Nonexpansive Ergodic Stochastic Control

Characterization via Coupled HJB Equations

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8 - 12, September, 2014

Overview

- Ergodic Control

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- Nonexpansive Property

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- Coupled HJB Equations

Ergodic Control

- $b : \mathbb{R}^d \times U \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ satisfy

$$\|b(x,u) - b(y,u)\| + \|\sigma(x) - \sigma(y)\| \leq C\|x - y\|$$

for all $x, y \in \mathbb{R}^d$, $u \in U$ and for some $C > 0$.

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- Consider the controlled stochastic differential equation

$$\begin{cases} dX(t) = b(X(t), u(t))dt + \sigma(X(t))dW(t), t > 0 \\ X(0) = x \in \mathbb{R}^d, \end{cases} \tag{1}$$

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- $u(\cdot)$ is admissible control i.e.,

$$u(\cdot) \in \mathcal{A} = \{u : [0,\infty) \to U : \\ \text{for } t \geq s, W(t) - W(s) \perp \sigma\{u(r), W(r), r \leq s\}.$$

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- For each $u(\cdot) \in \mathcal{A}$, (1) admits a unique strong solution.

Ergodic Control

- The (running) cost $c : \mathbb{R}^d \times U \to \mathbb{R}$ satisfies

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- The cost to be minimised:

$$\rho(x) = \inf_{u(\cdot)\in\mathbb{A}} \liminf_{T\to\infty} \mathbb{E}\frac{1}{T}\int_0^T c(X(t), u(t))\, dt$$

Ergodic Control

- Under "appropriate" assumptions the ergodic cost $\rho(x)$ is constant and is characterised as the unique constant such that

$$\rho = \min_{u \in U} \{ \frac{1}{2} tr(a(x) D^2 V(x) + b(x,u) \cdot DV(x) + r(x,u) \}$$

admits a viscosity solution $V(\cdot)$.

[1] A. Arapostathis, V.S. Borkar, and M.K. Ghosh, Ergodic Control of Diffusion Processes, Cambridge University Press, 2012

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- See "THE RED BOOK"[1]

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Ergodic Control

Some of the crucial assumptions in the literature are:

- The uniform ellipticity i.e.,

$$\sigma(x)\sigma^T(x) \geq \kappa I$$

  for each $x \in \mathbb{R}^d$ and for some $\kappa > 0$.
  Allows to construct feedback policies and to use pde theory.

Ergodic Control

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- Asymptotic Flatness
  Ensures that the solutions satisfy
  $$\mathbb{E}\|X(t) - Y(t)\| \leq e^{-Ct}\|x - y\|$$
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- ETC...

Ergodic Control

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Ergodic Control

The open issues are

- Degenrate Diffusions
- General Cost
- $\rho(x)$ is not constant

Ergodic Control

An Example (from the The Red Book)

Non-uniqueness of soutions to HJB Equations
Consider $b(x, u) = u$, $U = [-1, 1]$, $\sigma(x) = 1$. The cost function $c(x, u) = 1 - e^{-|x|}$.
The HJB equation becomes

$$\rho = \frac{1}{2}V''(x) - |V'(x)| + c(x)$$

For each $\rho \in [\frac{1}{3}, 1)$, there is a classical solution. And $\rho = \frac{1}{3}$ is the ergodic cost.

Ergodic Control

An Interesting Observation

Consider $b(x,u) = u$, $U = [-1,1]$, $\sigma(x) = \epsilon$. The cost function $c(x,u) = 1 - e^{-|x|}$.
The HJB equation becomes

$$\rho = \frac{\epsilon}{2} V''(x) - |V'(x)| + c(x)$$

For each ϵ, there are infinitely many solutions, but as $\epsilon \to 0$, the limiting equation has unique solution.

Ergodic Control

One More Example

Let $d = 2$, $b(x_1, x_2) = (x_2, -x_1)$ and $\sigma = 0$. There is no control. Then it is is easy to verify that $\rho(x)$ can not be constant. In fact,

$$\rho(x) = \frac{1}{2\pi} \int_0^{2\pi} c(|x|e^{ri\theta})\, d\theta.$$

This, clearly, shows that $\rho(x)$ is not constant.

Ergodic Control

Coupled HJB Equation

When $\rho(x)$ is not constant, it is expected to be solution of

$$0 = \inf_{u \in U} \left\{ \frac{1}{2} \operatorname{tr}(a(x) D^2 \rho(x)) + b(x,u) \cdot D\rho(x) \right\}$$

such that there is a solution $V(x)$ to

$$\rho = \inf_{u \in U} \left\{ \frac{1}{2} \operatorname{tr}(a(x) D^2 V(x)) + b(x,u) \cdot DV(x) + c(x,u) \right\}.$$

Ergodic Control

Coupled HJB Equation

- The motivation comes from Markov decision processes, where a complete theory is available even when the underlying Markov chain is not recurrent.

Ergodic Control

Coupled HJB Equation

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- There is no literature on this system of HJB Equations. (see pp. 300, RED BOOK).

Ergodic Control

Possibly Degenerate Diffusions

In the degrease case, BBG[2] studied the ergodic control under the following assumption: there exists a symmetric positive definite matrix Q and a constant $\alpha > 0$ such that for every $x, y \in \mathbb{R}^d$, $u \in U$,

$$2\langle b(x,u) - b(y,u), Q(x-y)\rangle - \frac{\langle Q(x-y), a(x,y)Q(x,y)\rangle}{\langle x-y, Q(x-y)\rangle} + \operatorname{tr}(a(x,y)Q) \leq -\alpha \|x-y\|^2. \quad (2)$$

The main consequence is that the associated flow is asymptotic flat and hence the ergodic value is constant.

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[2] G.K. Basak, V.S. Borkar and M.K. Ghosh, Ergodic control of degenerate diffusions, Stochastic Anal. Appl., 15(1997), 1 - 17.

Ergodic Control

Partially Degenerate Diffusions

Ergodic control of partially degenerate diffusions is studied by Borkar and Ghosh [3]

[3] V.S. Borkar, and M.K. Ghosh, Ergodic control of partially degenerate diffusions in a compact domain. Stochastics, 75(2003), 221 - 231

Ergodic Control

Ergodic Value is not Constant

The first work to discuss the non-constant ergodic value is BGQ[4] The work depends mainly on the assumption that for each x, y, we have

$$\sup_{u\in U}\inf_{u\in U}\{\langle b(x,y)-b(y,v),x-y\rangle+\frac{1}{2}\|\sigma(x,u)-\sigma(y,v)\|^2\}\leq 0. \tag{3}$$

[4] R. Buckdahn, D. Goreac, and M. Quincampoix, Existence of asymptotic values for non expansive stochastic control systems, App. Math. Optimization, 70(2014), 1 - 28

Ergodic Control

Ergodic Value is not Constant

Under this assumption, they show the following nonexpansivity condition: for every $T > 0, \epsilon > 0, x, y \in \mathbb{R}^e$, and every $u(\cdot) \in \mathcal{A}$, there exists $v(\cdot) \in \mathcal{A}$ such that

$$\mathbb{E}\|X(t;x,u(\cdot))-X(t;y,v(\cdot))\|^2 \leq \|x-y\|^2+\epsilon \text{ for } t \in [0,T] \text{ a.e.}$$

Using this they prove the uniform Tauberian theorem: $\frac{1}{T}V^T(t,x) \to \eta(x)$ uniformly as $T \to \infty$ if and only if $\lambda V_\lambda(x) \to \eta(x)$ uniformly as $\lambda \to 0$.

Ergodic Control

Ergodic Value is not Constant

The following issues are not studied in this work.

- Identifying $\eta(x)$ as ergodic value. What they have proved is the existence of uniform value.

In this work, we attempt to address these questions.

Ergodic Control

Ergodic Value is not Constant

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- Identifying $\eta(x)$ as ergodic value. What they have proved is the existence of uniform value.
- PDE characterization.

In this work, we attempt to address these questions.

Ergodic Control

Comparison of the Two Assumptions

- Both the assumptions consider degenerate case.

[5](see M.K. Ghosh and K.S. Mallikarjuna Rao, Differential Games with Ergodic Payoff, SIAM J. Control Optim., 43(2005), 2020 - 2035; and M. Quincampoix and J. Renault, On the existence of a limit value in some non expansive optimal control problems, SIAM J. Control Optim., 49(2011), 2118 - 2132.

Ergodic Control

Comparison of the Two Assumptions

- Both the assumptions consider degenerate case.
- In one dimensional case, both essentially reduces to the following:

$$\sup_{u\in U}\inf_{u\in U}\{\langle b(x,y)-b(y,v),\ x-y\rangle \leq -\alpha\|x-y\|^2.$$

for some $\alpha > 0$. In the case of (2), there will not be any sup or inf. Thus if we assume $\sigma \equiv 0$, then they imply the asymptotic flatness[5]

[5](see M.K. Ghosh and K.S. Mallikarjuna Rao, Differential Games with Ergodic Payoff, SIAM J. Control Optim., 43(2005), 2020 - 2035; and M. Quincampoix and J. Renault, On the existence of a limit value in some non expansive optimal control problems, SIAM J. Control Optim., 49(2011), 2118 - 2132.

Ergodic Control

Comparison of the Two Assumptions

- Note that the assumption (3) is weaker than the assumption (2), for higher dimensions, which can be proved by considering the relation between trace of the matrix and the norm.

Ergodic Control

Our assumption

We carry out our analysis under the assumption that

- There exists a symmetric positive definite matrix Q such that for every $x, y \in \mathbb{R}^d$, $u \in U$,

$$2\langle b(x,u)-b(y,u), Q(x-y)\rangle - \frac{\langle Q(x-y),\, a(x,y)Q(x,y)\rangle}{\langle x-y,\, Q(x-y)\rangle} + \operatorname{tr}(a(x,y)Q) \le 0. \quad (4)$$

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- We can consider sup and inf as in (3). However, we refrain from doing so as it does not pose any additional technical difficulties in the arguments that follow.
- Note that (4) is weaker than (3) without sup and inf.

We have the following result.

Lemma

For $x, y \in \mathbb{R}^d$ and any admissible control $u(\cdot) \in \mathcal{A}$, we have

$$\mathbb{E}\|X(t; x, u(\cdot)) - X(t; y, u(\cdot))\| \leq \|x - y\|.$$

Theorem

Assume (4) Then there exists a constant $C > 0$ such that

$$|V^T(t,x) - VTT(t,y)| \leq CT\|x-y\|$$

and

$$|V_\lambda(x) - V_\lambda(y)| \leq \frac{C}{\lambda}\|x-y\|$$

and C can be chosen to be independent of T and λ.

As a corollary, we have the following result.

Theorem

Assume (4). Then there exists Lipschitz continuous functions $\hat{\rho}(x)$ and $\bar{\rho}(x)$ such that

$$\frac{1}{T}V^T(T,x) \to \hat{\rho}(x)$$

as $T \to \infty$ along a subsequence and

$$\lambda V_\lambda(x) \to \bar{\rho}(x)$$

as $\lambda \to 0$ along a subsequence. Also both $\hat{\rho}$ and $\bar{\rho}$ are Lipschitz continuous.

Coupled HJB Equations

Assume that there is an invariant set K (compact) with respect to the dynamics (1). Then we have the following result.

Theorem

The ergodic value $\rho(x)$ is a viscosity solution of

$$0 = \inf_{u \in U} \left\{ \frac{1}{2} \operatorname{tr}(a(x) D^2 \rho(x) + b(x,u) \cdot D\rho(x) \right\}. \qquad (5)$$

and it satisfies

$$\rho(x) = \inf_{u(\cdot)} \mathbb{E}\rho(X(t)) = \inf_{u(\cdot)} \liminf_{T \to \infty} \frac{1}{T} \mathbb{E} \int_0^T \rho(X(t))\, dt.$$

Coupled HJB Equations

Idea of the proof is to use Ito formula, with a carefully estimating the error terms.
Assume $\rho(x)$ is a smooth function. Let $u(\cdot) \in \mathcal{A}$. Applying the Ito formula for $\rho(X(t))$, we have

$$\rho(X(t)) - \rho(x) = \int_0^t \{\frac{1}{2}\operatorname{tr}(a(X(s))D^2\rho(X(s))) + b(X(s), u(s)) \cdot D\rho(X(s))\}ds + M_t$$

where M_t is a local martingale. By the sub-solution property, we have

$$\rho(X(t)) - \rho(x) \geq M_t$$

Hence

$$\rho(x) \leq \inf_{u(\cdot)} \mathbb{E}\rho(X(t))$$

Coupled HJB Equations

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Then extension non-smooth case involves in regularising $\rho(x)$ together with careful error estimates.

Coupled HJB Equations

Our final result is the following:

Theorem

There exists a unique pair $(\rho(x), V(x))$ satisfying $V(0) = 0$ such that the pair satisfies

$$
\begin{aligned}
0 &= \inf_{u \in U} \left\{ \frac{1}{2} \operatorname{tr}(a(x) D^2 \rho(x)) + b(x,u) \cdot D\rho(x) \right\} \\
\rho &= \inf_{u \in U} \left\{ \frac{1}{2} \operatorname{tr}(a(x) D^2 V(x)) + b(x,u) \cdot DV(x) + c(x,u) \right\}.
\end{aligned}
$$

Further Results and Extensions

When there is no invariant set, we have the following result under *appropriate assumptions*.

Theorem

$(\rho(x), V(x))$ *is a pair of functions such that* $\rho(x)$ *is "largest" such that there is a solution to the coupled system.*

Further Results and Extensions

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This is not the end of the story. (Compare the above result with the example!)

And many more issues are open...

Thank You