Part 3
Consistency-Around-the-Cube

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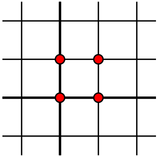
Bangalore 9-14.6.2014

Quadrilateral equations

The consistency approach is usually applied to quadrilateral equations defined on the following stencil



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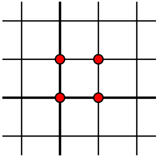


The equation must be linear in all corner values, i.e. multi-linear. The most general form is:

$$
\begin{aligned}
& k\, x\widetilde{x}\widehat{x}\widehat{\widetilde{x}} + l_1\, x\widetilde{x}\widehat{x} + l_2\, x\widetilde{x}\widehat{\widetilde{x}} + l_3\, x\widehat{x}\widehat{\widetilde{x}} + l_4\, \widetilde{x}\widehat{x}\widehat{\widetilde{x}} \\
& + s_1\, x\widetilde{x} + s_2\, \widetilde{x}\widehat{x} + s_3\, \widehat{x}\widehat{\widetilde{x}} + s_4\, \widehat{\widetilde{x}}x + s_5\, x\widehat{x} + s_6\, \widetilde{x}\widehat{\widetilde{x}} \\
& + q_1\, x + q_2\, \widetilde{x} + q_3\, \widehat{x} + q_4\, \widehat{\widetilde{x}} + u \equiv Q(x, \widetilde{x}, \widehat{x}, \widehat{\widetilde{x}}; p_1, p_2) = 0.
\end{aligned}
$$

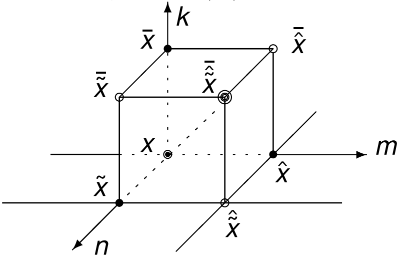
Coefficients k, l_i, s_i, q_i, u may depend on lattice parameters p_j.

CAC - Consistency Around a Cube

Definition of integrability: multidimensional consistency.
(This corresponds to the hierarchy of commuting flows.)

Adjoin a third direction $x_{n,m} \to x_{n,m,k}$ and construct a cube.

$$x_{n,m,k+1} = \bar{x}$$
$$x_{n,m+1,k} = \hat{x}$$
$$x_{n+1,m,k} = \tilde{x}$$



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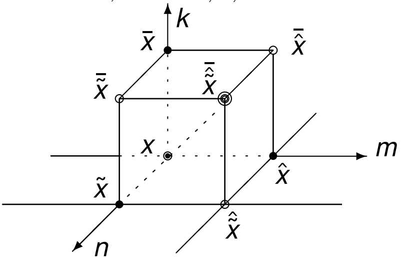

Use *the same map* (with different parameters) also in the (n, k) and (m, k) planes. Identical map on any parallel-shifted plane.

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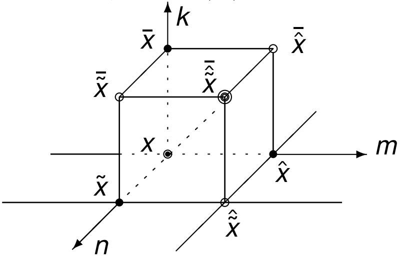
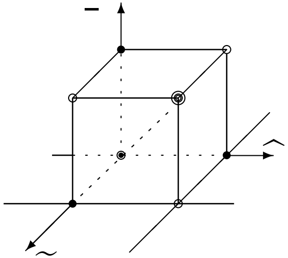


Use *the same map* (with different parameters) also in the (n, k) and (m, k) planes. Identical map on any parallel-shifted plane.

Idea: If x, $\tilde{x}$, $\hat{x}$, $\bar{x}$ are given, can solve for $\hat{\tilde{x}}$, $\bar{\tilde{x}}$, $\bar{\hat{x}}$, uniquely. But $\bar{\hat{\tilde{x}}}$ can be computed in 3 different ways and they must agree!

Consistency of pdKdV

As an example consider the pdKdV equation



$$(x - \hat{\tilde{x}})(\tilde{x} - \hat{x}) + q - p = 0,$$

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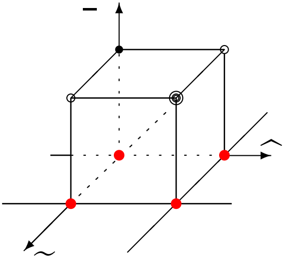

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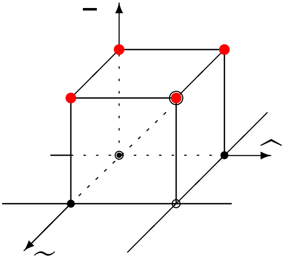

$$\begin{aligned} bottom: (x - \hat{\tilde{x}})(\tilde{x} - \hat{x}) + q - p &= 0, \\ top: (\bar{x} - \bar{\hat{\tilde{x}}})(\bar{\tilde{x}} - \bar{\hat{x}}) + q - p &= 0, \end{aligned}$$

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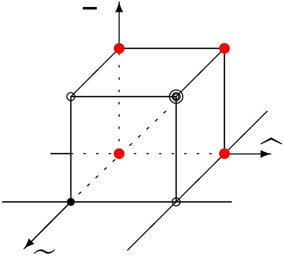

$$\begin{aligned} bottom: (x - \hat{\tilde{x}})(\tilde{x} - \hat{x}) + q - p &= 0,\\ top: (\bar{x} - \bar{\hat{\tilde{x}}})(\bar{\tilde{x}} - \bar{\hat{x}}) + q - p &= 0,\\ back: (x - \bar{\hat{x}})(\bar{x} - \hat{x}) + q - r &= 0, \end{aligned}$$

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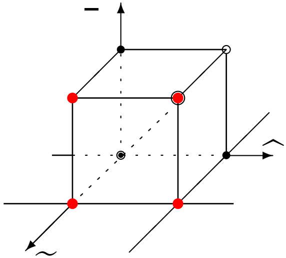

$$\begin{aligned} bottom : (x - \hat{\tilde{x}})(\tilde{x} - \hat{x}) + q - p &= 0, \\ top : (\bar{x} - \bar{\hat{\tilde{x}}})(\bar{\tilde{x}} - \bar{\hat{x}}) + q - p &= 0, \\ back : (x - \hat{\bar{x}})(\bar{x} - \hat{x}) + q - r &= 0, \\ front : (\tilde{x} - \tilde{\hat{\bar{x}}})(\tilde{\bar{x}} - \tilde{\hat{x}}) + q - r &= 0, \end{aligned}$$

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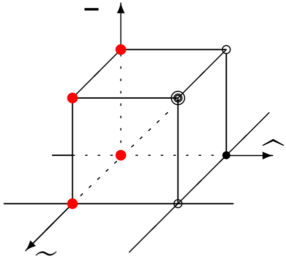

$$
\begin{aligned}
bottom: (x-\hat{\tilde{x}})(\tilde{x}-\hat{x})+q-p &= 0,\\
top: (\bar{x}-\bar{\hat{\tilde{x}}})(\bar{\tilde{x}}-\bar{\hat{x}})+q-p &= 0,\\
back: (x-\hat{\bar{x}})(\bar{x}-\hat{x})+q-r &= 0,\\
front: (\tilde{x}-\tilde{\hat{\bar{x}}})(\tilde{\bar{x}}-\tilde{\hat{x}})+q-r &= 0,\\
left: (x-\bar{\tilde{x}})(\tilde{x}-\bar{x})+r-p &= 0,
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$$

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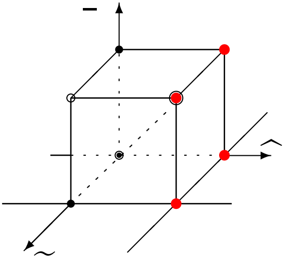

$$
\begin{aligned}
\mathit{bottom}: (x-\hat{\tilde{x}})(\tilde{x}-\hat{x})+q-p &= 0,\\
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\mathit{back}: (x-\hat{\bar{x}})(\bar{x}-\hat{x})+q-r &= 0,\\
\mathit{front}: (\tilde{x}-\tilde{\hat{\bar{x}}})(\tilde{\bar{x}}-\tilde{\hat{x}})+q-r &= 0,\\
\mathit{left}: (x-\bar{\tilde{x}})(\tilde{x}-\bar{x})+r-p &= 0,\\
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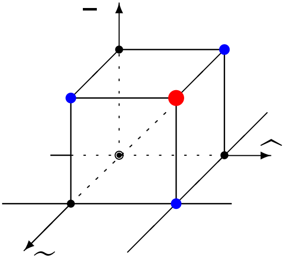

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left : (x - \bar{\tilde{x}})(\tilde{x} - \bar{x}) + r - p &= 0,\\
right : (\hat{x} - \hat{\bar{\tilde{x}}})(\hat{\tilde{x}} - \hat{\bar{x}}) + r - p &= 0,
\end{aligned}
$$

Solve first for the blue variables, then remaining eqs. all yield

$$
\bar{\hat{\tilde{x}}} = \frac{\tilde{x}\hat{x}(p-q) + \hat{x}\bar{x}(q-r) + \bar{x}\tilde{x}(r-p)}{\tilde{x}(r-q) + \hat{x}(p-r) + \bar{x}(q-p)}
$$

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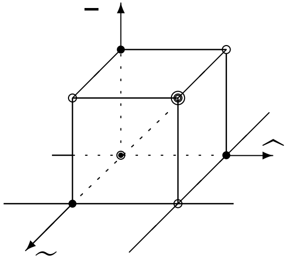

$$
\begin{aligned}
bottom: (x-{\color{blue}\hat{\tilde{x}}})(\tilde{x}-\hat{x})+q-p &= 0,\\
top: (\bar{x}-{\color{red}\bar{\hat{\tilde{x}}}})(\bar{\tilde{x}}-\bar{\hat{x}})+q-p &= 0,\\
back: (x-{\color{blue}\hat{\bar{x}}})(\bar{x}-\hat{x})+q-r &= 0,\\
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$$

Note the *tetrahedron property*: there is no unshifted x.

CAC provides a Lax pair

Recipe given by FW Nijhoff, in Phys. Lett. **A297** 49 (2002).

The third direction is taken as the spectral direction.

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This means: the auxiliary functions are generated from x_{**1}:

$x_{001} = f_{00}/g_{00}$, $x_{101} = f_{10}/g_{10}$, $x_{011} = f_{01}/g_{01}$, $x_{111} = f_{11}/g_{11}$.

Also new name for the lattice parameter $\lambda = r$.

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Solve x_{101} from the left side-equation and x_{011} from the back side-equation and express the result in terms of f, g.

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For the discrete KdV
$(x_{n,m+1} - x_{n+1,m})(x_{n,m} - x_{n+1,m+1}) = p^2 - q^2$, we have:

Left equation: $(x_{001} - x_{100})(x_{000} - x_{101}) = p^2 - r^2,$

Back equation: $(x_{001} - x_{010})(x_{000} - x_{011}) = q^2 - r^2,$

Solving for doubly shifted x we get

$$
\begin{aligned}
x_{101} &= x_{000} - \frac{p^2 - \lambda^2}{x_{001} - x_{100}}, \\
x_{011} &= x_{000} - \frac{q^2 - \lambda^2}{x_{001} - x_{010}},
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$$\begin{aligned} x_{101} &= x_{000} - \frac{p^2 - \lambda^2}{x_{001} - x_{100}}, \\ x_{011} &= x_{000} - \frac{q^2 - \lambda^2}{x_{001} - x_{010}}, \end{aligned}$$

and after changing variables

$$\begin{aligned} \frac{f_{10}}{g_{10}} &= \frac{x_{00} f_{00} + (\lambda^2 - p^2 - x_{10}\, x_{00}) g_{00}}{f_{00} - x_{10} g_{00}}, \\ \frac{f_{01}}{g_{01}} &= \frac{x_{00} f_{00} + (\lambda^2 - q^2 - x_{01}\, x_{00}) g_{00}}{f_{00} - x_{01} g_{00}}. \end{aligned}$$

Define

$$\phi = \begin{pmatrix} f \\ g \end{pmatrix}$$

and write the result

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as a matrix relation

$$\phi_{10} = L_{00}\phi_{00}, \quad \phi_{12} = M_{00}\phi_{00}$$

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For the KdV-map one finds

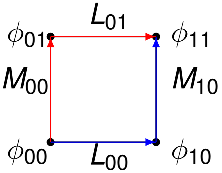
$$L_{00} = \gamma \begin{pmatrix} x_{00} & \lambda^2 - p^2 - x_{00}x_{10} \\ 1 & -x_{10} \end{pmatrix},\ M_{00} = \gamma' \begin{pmatrix} x_{00} & \lambda^2 - q^2 - x_{00}x_{01} \\ 1 & -x_{01} \end{pmatrix}.$$

where γ, γ' are separation constants.

Consistency of Lax pair: using the matrix equations

$\phi_{n+1,m} = L_{n,m}\phi_{n,m}, \quad \phi_{n,m+1} = M_{n,m}\phi_{n,m}$

we can transport ϕ from $(0,0)$ to $(1,1)$ via two routes:



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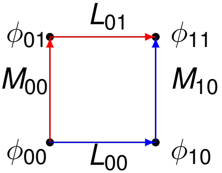

Therefore the consistency condition for well defined ϕ_{11} is $L_{01}(M_{00}\phi_{00}) = M_{10}(L_{00}\phi_{00})$ or as a matrix relation

$$L_{01}\,M_{00} = M_{10}\,L_{00}.$$

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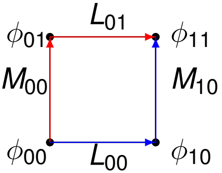

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$$L_{01}\, M_{00} = M_{10}\, L_{00}.$$

Since L, M are 2×2 matrices, this looks like 4 conditions.
They yield the parameters γ, γ' and the equation on the bottom quadrilateral.

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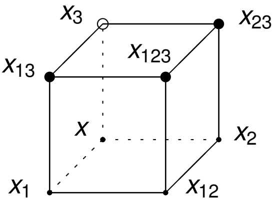

Bäcklund transformation

Take the side equations and compute x_{13}, x_{23}, x_{123} from left, back and front. This leaves right equation, which is quadratic polynomial in x_3.

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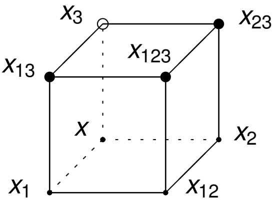

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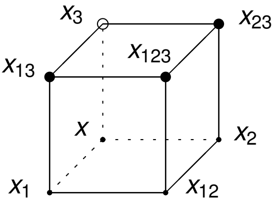

Bäcklund transformation

Take the side equations and compute $x_{13},\ x_{23},\ x_{123}$ from left, back and front. This leaves right equation, which is quadratic polynomial in x_3.

The greatest common divisor of its coefficients yields the bottom equation.

Similar computations on the bottom variables x, x_1, x_2, x_{12} yield the top equation.

The computations are equivalent to those for the Lax pair.

CAC as a search method

CAC has been used as a method to search and classify lattice equations:

Adler, Bobenko, Suris, Commun. Math. Phys. **233**, 513 (2003)

with 2 additional assumptions:

- symmetry ($\varepsilon, \sigma = \pm 1$):
$$Q(x_{000}, x_{100}, x_{010}, x_{110}; p_1, p_2) = \varepsilon Q(x_{000}, x_{010}, x_{100}, x_{110}; p_2, p_1) = \sigma Q(x_{100}, x_{000}, x_{110}, x_{010}; p_1, p_2)$$
- "tetrahedron property": x_{111} does not depend on x_{000}.

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- "tetrahedron property": x_{111} does not depend on x_{000}.

Result: complete classification under these assumptions, 9 models.

ABS results:

List H:

(H1) $$(x-\hat{\tilde{x}})(\tilde{x}-\hat{x})+q-p=0,$$

(H2) $$(x-\hat{\tilde{x}})(\tilde{x}-\hat{x})+(q-p)(x+\tilde{x}+\hat{x}+\hat{\tilde{x}})+q^2-p^2=0,$$

(H3) $$p(x\tilde{x}+\hat{x}\hat{\tilde{x}})-q(x\hat{x}+\tilde{x}\hat{\tilde{x}})+\delta(p^2-q^2)=0.$$

List A:

(A1) $$p(x+\hat{x})(\tilde{x}+\hat{\tilde{x}})-q(x+\tilde{x})(\hat{x}+\hat{\tilde{x}})-\delta^2pq(p-q)=0,$$

(A2)
$$(q^2-p^2)(x\tilde{x}\hat{x}\hat{\tilde{x}}+1)+q(p^2-1)(x\hat{x}+\tilde{x}\hat{\tilde{x}})-p(q^2-1)(x\tilde{x}+\hat{x}\hat{\tilde{x}})=0.$$

The main list

(Q1) $$p(x-\widehat{x})(\widetilde{x}-\widehat{\widetilde{x}})-q(x-\widetilde{x})(\widehat{x}-\widehat{\widetilde{x}})=\delta^2 pq(q-p)$$

(Q2)

$$p(x-\widehat{x})(\widetilde{x}-\widehat{\widetilde{x}})-q(x-\widetilde{x})(\widehat{x}-\widehat{\widetilde{x}})+pq(p-q)(x+\widetilde{x}+\widehat{x}+\widehat{\widetilde{x}})= pq(p-q)(p^2-pq+q^2)$$

(Q3) $$p(1-q^2)(x\widehat{x}+\widetilde{x}\widehat{\widetilde{x}})-q(1-p^2)(x\widetilde{x}+\widehat{x}\widehat{\widetilde{x}})= (p^2-q^2)\left((\widehat{x}\widetilde{x}+x\widehat{\widetilde{x}})+\delta^2\frac{(1-p^2)(1-q^2)}{4pq}\right)$$

(Q4) $$\mathrm{sn}(\alpha)(x\widetilde{x}+\widehat{x}\widehat{\widetilde{x}})-\mathrm{sn}(\beta)(x\widehat{x}+\widetilde{x}\widehat{\widetilde{x}})-\mathrm{sn}(\alpha-\beta)(\widetilde{x}\widehat{x}+x\widehat{\widetilde{x}})+ k\,\mathrm{sn}(\alpha)\mathrm{sn}(\beta)\mathrm{sn}(\alpha-\beta)(1+x\widetilde{x}\widehat{x}\widehat{\widetilde{x}})=0.$$ (JH 2005)

Beyond the “ABS-list”

- CAC but no tetrahedron property, e.g. $\widehat{\widetilde{x}} - \widehat{x} - \widetilde{x} + x = 0$
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- CAC but multicomponent, e.g. Boussinesq equations (partial classification JH 2011)

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- On a bigger stencil, typical for bilinear equations (recall part 2)

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Here: apply this to H1 (KdV). [JH, Zhang, J. Phys. A: Math. Theor. **42**, 404006 (2009).]

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$$(\widetilde{u} - u)^2 = r - p, \quad (\widehat{u} - u)^2 = r - q.$$

For convenience we reparametrize $(p, q) \to (a, b)$ by

$$p = r - a^2, \quad q = r - b^2.$$

One solution is: $u_0(n, m) = an + bm + \gamma$

1SS

Next construct the 1SS by considering side equations of the consistency cube as Bäcklund transformation:

$$(u - \bar{\widetilde{u}})(\widetilde{u} - \bar{u}) = p - \varkappa,$$

$$(u - \bar{\widehat{u}})(\bar{u} - \widehat{u}) = \varkappa - q.$$

– Here u is the background solution $u_0 = an + bm + \gamma$,
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We search for a new solution $\bar{u}$ of the form

$$\bar{u}=\bar{u}_0+v,$$

where $\bar{u}_0$ is the bar-shifted background solution

$$\bar{u}_0=an+bm+k+\lambda,$$

v the unknown and k is the new soliton parameter, $\varkappa=r-k^2$.

For v the side equations imply:

$$\widetilde{v} = \frac{Ev}{v+F}, \quad \widehat{v} = \frac{Gv}{v+H},$$

where $\varkappa = r - k^2$.

$$E = -(a+k), \quad F = -(a-k), \quad G = -(b+k), \quad H = -(b-k),$$

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$$E = -(a+k), \quad F = -(a-k), \quad G = -(b+k), \quad H = -(b-k),$$

The equations can be solved easily by writing them as matrix equations using $v = g/f$ and $\Phi = (g, f)^T$:

$$\Phi(n+1, m) = \mathcal{N}(n, m)\Phi(n, m), \quad \Phi(n, m+1) = \mathcal{M}(n, m)\Phi(n, m),$$

where

$$\mathcal{N}(n, m) = \Lambda \begin{pmatrix} E & 0 \\ 1 & F \end{pmatrix}, \quad \mathcal{M}(n, m) = \Lambda' \begin{pmatrix} G & 0 \\ 1 & H \end{pmatrix},$$

In this case E, F, G, H are constants and we can choose $\Lambda = \Lambda' = 1$.

Since the matrices $\mathcal{N}, \mathcal{M}$ commute it is easy to find

$$\Phi(n,m) = \begin{pmatrix} E^n G^m & 0 \\ \frac{E^n G^m - F^n H^m}{-2k} & F^n H^m \end{pmatrix} \Phi(0,0).$$

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If we define discrete plane-wave-factors by

$$\rho_{n,m} = \left(\frac{E}{F}\right)^n \left(\frac{G}{H}\right)^m \rho_{0,0} = \left(\frac{a+k}{a-k}\right)^n \left(\frac{b+k}{b-k}\right)^m \rho_{0,0},$$

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then we obtain (with suitable $v_{0,0}$)

$$v_{n,m} = \frac{-2k\rho_{n,m}}{1+\rho_{n,m}}.$$

Finally we obtain the 1SS for H1:

$$u_{n,m}^{(1SS)} = (an + bm + \lambda) + k + \frac{-2k\rho_{n,m}}{1+\rho_{n,m}}.$$

Bilinearizing transformation

The form of the 1SS

$$u_{n,m}^{1SS} = an + bm + \lambda + \frac{k(1-\rho_{n,m})}{1+\rho_{n,m}}$$

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When we use this in H1 we find

$$\begin{aligned}\mathrm{H1} &\equiv (u-\hat{\tilde{u}})(\tilde{u}-\hat{u}) - p + q \\ &= -\left[\mathcal{H}_1 + (a-b) f\hat{\tilde{f}}\right]\left[\mathcal{H}_2 + (a+b)\hat{f}\tilde{f}\right] / (f\hat{f}\tilde{f}\hat{\tilde{f}}) + (a^2-b^2),\end{aligned}$$

where

$$\begin{aligned}\mathcal{H}_1 &\equiv \hat{g}\tilde{f} - \tilde{g}\hat{f} + (a-b)(\hat{f}\tilde{f} - f\hat{\tilde{f}}) = 0, \\ \mathcal{H}_2 &\equiv g\hat{\tilde{f}} - \hat{\tilde{g}}f + (a+b)(f\hat{\tilde{f}} - \hat{f}\tilde{f}) = 0.\end{aligned}$$

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For continuous equations soliton solutions are given by Wronskians, for discrete equations we use Casoratians.

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and then compose the $N \times N$ Casorati matrix from columns with different shifts h_i, and then the determinant

$$C_{n,m}(\varphi; \{h_i\}) = |\varphi(n, m, h_1), \varphi(n, m, h_2), \cdots, \varphi(n, m, h_N)|.$$

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For example

$$\begin{aligned} C^2_{n,m}(\varphi) &:= |\varphi(n, m, 0), \cdots, \varphi(n, m, N-2), \varphi(n, m, N)| \\ &\equiv |0, 1, \cdots, N-2, N| \equiv |\widehat{N-2}, N|. \end{aligned}$$

Main result

The bilinear equations $\mathcal{H}_i$ are solved by Casoratians $f = |\widehat{N-1}|_{[h]}$, $g = |\widehat{N-2}, N|_{[h]}$, with φ_i given by

$$\varphi_i(n,m,h;k_i) = \varrho_i^+ k_i^h (a+k_i)^n (b+k_i)^m + \varrho_i^- (-k_i)^h (a-k_i)^n (b-k_i)^m.$$

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Similar results exist for H2,H3,Q1,Q3

J. Hietarinta and D.J. Zhang, *Soliton solutions for ABS lattice equations: II Casoratians and bilinearization* J. Phys. A: Math. Theor. **42**, 404006 (2009). `arXiv:0903.1717`

J. Atkinson, J. Hietarinta and F. Nijhoff, *Soliton solutions for Q3*, J. Phys. A: Math. Theor., **41** 142001 (2008). `arXiv:0801.0806`

The structure of the soliton solution is similar to those of the Hirota-Miwa equation

Boussinesq class of multi-component equations

Assume three dependent variables x, y, z related on the elementary square by

$$\begin{gathered}\tilde{y} = x\tilde{x} - z, \quad \hat{y} = x\hat{x} - z, \\ \hat{\tilde{y}} = \hat{x}\hat{\tilde{x}} - \hat{z}, \quad \hat{\tilde{y}} = \tilde{x}\hat{\tilde{x}} - \tilde{z}, \\ \hat{\tilde{z}} = x\hat{\tilde{x}} - y + \frac{p^3 - q^3}{\hat{x} - \tilde{x}}.\end{gathered}$$

This is the lattice Boussinesq equation (Togas, Nijhoff, 2005).

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\end{aligned}
$$

This is the lattice Boussinesq equation (Togas, Nijhoff, 2005).

The first four equations involve only points on the edges, while the last one is on the square.

One can now add the third direction and associated shifts, consistently.

Another equation of this type is

$$\widetilde{y}\,z = \widetilde{x} - x, \quad \widehat{y}\,z = \widehat{x} - x,$$

$$\widehat{\widetilde{z}} = \frac{z/y}{\widetilde{z} - \widehat{z}}(\widehat{z}\widetilde{y}p^3 - \widetilde{z}\widehat{y}q^3).$$

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One can write such equations also as one component equations but on a 3×3 stencil.

The mBSQ and SBSQ are obtained from the above equation, by eliminating x, z or y, z, respectively.

BSQ search

Search for BSQ-type equations with CAC
J.H.: J. Phys. A: Math. Theor. **44** (2011) 165204 (22pp)

Assume edge equations are linear separately in shifted and unshifted variables and do not depend on spectral parameters.

Then can classify edge equations into one of the following types

$$\widetilde{x}z = \widetilde{y} + x, \qquad \text{(A)}$$

$$\widetilde{x}\,x = \widetilde{y} + z, \qquad \text{(B)}$$

$$\widetilde{y}z = \widetilde{x} - x. \qquad \text{(C)}$$

The choice of variables: x appear as shifted an unshifted, y only shifted and z only unshifted.

Requirement of 3D consistency on the x, y edge evolutions leads to a condition on z:

Case A:

$$\widehat{\widetilde{z}}(\widetilde{z}-\widehat{z})+\bar{\widehat{z}}(\widehat{z}-\bar{z})+\widetilde{\bar{z}}(\bar{z}-\widetilde{z})=0$$

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Here $F_p=\phi(x,y,z,\widetilde{x},\widetilde{y},\widetilde{z},p)$ etc.

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Here $F_p=\phi(x,y,z,\widetilde{x},\widetilde{y},\widetilde{z},p)$ etc.

Next the 3D consistency of z gives the condition

$$\frac{\widehat{F}_p-\widehat{F}_r}{\widetilde{z}-\bar{z}}=\frac{\bar{F}_q-\bar{F}_p}{\widehat{z}-\widetilde{z}}=\frac{\widetilde{F}_r-\widetilde{F}_q}{\bar{z}-\widehat{z}} \qquad \text{(A*)}$$

For the other cases B and C one gets similar equations.

The difficult problem is to solve (A*), etc.

Main results:

Case A, i.e., $\widetilde{x}z = \widetilde{y} + x$, we found

$$\widehat{\widetilde{z}} = \frac{y}{x} + \frac{1}{x}\,\frac{p\,\widetilde{x} - q\,\widehat{x}}{\widetilde{z} - \widehat{z}}, \qquad \text{(A-2)}$$

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Case B, i.e., $x\widetilde{x} = \widetilde{y} + z$, a generalization of lBSQ

$$\widehat{\widetilde{z}} + y = b_0(\widehat{\widetilde{x}} - x) + x\,\widehat{\widetilde{x}} + \frac{p-q}{\widetilde{x} - \widehat{x}}, \tag{B-2}$$

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Case C, i.e., $z\widetilde{y} = \widetilde{x} - x$, we found two equations, which are kind of modifications of the lmBSQ/lSBSQ equation:

$$\widehat{\widetilde{z}} = \frac{d_2 x + d_1}{y} + \frac{z}{y}\,\frac{p\widetilde{y}\widehat{z} - q\widehat{y}\widetilde{z}}{\widetilde{z} - \widehat{z}}, \qquad \text{(C-3)}$$

and

$$\widehat{\widetilde{z}} = \frac{x\widehat{\widetilde{x}} + d_1}{y} + \frac{z}{y}\,\frac{p\widetilde{y}\widehat{z} - q\widehat{y}\widetilde{z}}{\widetilde{z} - \widehat{z}}. \qquad \text{(C-4)}$$

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- Soliton solutions can be constructed systematically
- We have classification results

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Generalizations still under development:

- Different equations on the sides
- Multi-component equations