Part 2
Hirota's bilinear method for lattice equations

Jarmo Hietarinta

Department of Physics and Astronomy, University of Turku
FIN-20014 Turku, Finland

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Why discrete?

- Many mathematical constructs can be interpreted as difference relations, e.g., recursion relations.
- Discretization is needed for numerical analysis. In fact the best algorithms are integrable (e.g., Shanks-Wynn ϵ-algorithm and Rutishauser's qd-algorithm)

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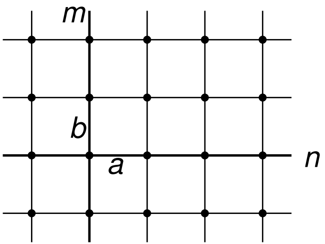
"Universal" definition: low growth of complexity under iterations.

But the practical and useful definition depends on the class of equations.

The lattice

We only consider equations on the Cartesian lattice, consisting of the points $\mathbb{Z}^2$ on $\mathbb{R}^2$.



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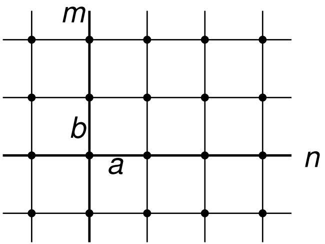

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Essentially different interpretation: Values defined everywhere on $\mathbb{R}^2$ but the equation relates points at a distance.

About notation

If the origin is fixed at $u(x_0, y_0) = u_{0,0}$, the correspondence between continuous and discrete variables at a generic point is

$$u(x, y) = u(x_0 + an, y_0 + bm) = u_{n,m}.$$

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This means that if we have functions with shifts, for example

$$f(x, y), g(x, y + b), h(x + 2a, y + b)$$

then we can think a, b as lattice parameters and use notation

$$f(x, y) = f_{n,m},\ g(x, y+b) = g_{n,m+1},\ h(x+2a, y+b) = h_{n+2,m+1}.$$

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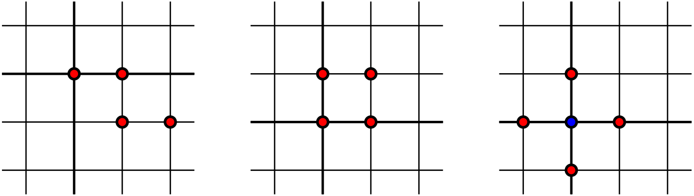
Several shorthand notations exist, e.g,. $f_{n,m} = f = f_{00}$,

$$g_{n,m+1} = \widehat{g} = g_{01} = g_m = g_{[2]}, \quad h_{n+2,m+1} = \widehat{\tilde{\tilde{h}}} = h_{21} = h_{nnm} = h_{[112]}.$$

Be careful when reading articles!

Stencil

A lattice equation is a relation between points on a *stencil*.
The same stencil is used everywhere on the lattice.



a) b) c)

In the 1-component case there is one equation and usually we can compute the value at any perimeter point once the other values are known.

Examples

The discrete KdV can be given as (stencil a)

$$\alpha(y_{n+2,m-1} - y_{n,m}) = \left(\frac{1}{y_{n+1,m-1}} - \frac{1}{y_{n+1,m}}\right)$$

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$$(u_{n,m+1} - u_{n+1,m})(u_{n,m} - u_{n+1,m+1}) = p^2 - q^2$$

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The "similarity constraint" for KdV is (stencil c)

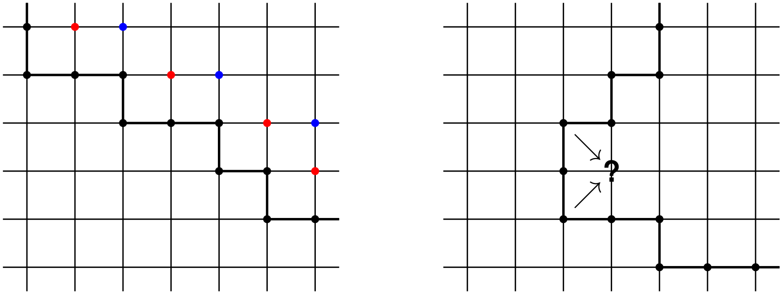
$$(\lambda(-1)^{n+m} + \tfrac{1}{2})u_{n,m} + \frac{np^2}{u_{n-1,m}-u_{n+1,m}} + \frac{mq^2}{u_{n,m-1}-u_{n,m+1}} = 0$$

There are still other kinds of possible stencils.

Evolution

What kind of initial values can we have?

In the quadrilateral case (stencil b) steplike initial values OK, but any overhang would lead into trouble.



Straight and skew limits

What is the continuum limit of a lattice equation?

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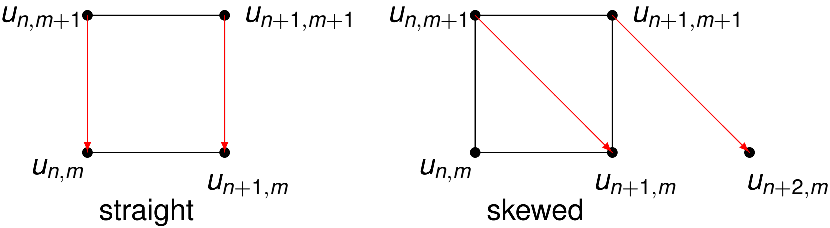
For a quadrilateral equation we can have two kinds of semi-continuous limits (= flattening the square):



- Straight: $u_{n+\delta,m+\kappa} = y_{n+\delta}(\xi + \epsilon\kappa) \quad (\xi = \xi_0 + m\epsilon)$.
- Skewed: $u_{n+\delta,m+\kappa} = w_{n+m+\delta+\kappa-1}(\tau + \epsilon\kappa)$, with $n + m = N$.

Continuum limit of dpKdV

The Korteweg-de Vries equation in potential form is

$$v_t = v_{xxx} + 3v_x^2,$$

how is this related to the dpKdV given by
$(u'_{n,m} = u_{n,m} + np + mq)$

$$(p - q + u'_{n,m+1} - u'_{n+1,m})(p + q + u'_{n,m} - u'_{n+1,m+1}) = p^2 - q^2$$

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In the "straight" continuum limit we take

$$u'_{n,m+k} = y_n(\xi + \epsilon k), \qquad q = 1/\epsilon$$

and expand, obtaining in leading order

$$\partial_\xi(y_n + y_{n+1}) = 2p(y_{n+1} - y_n) - (y_{n+1} - y_n)^2$$

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$$u'_{n,m} = w_{N-1}(\tau), \quad u'_{n+1,m} = w_N(\tau),$$
$$u'_{n,m+1} = w_N(\tau + \epsilon), \quad u'_{n+1,m+1} = w_{N+1}(\tau + \epsilon)$$

and then expand in ϵ. The result is (at order ϵ)

$$\partial_\tau w_N = \frac{2p}{2p + w_{N-1} - w_{N+1}} - 1.$$

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If we let $W_n = 2p + w_{N-2} - w_N$ then we get

$$\dot{W}_N = 2p\left(\frac{1}{W_{N+1}} - \frac{1}{W_{N-1}}\right)$$

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$$2v_\xi + \epsilon v_{\xi\tau} + \tfrac{1}{2}\epsilon^2 v_{\xi\tau\tau} \cdots = 2v_\tau + \epsilon v_{\tau\tau} + \tfrac{1}{3}\epsilon^2 v_{\tau\tau\tau} - \epsilon^2 v_\tau^2 + \dots$$

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Now we need to redefine the independent variables from ξ, τ to x, t using

$$\partial_\tau = \partial_x + \tfrac{1}{12}\epsilon^2 \partial_t, \quad \partial_\xi = \partial_x$$

and then we get

$$v_t = v_{xxx} + 6v_x^2$$

which is the potential form of KdV.

The discrete equation was symmetric, the continuum limit is not!

The skew limit gave

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leading to

$$2v_\tau - (\epsilon^2 v_x + \tfrac{1}{6}\epsilon^4 v_{xxx})(v_\tau + 1) + \cdots = 0.$$

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As before we need to change "time", now by

$$\partial_\tau = \tfrac{1}{2}\epsilon^2\partial_x + \tfrac{1}{12}\epsilon^4\partial_t.$$

Then at the lowest nontrivial order (ϵ^4) we find

$$v_t = v_{xxx} + 3v_x^2.$$

Bilinear difference equations

Differences as derivatives: By Taylor series

$$\begin{aligned} f(x+a) &= f(x) + a\partial_x f(x) + \tfrac{1}{2}a^2\partial_x^2 f(x) + \tfrac{1}{3!}a^3\partial_x^3 f(x) + \dots \\ &= \left[1 + a\partial_x + a^2\tfrac{1}{2}\partial_x^2 + \tfrac{1}{3!}a^3\partial_x^3 + \dots\right] f(x) \\ &= e^{a\partial_x} f(x). \end{aligned}$$

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Therefore

$$\begin{aligned} e^{aD_x} f(x)\cdot g(x) &= e^{a(\partial_x - \partial_y)} f(x)g(y)|_{y=x} \\ &= f(x+a)g(x-a). \end{aligned}$$

Thus bilinear difference equations are obtained if we use exponentials of Hirota derivatives.

Gauge invariance

Gauge-invariance is equivalent to being in Hirota bilinear form.

In continuous case equation should be invariant under $f_i \to f_i' := e^{ax+bt} f_i$:

$$P(D)(e^{ax+bt} f) \cdot (e^{ax+bt} g) = e^{2(ax+bt)} P(D) f \cdot g$$

Discrete gauge transform: $f_j(n,m) \to f_j'(n,m) = A^n B^m f_j(n,m)$.

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We say an equation is in **discrete Hirota bilinear form** if it can be written as

$$\sum_j c_j f_j(n + \nu_j^+, m + \mu_j^+) g_j(n + \nu_j^-, m + \mu_j^-) = 0$$

where the index sums $\nu_j^+ + \nu_j^- = \nu^s, \mu_j^+ + \mu_j^- = \mu^s$ do not depend on j.

In 1970's Hirota discretized many continuous bilinear equations while keeping their multisoliton structure.

For discrete KdV Hirota proposed the *symmetric* form

$$[\sinh(D_n + D_m)(2\delta^{-1}\sinh(2D_m) + 2\sinh(2D_n))]f_{n,m}\cdot f_{n,m} = 0.$$

Writing the exponentials as shifts we get

$$f_{n+3,m+1}f_{n-3,m-1}+\delta^{-1}f_{n+1,m+3}f_{n-1,m-3}-(1+\delta^{-1})f_{n+1,m-1}f_{n-1,m+1} = 0,$$

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This can be nonlinearized using

$$W_{n,m} = \frac{f_{n+2,m}f_{n-2,m}}{f_{n,m+2}f_{n,m-2}} - 1.$$

which yields

$$\frac{1}{1+W_{n,m+2}} - \frac{1}{1+W_{n,m-2}} = \delta(W_{n+2,m} - W_{n-2,m}),$$

The perturbative method for soliton solutions

Discrete bilinear one-component equations still have the form

$$P(D)f \cdot f = 0,$$

but P is now a sum of exponentials.

If we try for the discrete bilinear KdV the one-soliton solution of the form $f = 1 + A^n B^m$ the dispersion relation is complicated.

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If we try for the discrete bilinear KdV the one-soliton solution of the form $f = 1 + A^n B^m$ the dispersion relation is complicated.

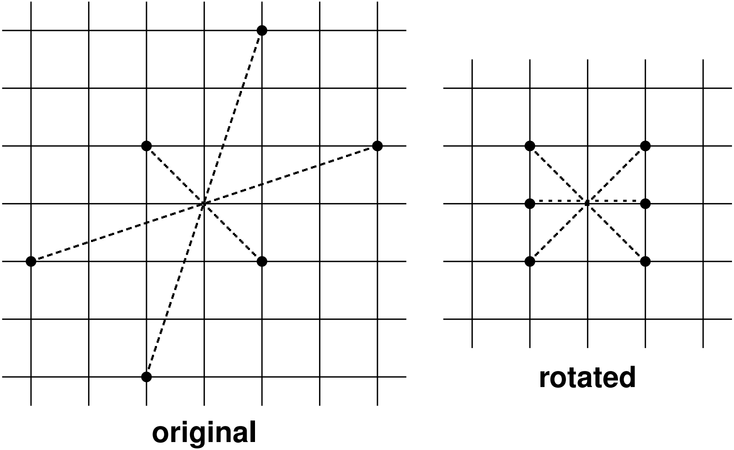
Let us make the rotation

$$n = 2\mu + \nu,\; m = 2\mu - \nu,\; \mu = \tfrac{1}{4}(n+m),\; \nu = \tfrac{1}{2}(n-m)$$

then the bilinear lattice KdV equation becomes

$$f_{\mu+1,\nu+1} f_{\mu-1,\nu-1} + \delta^{-1} f_{\mu+1,\nu-1} f_{\mu-1,\nu+1} - (1+\delta^{-1}) f_{\mu,\nu+1} f_{\mu,\nu-1} = 0.$$

The equations in pictures:



Now the 1SS $f = 1 + A^\nu B^\mu$ yields the DR

$$A(B^2\delta + 1) = (B^2 + \delta).$$

We can parametrise the solution using p as follows:

$$B = \frac{p-1}{p+1}, \quad A = \frac{p^2 + 2pd + 1}{p^2 - 2pd + 1}, \quad \delta = \frac{1+d}{1-d}.$$

or in terms of k

$$k := \frac{2\alpha p}{1+p^2}, \quad A = \frac{\beta - k}{\beta + k}, \quad B = \left(\frac{\alpha - k}{\alpha + k}\right)^2, \quad d = -\beta/\alpha.$$

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If we next compute the 2SS we get the phase factor

$$A_{ij} = \left(\frac{k_i - k_j}{k_i + k_j}\right)^2$$

so it really is a discretized KdV equation.

Classifying equations

A derivative can be discretized in many ways

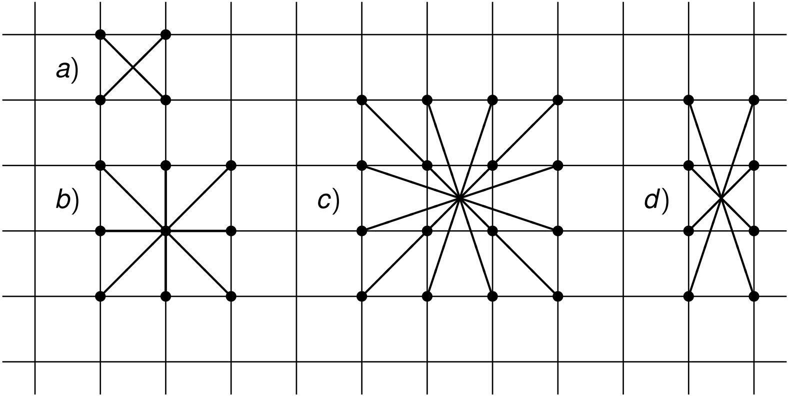
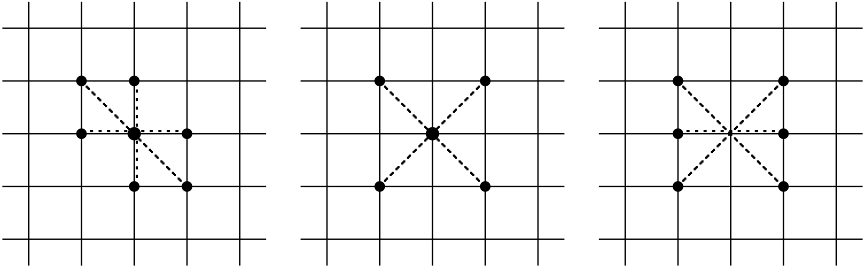


Figure: Possible stencils for 2D bilinear equations.

Integrable configurations

If we restrict to the 3×3 stencil then the following 1-component configurations have 3SS



See: JH and D.J. Zhang, J. Diff. Eq. Appl. **19**, 1292 (2013)

Examples

Four term equation:

$$a\, f_{n+1,m-1} f_{n-1,m+1} + b\, f_{n+1,m} f_{n-1,m} + c\, f_{n,m+1} f_{n,m-1} \\ -(a+b+c) f_{n,m}^2 = 0.$$

Three term equation with center point (Toda lattice)

$$a\, f_{n+1,m+1} f_{n-1,m-1} + b\, f_{n+1,m-1} f_{n-1,m+1} - (a+b) f_{n,m}^2 = 0.$$

Three term equation (KdV)

$$a\, f_{n+1,m+1} f_{n-1,m-1} + b\, f_{n+1,m-1} f_{n-1,m+1} - (a+b) f_{n,m+1} f_{n,m-1} = 0.$$

Hirota's DAGTE equation

In 1981 Hirota's discretisation program culminated in the paper J. Phys. Soc. Jpn. **50**, 3785 (1981) where he presented the "Discrete Analogue of the Generalised Toda Equation" given by

$$[Z_1 \exp(D_1) + Z_2 \exp(D_2) + Z_3 \exp(D_3)]\, f \cdot f = 0$$

Hirota showed that many soliton equations follow from this by specific choices of D_i. Usually $Z_1 + Z_2 + Z_3 = 0$.

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Then Miwa gave (Proc. Japan. Acad. **58**, Ser. A. 9, (1982)) a particular parametrisation for DAGTE:

$$a(b-c)\tau_{l+1,m,n}\tau_{l,m+1,n+1} + b(c-a)\tau_{l,m+1,n}\tau_{l+1,m,n+1} + c(a-b)\tau_{l,m,n+1}\tau_{l+1,m+1,n} = 0.$$

This is then called the "Hirota-Miwa" equation.
This parametrisation is convenient for soliton solutions.

Soliton solutions to the Hirota-Miwa equation

The soliton solutions were given in Miwa's paper as

$$\tau_{l,m,n} = \sum_{\mu_i \in \{0,1\}} \exp\left[\sum_{\substack{i,j=1\\ i<j}}^{N} a_{ij}\,\mu_i\,\mu_j + \sum_{i=1}^{N} \mu_i\,\eta_i\right],$$

where

$$\begin{aligned} e^{\eta_j} &= \left(\frac{1-aq_j}{1-ap_j}\right)^l \left(\frac{1-bq_j}{1-bp_j}\right)^m \left(\frac{1-cq_j}{1-cp_j}\right)^n, \\ \exp(a_{ij}) = A_{ij} &= \frac{(p_i-p_j)(q_i-q_j)}{(p_i-q_j)(q_i-p_j)}. \end{aligned}$$

From the form of A_{ij} we conclude that this is a discretisation of the bilinear KP equation.

Discrete KdV as 2-reduction of KP

Recall that in the continuous case we used $q = -p$. Now this implies

$$\begin{aligned} e^{\eta_i} &= \left(\frac{1+ap_i}{1-ap_i}\right)^n \left(\frac{1+bp_i}{1-bp_i}\right)^m \left(\frac{1+cp_i}{1-cp_i}\right)^k, \\ A_{ij} &= \frac{(p_i-p_j)^2}{(p_i+p_j)^2}. \end{aligned}$$

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But in addition we must <u>reduce the dimension</u>.

The parameters a, b, c must be chosen so that the solution is invariant in some direction. i.e., $\tau_{n+\nu,m+\mu,k+\kappa} = \tau_{n,m,k}$

$$\left(\frac{1+ap_i}{1-ap_i}\right)^\nu \left(\frac{1+bp_i}{1-bp_i}\right)^\mu \left(\frac{1+cp_i}{1-cp_i}\right)^\kappa = 1.$$

For 2-reduction we take $\kappa = 0, \nu = 1, \mu = 1, b = -a$.

With $b = -a$,

$$e^{\eta_i} = \left(\frac{1 + ap_i}{1 - ap_i}\right)^{n-m} \left(\frac{1 + cp_i}{1 - cp_i}\right)^{k}.$$

We use the reduction condition

$$\tau_{n,m+1,k} = \tau_{n-1,m,k}, \forall n, k$$

to change all $m + 1$ to m (after which we omit m)

The resulting equation is

$$(a + c)\,\tau_{n+1,k}\,\tau_{n-1,k+1} + (c - a)\,\tau_{n-1,k}\,\tau_{n+1,k+1} + 2c\,\tau_{n,k+1}\,\tau_{n,k} = 0.$$

This is then a bilinear discrete KdV and its doubly continuous limit is $(D_x^4 - 3D_x D_t)F \cdot F = 0$.

3-reduction

For the 3-reduction $p^3 - q^3 = 0$, let $q = \omega p$, with $\omega^3 = 1, \omega \neq 1$

$$
\begin{aligned}
e^{\eta_i} &= \left(\frac{1-\omega a p_i}{1-a p_i}\right)^n \left(\frac{1-\omega b p_i}{1-b p_i}\right)^m \left(\frac{1-\omega c p_i}{1-c p_i}\right)^k, \\
A_{ij} &= \frac{(p_i - p_j)^2}{p_i^2 + p_i p_j + p_j^2}.
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But in addition we must reduce the dimension.

We choose parameters a, b, c so that $\tau_{n+1,m+1,k+1} = \tau_{n,m,k}$, i.e.,

$$\left(\frac{1-\omega a p_i}{1-a p_i}\right) \left(\frac{1-\omega b p_i}{1-b p_i}\right) \left(\frac{1-\omega c p_i}{1-c p_i}\right) = 1.$$

for all p_i. This is accomplished with $b = \omega a,\ c = \omega^2 a$.

Thus

$$e^{\eta_i} = \left(\frac{1-\omega a p_i}{1-a p_i}\right)^{n-k} \left(\frac{1-\omega^2 a p_i}{1-\omega a p_i}\right)^{m-k}$$

We use the reduction condition

$$\tau_{n,m,k+1} = \tau_{n-1,m-1,k}, \quad \forall n, m$$

to change all $k+1$ to k (and then omit k)

The resulting equation is (Date Jimbo Miwa, JPSJ (1983))

$$\tau_{n+1,m}\,\tau_{n-1,m} + \omega^2\,\tau_{n,m+1}\,\tau_{n,m-1} + \omega\,\tau_{n-1,m-1}\,\tau_{n+1,m+1} = 0.$$

This is a bilinear discrete BSQ and its doubly continuous limit with

$$\tau_{n+\nu,m+\mu} = F(x + (\nu + \omega\mu)\epsilon, y + i\sqrt{3}(\nu + \tfrac{1}{3}(\omega - 1)\mu)\epsilon^2), \quad \epsilon \to 0$$

is $(D_x^4 - 4D_y^2)F \cdot F = 0$.

Miwa's 4-term equation (BKP)

The equation is

$$
\begin{aligned}
&(a+b)(a+c)(b-c)\,\tau_{n+1,m,k}\tau_{n,m+1,k+1}\\
&+(b+c)(b+a)(c-a)\,\tau_{n,m+1,k}\tau_{n+1,m,k+1}\\
&+(c+a)(c+b)(a-b)\,\tau_{n,m,k+1}\tau_{n+1,m+1,k}\\
&+(a-b)(b-c)(c-a)\,\tau_{n+1,m+1,k+1}\tau_{n,m,k}=0.
\end{aligned}
$$

Its soliton solutions have the form (Miwa, 1982)

$$
\begin{aligned}
e^{\eta_i} &= \left(\frac{(1-ap_i)(1-aq_i)}{(1+ap_i)(1+aq_i)}\right)^n \left(\frac{(1-bp_i)(1-bq_i)}{(1+bp_i)(1+bq_i)}\right)^m \left(\frac{(1-cp_i)(1-cq_i)}{(1+cp_i)(1+cq_i)}\right)^k,\\
A_{ij} &= \frac{(p_i-p_j)(p_i-q_j)(q_i-p_j)(q_i-q_j)}{(p_i+p_j)(p_i+q_j)(q_i+p_j)(q_i+q_j)}.
\end{aligned}
$$

4-term discrete BSQ

If we now apply the reduction $\tau_{n+1,m+1,k+1} = \tau_{n,m,k}$ to Miwa's BKP equation we obtain [JH,D-j Zhang, JDEA (2013)]

$$f_{n-1,m+1} f_{n+1,m-1}\, o_1 o_3 - f_{n-1,m} f_{n+1,m}\, o_3(o_1 + o_3) \\ - f_{n,m-1} f_{n,m+1}\, o_1(o_1 + o_3) + f_{n,m}^2\,(o_1^2 + o_1 o_3 + o_3^2) = 0.$$

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After a change of variables $(p,q) \to k'$, we can write

$$e^{\eta_j} = \left(\frac{k_i' + (o_1 + 2o_3) - 3\sigma_j o_1}{k_i' + (o_1 + 2o_3) + 3\sigma_j o_1}\right)^n \left(\frac{k_i' - (2o_1 + o_3) + 3\sigma_j o_3}{k_i' - (2o_1 + o_3) - 3\sigma_j o_3}\right)^m,$$

$$(A_{ij})^{\sigma_i\sigma_j} = \frac{(k_i' - k_j')^2}{{k_1'}^2 + k_1' k_2' + {k_2'}^2 - 12(o_1^2 + o_1 o_3 + o_3^2)}.$$

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$$f_{n-1,m+1} f_{n+1,m-1}\, o_1 o_3 - f_{n-1,m} f_{n+1,m}\, o_3(o_1 + o_3)$$
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Its doubly continuous limit with

$$f_{n+\nu, m+\mu} = F(x + o_1\nu - o_3\mu,\, y + o_1^2\nu + o_3^2\mu), \quad o_i \to 0$$

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In the discrete case there a more possibilities, because there are many ways to discretize a derivative.

The main 1-component equations are Hirota's DAGTE equation (3 components) and Miwa's BKP equation (4 terms).

The Sato theory extends to the discrete case (Miwa's transform).