Two approaches to integrability: Hirota's bilinear method and 3SS Multidimensional consistency

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Bangalore 9-14.6.2014

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- for maps: existence of sufficient number of **conserved quantities**

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Hirota's bilinear method is an effective tool for studying the existence of multisoliton solutions but there are also deep mathematical connections (Sato theory).

These lectures are in three parts:

1. Introduction to Hirota's bilinear method for continuous systems.

2. Hirota's bilinear method for integrable difference equations.

3. Multidimensional consistency of lattice equations.

Part 1.

Introduction to Hirota's method for continuous systems.

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Hirota: Let us define a new dependent variable F by

$$u = 2\partial_x^2 \log F. \tag{1}$$

With F it should be easy to construct soliton solutions.

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How do soliton equations look in terms of F?

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Let us do the change of variables step by step.
First introduce v by

$$u = \partial_x v.$$

After this (2) can be written as

$$\partial_x[v_{xxx} + 3v_x^2 + v_t] = 0,$$

which can be integrated to the *potential KdV*.

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Next substituting

$$v = \alpha \partial_x \log F,$$

into $v_{xxx} + 3v_x^2 + v_t = 0$ yields

$$F^2 \times (\text{something quadratic}) + 3\alpha(2-\alpha)(2FF'' - F'^2)F'^2 = 0.$$

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Thus we get a quadratic equation if we choose $\alpha = 2$:

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This can be written as

$$(D_x^4 + D_xD_t)F \cdot F = 0,$$

where the Hirota's derivative operator D is defined by

$$\begin{aligned} D_x^n f \cdot g &= (\partial_{x_1} - \partial_{x_2})^n f(x_1)g(x_2)\big|_{x_2=x_1=x} \\ &\equiv \partial_y^n f(x+y)g(x-y)\big|_{y=0}. \end{aligned}$$

We say that an equation is in the Hirota bilinear form if all its derivatives appear trough Hirota's D-operator

$$D_x^n\, f\cdot g = (\partial_{x_1} - \partial_{x_2})^n f(x_1)g(x_2)\big|_{x_2=x_1=x}.$$

Thus D operates on a product of two functions like the Leibniz rule, except for a crucial sign difference. For example

$$\begin{aligned} D_x\, f\cdot g &= f_x g - f g_x, \\ D_x D_t\, f\cdot g &= f g_{xt} - f_x g_t - f_t g_x + f g_{xt} \\ P(D) f\cdot g &= P(-D) g\cdot f. \end{aligned}$$

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For later use note also that

$$\begin{aligned} P(D) f \cdot 1 &= P(\partial) f, \\ P(D) e^{px} \cdot e^{qx} &= P(p-q) e^{(p+q)x}. \end{aligned}$$

Bilinear form of KP

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The Hirota-Satsuma shallow water-wave equation

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becomes with (1) and one integration

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The Sawada-Kotera equation (SK)

$$u_{xxxxx} + 15uu_{xxx} + 15u_x u_{xx} + 45u^2 u_x + u_t = 0,$$

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Soliton solutions

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During the derivation we will often use the gauge invariance of bilinear equations:

$$P(D)(e^{\kappa} F \cdot e^{\kappa} G) = e^{2\kappa} P(D) F \cdot G, \qquad \text{if } \kappa = \vec{c} \cdot \vec{x}.$$

For the 1SS try

$$F = 1 + \varepsilon f_1.$$

This implies

$$P(D_x, \dots)\{1\cdot 1 + \varepsilon\, 1\cdot f_1 + \varepsilon\, f_1\cdot 1 + \varepsilon^2\, f_1\cdot f_1\} = 0.$$

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Since P is even, the order ε^1 yields

$$P(\partial_x, \partial_y, \dots) f_1 = 0.$$

which is solved by

$$f_1 = e^\eta, \quad \eta = px + qy + \omega t + \cdots + \text{ const},$$

where the parameters $p, q, \dots$ satisfy the dispersion relation

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Then order ε^2 term vanishes: $P(\vec{D})e^{\eta}\cdot e^{\eta} = e^{2\eta}P(D)1\cdot 1 = 0.$

The solution

$$F = 1 + e^{\eta}, \quad \eta = \vec{x} \cdot \vec{p} + \eta^0, \quad P(\vec{p}_i) = 0,$$

corresponds to a soliton:

$$\begin{aligned} u &= 2\partial_x^2(\log(F)) \\ &= \frac{2p^2 e^{\eta}}{(1+e^{\eta})^2} = \frac{p^2/2}{\cosh(\frac{1}{2}\eta)^2} \end{aligned}$$

Ansatz for the two-soliton solution (perturbatively!):

$$F = 1 + \varepsilon\,(e^{\eta_1} + e^{\eta_2}) + \varepsilon^2 A_{12} e^{\eta_1+\eta_2}, \quad \eta_i = \vec{x}\cdot\vec{p}_i + \eta_i^0,$$

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Substituting this into the equation gives:

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KdV: $P(D_x, D_t) = D_x^4 + D_x D_t$, $\eta = px + \omega t + \eta_0$, $\omega = -p^3$:

$$A_{12} = -\frac{(p_1 - p_2)^4 + (p_1 - p_2)(\omega_1 - \omega_2)}{(p_1 + p_2)^4 + (p_1 + p_2)(\omega_1 + \omega_2)} = \frac{(p_1 - p_2)^2}{(p_1 + p_2)^2}.$$

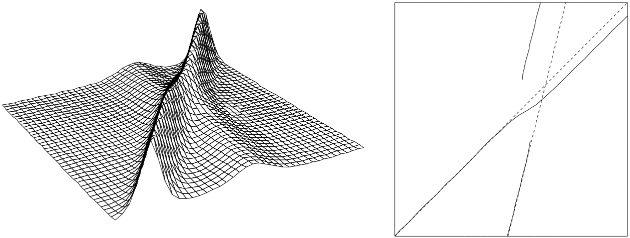

Figure: Scattering of Korteweg–de Vries solitons. On the left a profile view, on the right the locations of the maxima, along with the free soliton trajectory as a dotted line. ($p_1 = \frac{1}{2}$, $p_2 = 1$.)

Result: Any equation of type

$$P(\vec{D}_x)F \cdot F = 0$$

has two-soliton solutions

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This is a level of partial integrability: we can have elastic scattering of two solitons, for any dispersion relation, if the nonlinearity is suitable.

Clearly all of these equations cannot be integrable.
What distinguishes the integrable equations?

Hirota integrability:

If the 1SS is given by

$$F = 1 + \varepsilon e^{\eta}, \quad \eta_i = \vec{x} \cdot \vec{p}_i + \eta_i^0, \quad P(\vec{p}_i) = 0,$$

then there should be an NSS of the form

$$F = 1 + \varepsilon \sum_{j=1}^{N} e^{\eta_j} + \text{(finite number of h.o. terms)}$$

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Almost all equations have multisoliton solutions for some restricted set of parameters, it does not imply integrability.

Apply this principle to the three-soliton solution:

$$\begin{aligned} F_{3SS} = & 1 + \varepsilon \left(e^{\eta_1} + e^{\eta_2} + e^{\eta_3} \right) \\ & + \varepsilon^2 \left(A_{12} e^{\eta_1 + \eta_2} + A_{23} e^{\eta_2 + \eta_3} + A_{31} e^{\eta_3 + \eta_1} \right) \\ & + \varepsilon^3 A_{123} e^{\eta_1 + \eta_2 + \eta_3} \end{aligned}$$

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No freedom left: parameters restricted only by the DR, phase factors given already.

Existence of a 3SS is a condition on the equation, i.e., on P !

Substituting F_{3SS} into $P(D)F \cdot F = 0$ yields the "three-soliton-condition"

$$\sum_{\sigma_i=\pm} P(\sigma_1\vec{p}_1 + \sigma_2\vec{p}_2 + \sigma_3\vec{p}_3)P(\sigma_1\vec{p}_1 - \sigma_2\vec{p}_2) \times P(\sigma_2\vec{p}_2 - \sigma_3\vec{p}_3)P(\sigma_1\vec{p}_1 - \sigma_3\vec{p}_3) = 0.$$

or

$$\sum_{\sigma_i=\pm} \frac{P(\sigma_1\vec{p}_1 + \sigma_2\vec{p}_2 + \sigma_3\vec{p}_3)}{P(\sigma_1\vec{p}_1 + \sigma_2\vec{p}_2)P(\sigma_2\vec{p}_2 + \sigma_3\vec{p}_3)P(\sigma_1\vec{p}_1 + \sigma_3\vec{p}_3)} = 0.$$

Those polynomials P that satisfy this condition yield equations that are integrable also by other criteria.

1-component results

The complete list of 1-component nonlinear Hirota bilinear equations with 3SS is (JH, J. Math. Phys. (1987-1988)):

$$
\begin{aligned}
(D_x^4 - 4D_xD_t + 3D_y^2)F\cdot F &= 0,\\
(D_x^3D_t + aD_x^2 + D_tD_y)F\cdot F &= 0,\\
(D_x^4 - D_xD_t^3 + aD_x^2 + bD_xD_t + cD_t^2)F\cdot F &= 0,\\
(D_x^6 + 5D_x^3D_t - 5D_t^2 + D_xD_y)F\cdot F &= 0.
\end{aligned}
$$

These are the Kadomtsev-Petviashvili, Hirota-Satsuma-Ito, new, and Sawada-Kotera-Ramani equations.

Hierarchies

Higher order equations in a hierarchy are obtained from a set of equations containing some dummy variables.
The bilinear KP hierarchy starts as (Jimbo-Miwa 1983)

$$\begin{aligned}
(D_1^4 - 4D_1D_3 + 3D_2^2)\, f\cdot f &= 0,\\
((D_1^3 + 2D_3)D_2 - 3D_1D_4)\, f\cdot f &= 0,\\
(D_1^6 - 20D_1^3D_3 - 80D_3^2 + 144D_1D_5 - 45D_1^2D_2^2)\, f\cdot f &= 0,\\
&\vdots
\end{aligned}$$

– Infinite number of variables x_n, $n = 1, 2, 3, \ldots$ $(D_k \equiv D_{x_k})$

– obeying infinite number of equations

– that are weight homogeneous, if D_k is given weight k.

Multisoliton solutions to the whole hierarchy

The formula for the NSS is

$$F = \sum_{\substack{\mu_i=0,1\\1\le i\le N}} \exp\left(\sum_{1\le i<j\le N} a_{ij}\mu_i\mu_j + \sum_{i=1}^{N}\mu_i\eta_i\right),$$

where the plane wave factor (PWF) is

$$e^{\eta_j} = \exp\left[(p_j - q_j)x_1 + (p_j^2 - q_j^2)x_2 + (p_j^3 - q_j^3)x_3 + \dots\right]$$

and the phase factor

$$\exp(a_{ij}) = A_{ij} = \frac{(p_i - p_j)(q_i - q_j)}{(p_1 - q_j)(q_i - p_j)}.$$

KdV as 2-reduction KP

2+1 dimensional solitons have 2 soliton parameters p_i, q_i.

1+1 dimensional solitons have only one soliton parameter.

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When this is applied to KP solution it yields:
Phase factor

$$A_{ij} = \frac{(p_i - p_j)(q_i - q_j)}{(p_i - q_j)(q_i - p_j)} \quad \longrightarrow \quad \frac{(p_i - p_j)^2}{(p_i + p_j)^2},$$

plane wave factor

$$e^{\eta_j} = e^{(p_j - q_j)x + \cancel{(p_j^2 - q_j^2)y} + (p_j^3 - q_j^3)t + \cdots} \quad \longrightarrow \quad e^{2p_j x + 2p_j^3 t + \cdots}$$

and equation

$$(D_x^4 + 3D_y^2 - 4\,D_x D_t) f \cdot f = 0 \quad \longrightarrow \quad (D_x^4 - 4\,D_x D_t) f \cdot f = 0.$$

BSQ as 3-reduction of KP

The 3-reduction means $q_i^3 = p_i^3$, i.e.,
$q_i = \omega p_i$, where $\omega^3 = 1$, $\omega \neq 1$.

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Apply to KP yields:

$$A_{ij} = \frac{(p_i - p_j)(q_i - q_j)}{(p_i - q_j)(q_i - p_j)} \longrightarrow \frac{(p_i - p_j)^2}{p_i^2 + p_i p_j + p_j^2},$$

$$e^{\eta_j} = e^{(p_j - q_j)x + (p_j^2 - q_j^2)y + \cancel{(p_j^3 - q_j^3)t} + \cdots} \longrightarrow e^{(1-\omega)p_j x + (1-\omega^2)p_j^2 y}$$

Now scale p and y by

$$p_j = k_j/(1-\omega), \quad y = i\sqrt{3}y' \quad \Rightarrow \quad e^{\eta_j} = e^{k_j x + k_j^2 y'}$$

and we get the Boussinesq equation

$$(D_x^4 + 3D_y^2 - 4\,D_x D_t) f \cdot f = 0 \longrightarrow (D_x^4 - D_{y'}^2) f \cdot f = 0.$$

Higher order equations: Lax5

The 5th order KdV equation (Lax5)

$$u_{xxxxx} + 10uu_{xxx} + 20u_x u_{xx} + 30u^2 u_x - 16u_t = 0$$

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is bilinearized by

$$\begin{cases} (D_1^4 - 4D_1 D_3)\, f \cdot f &= 0, \\ (D_1^6 - 20D_1^3 D_3 - 80D_3^2 + 144D_1 D_5)\, f \cdot f &= 0, \end{cases}$$

These follow from the 1st and 3rd equation in the KP hierarchy, after the 2-reduction $D_{2n} = 0$ (in solutions, $q_j = -p_j$). Also:

– have to eliminate the dummy variable D_3

– change names $D_1 = D_x,\ D_5 = D_t$

– use substitution $u = 2\partial_x^2 \log(f)$.

The modified KdV (mKdV) equation

$$u_{xxx} + \epsilon 6 u^2 u_x + u_t = 0, \tag{3}$$

with travelling wave solutions

$$\begin{aligned} u &= \frac{\pm p}{\cosh(px - p^3 t + c)}, \text{ if } \epsilon = 1, \\ u &= \frac{\pm p}{\sinh(px - p^3 t + c)}, \text{ if } \epsilon = -1. \end{aligned}$$

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First make the equation scale invariant with

$$u = \partial_x w,$$

after which we get from (3) (note: $\epsilon = +1$)

$$\partial_x [w_{xxx} + 2w_x^3 + w_t] = 0,$$

integrate once to get the potential mKdV equation.

Bilinearization

New kind of substitution:

$$w = 2\arctan(G/F), \text{ i.e., } u = 2\frac{D_x G \cdot F}{F^2 + G^2},$$

and then the potential mKdV becomes

$$\begin{aligned}&(F^2 + G^2)[(D_x^3 + D_t)G \cdot F] \\ &+3(D_x F \cdot G)[D_x^2(F \cdot F + G \cdot G)] = 0 \qquad .\end{aligned}$$

Two free functions, G and F, need two equations.

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Two free functions, G and F, need two equations.
For solitons we can take

$$\begin{cases}(D_x^3+D_t)(G\cdot F) &= 0,\\ D_x^2(F\cdot F+G\cdot G) &= 0.\end{cases}$$

More general possibility (λ is an arbitrary function of x, t):

$$\begin{cases}(D_x^3+D_t+3\lambda D_x)(G\cdot F) &= 0,\\ (D_x^2+\lambda)(F\cdot F+G\cdot G) &= 0,\end{cases}$$

The sine-Gordon (sG) equation

$$\phi_{xx} - \phi_{tt} = \sin\phi.$$

The substitution

$$\phi = 4\arctan(G/F),$$

yields

$$[(D_x^2 - D_t^2 - 1)G\cdot F](F^2 - G^2) \\ -FG[(D_x^2 - D_t^2)(F\cdot F - G\cdot G)] = 0.$$

The usual splitting is by

$$\begin{cases} (D_x^2 - D_t^2 - 1)G\cdot F = 0, \\ (D_x^2 - D_t^2)(F\cdot F - G\cdot G) = 0. \end{cases}$$

Soliton solutions for the mKdV/sG class

The mKdV and sG equations belong to the class

$$\begin{cases} B(D_{\vec{x}})\, G \cdot F = 0, \\ A(D_{\vec{x}})(F \cdot F + \epsilon G \cdot G) = 0, \end{cases} \tag{4}$$

where A is even and B either odd (mKdV) or even (sG).

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For the vacuum we choose $F = 1, G = 0$ and therefore we must have $A(0) = 0$.

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where A is even and B either odd (mKdV) or even (sG).
For the vacuum we choose $F = 1$, $G = 0$ and therefore we must have $A(0) = 0$. For the 1SS we may try

$$F = 1 + \alpha e^{\eta},\ G = \beta e^{\eta}.$$

Direct calculation yields from (4) the conditions

$$\alpha A(\vec{p}) = 0,\ \beta B(\vec{p}) = 0,\ \alpha\beta B(0) = 0.$$

Now we can in principle have two different kinds of solitons

type a: $F = 1 + e^{\eta_A},\ G = 0,$ DR: $A(\vec{p}) = 0,$
type b: $F = 1,\ G = e^{\eta_B},$ DR: $B(\vec{p}) = 0.$

Bilinearizing nlS

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$$f\,[(iD_t + D_x^2)g \cdot f] - g\,[D_x^2 f \cdot f - \epsilon 2|g|^2] = 0,$$

The splitting is

$$\begin{cases} (iD_t + D_x^2 - 2\rho^2)g \cdot f & = & 0, \\ (D_x^2 - 2\rho^2)f \cdot f & = & \epsilon 2|g|^2. \end{cases}$$

For bright solitons $\rho = 0$, for dark solitons $\rho \neq 0$, $\epsilon = -1$.

Soliton solutions

For bright solitons the vacuum soliton is given by $f = 1, g = 0$. In the formal expansion the 1SS is

$$f = 1 + \varepsilon f_1 + \varepsilon^2 f_2 + \dots, \; g = \varepsilon g_1 + \dots$$

One finds the solution

$$g = e^{\eta},\; f = 1 + a\, e^{\eta+\eta^*},\; \eta = px + \omega t \text{ complex.}$$

$$\text{where } i\omega + p^2 = 0, \quad a = \epsilon/(p+p^*)^2.$$

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For dark solitons the 0SS is given by a pure phase

$$f = 1, \; g \equiv g_0 = \rho e^{\theta}, \; \theta = i(kx - \omega t), \; \omega = k^2 + 2\rho^2.$$

and the 1SS by $g = g_0(1 + Ze^{\eta}), \; f = 1 + e^{\eta}$, where

$$\eta = px - \Omega t, \Omega = p(2k - \sigma), \; \sigma = \sqrt{4\rho^2 - p^2}, \; Z = \frac{\sigma + ip}{\sigma - ip}.$$

Bilinearization can be difficult. As an example let us consider the Sasa-Satsuma equation

$$q_t + q_{xxx} + 6|q|^2 q_x + 3q|q^2|_x = 0.$$

Try $q = G/F$, G complex, F real. The result can be separated to

$$\begin{cases} (D_x^3 + D_t) G \cdot F &= 0, \\ D_x G \cdot G^* &= 0, \\ D_x^2 F \cdot F &= 4|G|^2, \end{cases}$$

But this is wrong: 3 real functions, 4 eqs., too restrictive.

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But this is wrong: 3 real functions, 4 eqs., too restrictive.
Either one of the following is OK (see C. Gilson, J. Hietarinta, J. Nimmo, and Y. Ohta, Phys. Rev. E **68**, 016614 (2003))

$$\begin{cases} (D_x^3 + 4D_t)\, G \cdot F &= 3D_x\, H \cdot F, \\ (D_x^3 + 4D_t)\, G^* \cdot F &= 3D_x\, H^* \cdot F, \\ D_x^2\, G \cdot F &= -HF, \\ D_x^2\, G^* \cdot F &= -H^* F, \\ D_x^2\, F \cdot F &= 4|G|^2, \end{cases} \qquad \begin{cases} (D_x^3 + D_t)\, G \cdot F &= 3SG, \\ (D_x^3 + D_t)\, G^* \cdot F &= -3SG^*, \\ D_x\, G \cdot G^* &= SF, \\ D_x^2\, F \cdot F &= 4|G|^2, \end{cases}$$

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Hirota's bilinear method is effective for constructing soliton solutions.

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In order to apply it one must transform the nonlinear equation into bilinear form. This may be difficult.

A large class of equations in bilinear form have 2SS, but the existence of 3SS is a strict integrability criterion.

There is a deep mathematical theory behind the bilinear approach, developed by M. Sato and his collaborators in Kyoto.