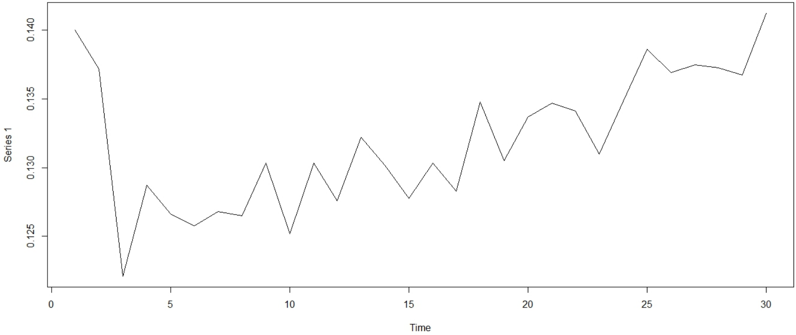


Figure: International interest rate

# Shock in $E_t(\eta_{jt})$

- Country specific shock.

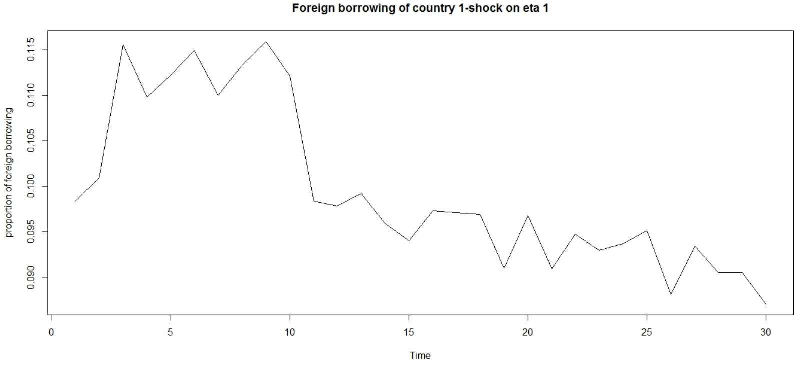


Figure: $\eta_{1t}$ shock on proportion of foreign borrowing of country 1

# Shock in $E_t(\eta_{jt})$

- Country specific shock.

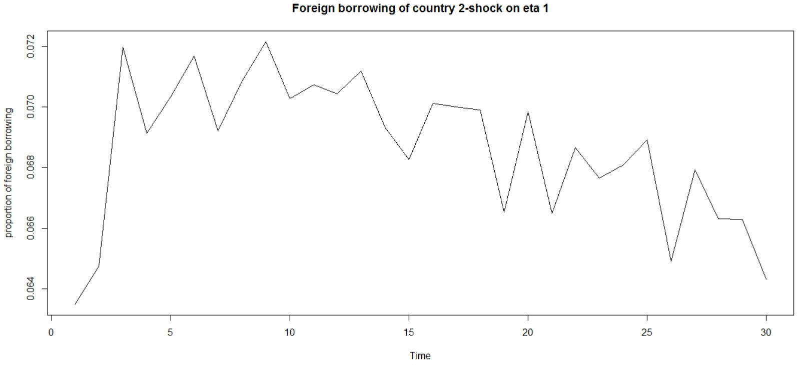


Figure: $\eta_{1t}$ shock on proportion of foreign borrowing of country 2

# Shock in $E_t(\eta_{jt})$

- Country specific shock.

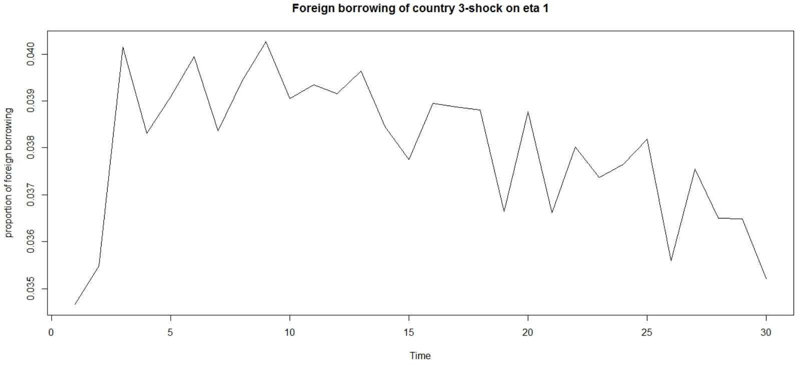


Figure: $\eta_{1t}$ shock on proportion of foreign borrowing of country 3

- Country specific shock.

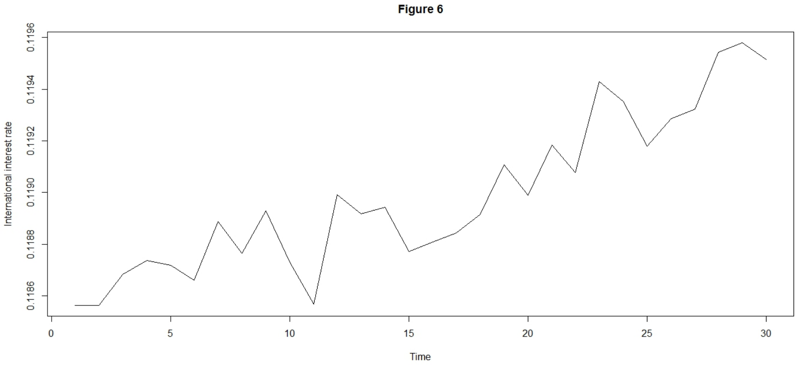


Figure: $\eta_{1t}$ shock on international interest rate

# Contagious effect of country's risk

- Change in $E_t(\epsilon_{1t})$ effects other countries too!
- Expression of $\mu_{1t}$ involves parameters of other countries.

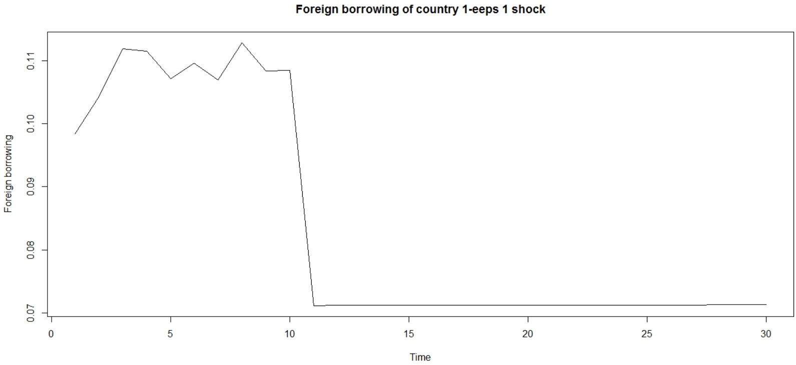


Figure: $E_t(\epsilon_{1t})$ shock on proportion of foreign borrowing of country 1

# Contagious effect of country's risk

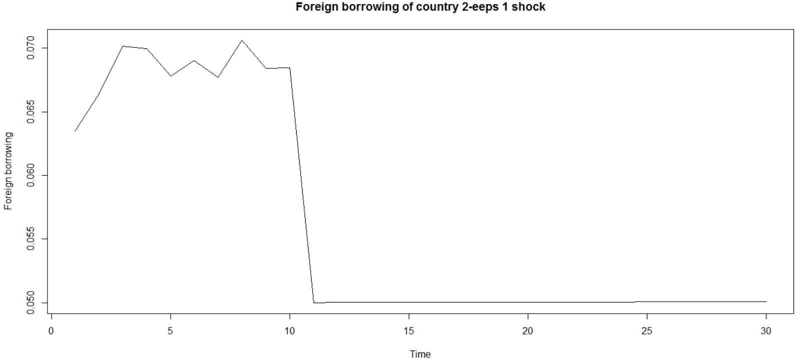


Figure: $E_t(\epsilon_{1t})$ shock on proportion of foreign borrowing of country 2

# Contagious effect of country's risk

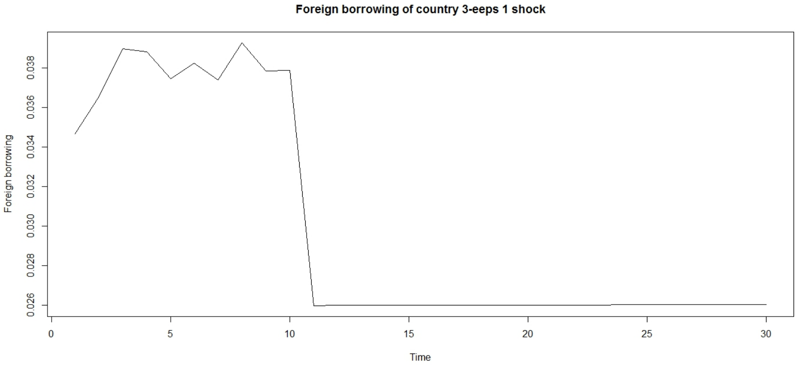


Figure: $E_t(\epsilon_{1t})$ shock on proportion of foreign borrowing of country 3

# Contagious effect of country's risk

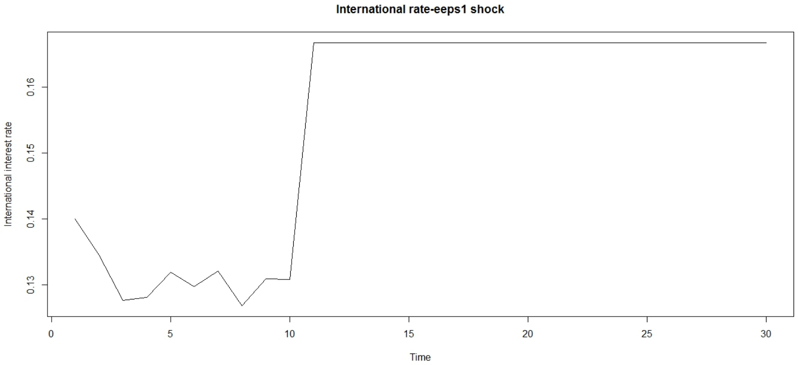


Figure: $E_t(\epsilon_{1t})$ shock on interest rate

# Improvements

- A better model to keep $F_t$ and $G_t$ bounded.
- Modelling of variables like $E_t(\frac{e_{t+1}}{e_t})$, instead of assuming to be constant over time.

Thank You All