Fundamentals: QRT maps and Degree

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QRT Maps Review:
Two kinds:

- Symmetric
- Asymmetric

Consider the 3×3 matrices

$$A[\mathrm{i}] = \begin{pmatrix} \alpha_i & \beta_i & \gamma_i \\ \delta_i & \epsilon_i & \xi_i \\ \kappa_i & \lambda_i & \mu_i \end{pmatrix}, i = 0, 1$$

Introduce the vectors

$$X_n = \begin{pmatrix} x_n^2 \\ x_n \\ 1 \end{pmatrix}, X_{n+1} = \begin{pmatrix} x_{n+1}^2 \\ x_{n+1} \\ 1 \end{pmatrix}$$

$$Y_n = \begin{pmatrix} y_n^2 \\ y_n \\ 1 \end{pmatrix}, Y_{n+1} = \begin{pmatrix} y_{n+1}^2 \\ y_{n+1} \\ 1 \end{pmatrix}$$

Construct f and g as follows

$$f\left(Y_n\right) = \left(A[0].Y_n\right) \times \left(A[1].Y_n\right)$$

$$f\left(Y_n\right) = \left(\begin{array}{c} \alpha_0 {y_n}^2 + \beta_0 y_n + \gamma_0 \\ \delta_0 {y_n}^2 + \epsilon_0 y_n + \zeta_0 \\ \kappa_0 {y_n}^2 + \lambda_0 y_n + \mu_0 \end{array}\right) \times \left(\begin{array}{c} \alpha_1 {y_n}^2 + \beta_1 y_n + \gamma_1 \\ \delta_1 {y_n}^2 + \epsilon_1 y_n + \zeta_1 \\ \kappa_1 {y_n}^2 + \lambda_1 y_n + \mu_1 \end{array}\right)$$

$$= \left(\begin{array}{c} f_1\left(y_n\right) \\ f_2\left(y_n\right) \\ f_3\left(y_n\right) \end{array}\right)$$

$$\mathrm{g}(X_{n+1}) = \left(A[0]^T.X_{n+1}\right) \times \left(A[1]^T.X_{n+1}\right)$$

$$\mathrm{g}(X_{n+1}) = \begin{pmatrix} \alpha_0 {x_{n+1}}^2 + \beta_0 x_{n+1} + \gamma_0 \\ \delta_0 {x_{n+1}}^2 + \epsilon_0 x_{n+1} + \zeta_0 \\ \kappa_0 {x_{n+1}}^2 + \lambda_0 x_{n+1} + \mu_0 \end{pmatrix} \times$$

$$\begin{pmatrix} \alpha_1 {x_{n+1}}^2 + \beta_1 x_{n+1} + \gamma_1 \\ \delta_1 {x_{n+1}}^2 + \epsilon_1 x_{n+1} + \zeta_1 \\ \kappa_1 {x_{n+1}}^2 + \lambda_1 x_{n+1} + \mu_1 \end{pmatrix} = \begin{pmatrix} g_1\left(x_{n+1}\right) \\ g_2\left(x_{n+1}\right) \\ g_3\left(x_{n+1}\right) \end{pmatrix}$$

Given f_i and g_i, we get the mapping as follows
$X_n.\left(f\left(Y_n\right) \times X_{n+1}\right) = 0$

$f_1 x_{1+n} - f_2 x_{1+n}^2 + x_n^2 \left(f_2 - f_3 x_{1+n}\right) + x_n \left(-f_1 + f_3 x_{1+n}^2\right) = 0$
Simplify:
$\left(-x_n + x_{1+n}\right)\left(f_1 - f_2 x_{1+n} + x_n \left(-f_2 + f_3 x_{1+n}\right)\right) = 0$
Divide by $\left(-x_n + x_{1+n}\right)$
$f_1 - f_2 x_{1+n} + x_n \left(-f_2 + f_3 x_{1+n}\right) = 0$

Solving for x_{1+n}
$x_{1+n} = \frac{f_1 - f_2 x_n}{f_2 - f_3 x_n}$

Similarly,

$$Y_n.\left(g\left(X_n\right)\times Y_{n+1}\right)=0$$

$$\mathrm{g1}y_{1+n}-\mathrm{g2}y_{1+n}^2+y_n^2\left(\mathrm{g2}-\mathrm{g3}y_{1+n}\right)+y_n\left(-\mathrm{g1}+\mathrm{g3}y_{1+n}^2\right)=0$$

$$\left(-y_n+y_{1+n}\right)\left(\mathrm{g1}-\mathrm{g2}y_{1+n}+y_n\left(-\mathrm{g2}+\mathrm{g3}y_{1+n}\right)\right)=0$$

$$\mathrm{g1}-\mathrm{g2}y_{1+n}+y_n\left(-\mathrm{g2}+\mathrm{g3}y_{1+n}\right)=0$$

This implies,

$$y_{n+1}=\frac{\mathrm{g1}-\mathrm{g2}y_n}{\mathrm{g2}-\mathrm{g3}y_n}$$

Invariant :

I[x,y]= $\dfrac{X[x]^T.A[0]Y[y]}{X[x]^T.A[1]Y[y]} = K(\text{Constant})$

The definition is given below

Let ϕ be a rational evolution map. The action of the map represents one
The iteration of the map is given by

$$\phi : (x_{n-1}, x_n) \rightarrow (x_n, x_{n+1})$$

is an evolution . If

$$I\, o\phi\, [x_{n-1}, x_n] = I\, [\phi\, [x_{n-1}, x_n]] = I\, [x_n, x_{n+1}] = I\, [x_{n-1}, x_n]$$

is true then I$[x_{n-1}, x_n]$ is called invariant.

For further classification the following form may be useful

1) Symmetric autonomous case

$$x_{n+1}x_{n-1}f_3(x_n) - (x_{n+1} + x_{n-1}) f_2(x_n) + f_1(x_n) = 0$$

2)Asymmetric autonomous case

$$x_{n+1}x_n f_3(y_n) - (x_{n+1} + x_n) f_2(y_n) + f_1(y_n) = 0$$
$$y_n y_{n-1} g_3(x_n) - (y_n + y_{n-1}) g_2(x_n) + g_1(x_n) = 0$$

Invariants are:

$X[x]^T.A[0]Y[y] - K(X[x]^T.A[1]Y[y]) = 0$

Invariant for asymmetric case :

$$\begin{aligned} x^2y^2(\alpha_0 - K\alpha_1) + x^2y(\beta_0 - K\beta_1) + x^2(\gamma_0 - K\gamma_1) \\ +xy^2(\delta_0 - K\delta_1) + xy(\epsilon_0 - K\epsilon_1) + y^2(\kappa_0 - K\kappa_1) \\ +y(\lambda_0 - K\lambda_1) + \mu_0 - K\mu_1 + x(\xi_0 - K\xi_1) &= 0 \end{aligned}$$

Invariant symmetric case :

$$\begin{aligned} x^2y^2(\alpha_0 - K\alpha_1) + x^2y(\delta_0 - K\delta_1) + xy^2(\delta_0 - K\delta_1) \\ +xy(\epsilon_0 - K\epsilon_1) + x^2(\kappa_0 - K\kappa_1) + y^2(\kappa_0 - K\kappa_1) \\ +x(\lambda_0 - K\lambda_1) + y(\lambda_0 - K\lambda_1) + \mu_0 - K\mu_1 &= 0 \end{aligned}$$

This means that,
Denote (x, y) as (x_n, x_{1+n}) in the above expression

$$\begin{aligned} x_n^2 x_{1+n}^2 (\alpha_0 - K\alpha_1) + x_n^2 x_{1+n} (\delta_0 - K\delta_1) + x_n x_{1+n}^2 (\delta_0 - K\delta_1) \\ + x_n x_{1+n} (\epsilon_0 - K\epsilon_1) + x_n^2 (\kappa_0 - K\kappa_1) + x_{1+n}^2 (\kappa_0 - K\kappa_1) \\ + x_n (\lambda_0 - K\lambda_1) + x_{1+n} (\lambda_0 - K\lambda_1) + \mu_0 - K\mu_1 &= 0 \end{aligned}$$

The invariant
$(\alpha_0 + K\alpha_1) {x_n}^2 {x_{n+1}}^2 + (\beta_0 + K\beta_1) \left({x_n}^2 x_{n+1} + x_n {x_{n+1}}^2\right) +$
$(\gamma_0 + K\gamma_1) \left({x_n}^2 + {x_{n+1}}^2\right)$
$+(\epsilon_0 + K\epsilon_1) x_n x_{n+1} + (\zeta_0 + K\zeta_1) (x_n + x_{n+1}) + (\mu_0 + K\mu_1) = 0$

Example : McMillan map :

x'=y

y'=-x- $\dfrac{\beta y^2+\epsilon y+\xi}{\alpha y^2+\beta y+\gamma}$

This implies

$y'=\frac{-x\left(\alpha y^2+\beta y+\gamma\right)-\left(\beta y^2+\epsilon y+\xi\right)}{\alpha y^2+\beta y+\gamma}$

Put$x=x_n, y=$

$x_{n+1}$and taking the down shift then the McMillan map becomes

x$_{n+1}=\frac{-x_{n-1}\left(\alpha_0 x_n{}^2+\beta_0 x_n+\gamma_0\right)-\left(\beta_0 x_n{}^2+\epsilon_0 x_n+\xi_0\right)}{\alpha_{\Box} x_n{}^2+\beta_0 x_n+\gamma_0}$

Therefore,

$f_1\left(x_n\right)=-\left(\beta_0 x_n{}^2+\epsilon_0 x_n+\xi_0\right)$

f$_2(y)=\alpha_0 x_n{}^2+\beta_0 x_n+\gamma_0$

$f_3(y)=0$

Invariant

$\alpha_0 x_n{}^2 x_{n+1}{}^2+\beta_0\left(x_n{}^2 x_{n+1}+x_n x_{n+1}{}^2\right)+\gamma_0\left(x_n{}^2+x_{n+1}{}^2\right)+$
$\epsilon_0 x_n x_{n+1}+\xi_0\left(x_n+x_{n+1}\right)+K=0$

Some more examples

- D ΔKdV

Mapping: $(x_{n-1} + x_{n+1}) = \frac{x_n - \omega}{2p^2 {x_n}^2}$

Invariant: $2\mathrm{p}^2 {x_n}^2 {x_{n+1}}^2 - x_n x_{n+1} + \omega (x_n + x_{n+1}) + K = 0$

- D $\Delta MK - dV$:

Mapping: $(x_{n-1} + x_{n+1}) = \frac{\omega - \frac{1}{2} - {x_n}^2}{x_n - {x_n}^2}$

Invariant: ${\mathrm{x}_n}^2 {x_{n+1}}^2 - {x_n}^2 x_{n+1} - x_n {x_{n+1}}^2 + \left(\omega - \frac{1}{2}\right)(x_n + x_{n+1}) + K = 0$

- D ΔNLS

Mapping: $(x_{n-1} + x_{n+1}) = \frac{(\omega+2) x_n}{\left(1 + \frac{1}{2}{x_n}^2\right)}$

Invariant: ${\mathrm{x}_n}^2 {x_{n+1}}^2 - 2(\omega + 2) x_n x_{n+1} + 2{x_n}^2 + 2{x_{n+1}}^2 + K = 0$

Algebraic curve:

An algebraic curve is a curve which is determined by some polynomial equation

$$f(x, y) = \sum a_{ij} x^i y^j = 0.$$

The degree of the curve is the degree of the polynomial $f(x, y)$.

Definition:

A affine space over the field $\mathbb{C}$ is a vector space $\mathbb{C}^n = \mathbb{A}^n$.

Definition:

Projective $n-$space $\mathbb{P}^n$ can be interpreted as the

$$\mathbb{P}^n = \frac{\mathbb{C}^{n+1} \setminus \{0\}}{\sim}.$$

where $\sim$ denotes the equivalence relation of points lying on the same line through the origin:

$$(x_0, x_1, \ldots, x_n) \sim (y_0, y_1, \ldots, y_n) \;\; \Leftrightarrow$$

there exists a complex number $\lambda \neq 0$ such that

$$(y_0, y_1, \ldots, y_n) \sim (\lambda x_0 \lambda, x_1, \ldots, \lambda x_n).$$

A point in projective space $\mathbb{P}^n$ can be thought of as an equivalence class

$$[(x_0, x_1, \ldots, x_n)] = \{(\lambda x_0, \lambda x_1, \ldots, \lambda x_n) | \lambda \in \mathbb{C}\}$$

in which at least one of the coordinates $x_0, x_1, \ldots, x_n$ must be nonzero.

Homogeneous coordinates of the point in $\mathbb{P}^n$ can be written as

$$[x_0 : x_1 : \ldots : x_n] \in \mathbb{P}^n$$

$$\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$$

$$[x_0 : x_1] \to \begin{cases} \frac{x_1}{x_0}, & \text{for } x_0 \neq 0; \\ \infty, & \text{for } x_0 = 0. \end{cases}$$

$$\mathbb{P}^1 = \{\text{finite}\} \cup \{\text{infinity part}\} = \mathbb{C} \cup \{[1:0]\}$$

where, $[1:0]$ is point at infinity. The line $x_0 = 0$, labeled ∞ is also a point at infinity.

$$\mathbb{P}^2 = \mathbb{C}^2 \cup \mathbb{P}^1.$$

Example:

Choose the coordinates $(x_0, x_1, x_2) \in \mathbb{C}^3$.
Reference plane $x_0 = 1$

$$[x_0 : x_1 : x_2] \to (\frac{x_1}{x_0}, \frac{x_2}{x_0})\text{for } x_0 \neq 0$$
$$= (x_1, x_2)$$

When $x_0 = 0$
$[x_0 : x_1 : x_2] \to [x_1 : x_2]$ a point in the projective line.

$\mathbb{P}^n$ is the set of all complex lines through the origin in $\mathbb{C}^{n+1}$
Let $[x_0 : x_1 : \ldots : x_n]$ and $[y_0 : y_1 : \ldots : y_n]$ be two distinct points, the corresponding vectors x and y. The line has a point $[a_0x_0 + b_0y_0 : a_1x_1 + b_1y_1 : \ldots : a_nx_n + b_ny_n]$, a, b not both zero.

Let $p = [1 : 0 : 0 : 0]$, $q = [a : b : c : d]$ the projective line has points with coordinates $[u + va : vb : vc : vd]$.

Numerical invariant:-Degree

Etienne Bézout (1779)

Bézout theorem is statement in algebraic geometry concerning the number of common points or intersection points of two plane curves.

Claim:
The number of intersection of two such curves is equal to the product of their degrees.

Let X and Y are two algebraic curves without any non-constant common points in $\mathbb{P}^2$ of degree m and n respectively, then the number of intersection points does not exceed mn provided that the curves are not tangent to each other at any of their intersection points.

Two parallel lines intersects at a unique point that lies at infinity.

Consider two parallel lines

$$x + 2y = 3$$

$$x + 2y = 5$$

In the projective space

$$x + 2y - 3z = 0$$

$$x + 2y - 5z = 0$$

Solving them, we get $x = -2y,\ z = 0$, $[-2 : 1 : 0]$ in homogeneous coordinate.
As $z = 0$, the point lies at infinity.
This implies two lines meet at infinity.

Special case:

An algebraic curve of degree n intersects a given line in n points.

Example:
$y - x^2 = 0$ has degree 2.
The line $y - ax = 0$ has degree 1.
They meet exactly two points when $a \neq 0$.

Example:
Let C is given generic irreducible curve defined by an irreducible polynomial of degree m intersecting a generic line L defined a linear polynomial of degree one.

Therefore, the number of intersection points of C with L is exactly m distinct points. (In the case of the line L is tangent of C then we have to count with multiplicity).

Homogenization of a Polynomial

If

$$\begin{aligned} f(x,y) &= \sum a_{ij}x^i y^j \\ F(X,Y,Z) &= \sum a_{ij}X^i Y^j Z^k \\ F(X,Y,Z) &= Z^{deg\ f} f(\frac{X}{Z},\frac{Y}{Z}) \end{aligned}$$

is the homogenization of f, where $k = deg f - i - j$
Circle $x^2 + y^2 = 1 \rightarrow X^2 + Y^2 - Z^2 = 0$, $Z = 0$ is the point at infinity.
Note that it is of homogeneous degree 2.

Remark:

If $(x_0, x_1, \ldots, x_n)$ are homogeneous coordinates, that is

$$f(\lambda x_0, \lambda x_1, \ldots, \lambda x_n) = \lambda^d f(x_0, x_1, \ldots, x_n)$$

then $f(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0})$ is well defined.

Consider cubic curve

$$y^2 - x^3 - x^2 = 0*$$

Homogeneous

$$(x, y) \to [X : Y : Z]$$

$$(x, y) \to (\frac{X}{Z} : \frac{Y}{Z}) **$$

On substitution $**$ into $*$, we get

$$\frac{Y^2}{Z^2} - \frac{X^3}{Z^3} - \frac{X^2}{Z^2} = 0$$

$$ZY^2 - X^3 - ZX^2 = 0$$

Let $F(X, Y, Z) = ZY^2 - X^3 - ZX^2$ then the solution of $F_X = F_Y = F_Z = 0$ are singular.

$$F_X = -3X^2 - 2XZ = 0$$

$$F_Y = 2ZY = 0$$

$$F_Z = Y^2 - X^2 = 0$$

If $Y = 0$, then

$$F_Z = 0 \Rightarrow X = 0$$

therefore,

$$[0 : 0 : Z] \sim [0 : 0 : 1] \in \mathbb{C}$$

If $Z = 0$, then

$$F_X = 0 \Rightarrow X = 0, F_Z = 0 \Rightarrow Y = 0$$

therefore,

$$[0 : 0 : 0] \text{ is not in } \mathbb{P}^2.$$

Rational map:

Let X be an irreducible variety. A rational map ϕ from $X \to \mathbb{P}^n$ is given by an n-tuple of rational functions.

$$\phi(x) = (h_1(x), h_2(x), \ldots, h_n(x)),\ h_i(x) \in \mathbb{C}[x]$$

If $X \in \mathbb{P}^n$

$$h_i(x) = \frac{F_i(x)}{G_i(x)}$$

where $F_i(x)$ and $G_i(x)$ are same degree of polynomials.
If ϕ and ϕ^{-1} are rational then the map is called birational.

Degree of the rational map:
The maximum degree of the common denominator and the various numerators is called the degree of the rational map.
$d_n \leq d^n$
However, in the projective space, it is the common degree of the homogeneous polynomials.

- The degree of the composition of two maps is bounded by the product of the degrees of the maps.
  $d_{m+n} \leq d_m d_n$
- When calculating composition of maps, common factors may appear which lower the degree of the resulting map. Therefore, one has to remove all common factors before taking degree.

If any common factor appears among these polynomials, we factor it out. the degree d is well defined.

$$\phi_2 \circ \phi_1 = m(\phi_2, \phi_1) * (\phi_2 \times \phi_1)$$

where,$(\phi_2 \times \phi_1)$ is the reduced composition.

The degree d_k of the $k-$ iterates of ϕ is bounded by d^k, the entropy is thus bounded by $\log(d)$.
Denote $\phi^{[n]}$ the n iterates of a map ϕ, after removing the common factors.

Algebraic entropy:(Bellon Viallet)
Let f and g are two birational map

$$deg(f \circ g) \leq \deg(f)\deg(g).$$

When the composition of the maps have some factors in common, there is a reduction of degree shift inequality holds.
Algebraic entropy:(Bellon Viallet)

$$\epsilon = \lim_{n \to \infty} (\frac{1}{n}) \log(deg(f^{\sim}))$$

is called algebraic entropy of f.
Let ϕ be a birational map. If $g = \phi^{-1} \circ f \circ \phi$ then f and g have the same algebraic entropy.
That is, Algebraic entropy is invariant under birational conjugacy.

Intersections:
The degree of the map is equal to the number of intersections between the image of a generic line in $\mathbb{P}^n$ and a generic hyperplane in $\mathbb{P}^n$.
Thus the algebraic entropy measures the growth rate of the number of intersections between one sub manifold and the image of another sub manifold.

In other words,
The intersection points of a fixed curve Γ_1 with the image of another curve Γ_2 under the k-th iteration of ϕ

$$\sharp(\Gamma_1 \cap \phi^k(\Gamma_2)),$$

Γ_1 and Γ_2 are algebraic curves in $\mathbb{P}^2$

$$\phi^k = \phi \circ \phi \circ \ldots \circ \phi, \text{k-times}$$

Generating Functions:
From the first few members of the degree sequence $\{d_n\}$ we can fit the generating function

$$f(x) = \sum_{n=0}^{\infty} d_n x^n.$$

Since the degrees are bounded, the above series always has a non-zero radius of convergence.
Usually, the generating function is rational with integer coefficients. By expending the generating function we can obtain the degree sequence for larger iteration.

THANK YOU!