# Linearizations and Exact Solutions of Discrete Systems

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with

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## Linearizable systems

- Properties
- Integration

## Continuous

A. Riccati:

$$y' = \alpha y^2 + \beta y + \gamma$$

B. Gambier:

$$y' = -y^2 + by + c$$

$$x' = ax^2 + nxy + \sigma$$

$a = a(t), b = b(t), c = c(t), \sigma = constant$

C. Third-kind:

We start from the linear second order equation in the form:

$$\frac{\alpha x'' + \beta x' + \gamma x + \delta}{\epsilon x'' + \zeta x' + \eta x + \theta} = K$$

where $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta, \theta$ are functions of $t$ with $K$ a constant.

A nonlinear second order equation of the form:

$$f(x'', x', x) = M$$

where $f$ is a polynomial of degree two in $x$ together with its derivatives, but linear in $x''$.

## Discrete

A. Riccati:

$$x_{n+1} = \frac{\alpha(n)x_n + \beta(n)}{\gamma(n)x_n + \delta(n)}$$

B. Gambier:

$$y_{n+1} = \frac{\alpha y_n + \beta}{\gamma y_n + \delta}$$

$$x_{n+1} = \frac{a y_n x_n + b x_n + c y_n + d}{f y_n x_n + g x_n + h y_n + k}$$

where $\alpha, \ldots, \delta$ and $a, \ldots, k$ are all functions of independent discrete variable $n$.

C. Third-kind:

$$\frac{\alpha x_{n+1} + \beta x_n + \gamma x_{n-1} + \delta}{\epsilon x_{n+1} + \zeta x_n + \eta x_{n-1} + \theta} = K,$$

where $\alpha, \ldots, \theta$ are all functions of $n$ with $K$ a constant.

A nonlinear mapping

$$f(x_{n+1}, x_n, x_{n-1}; n) = M$$

where $f$ is globally polynomial of degree two in all the $x'$s but not more then linear separately in each of $x_{n-1}$ and $x_{n+1}$.

In this talk we present results on linearizable discrete systems belongs to the above three family, we discuss:

a. Linearization

b. Singularity properties

c. Degree growth properties

d. Integration

# Linearization

A. Riccati:

We recall that Riccati equation

$$y' = \alpha y^2 + \beta y + \gamma$$

can be transformed into a linear second-order ordinary differential equation through a simple Cole-Hopf transformation:

$$y = -\frac{w'}{\alpha w}$$

The second-order linear ordinary differential equation:

$$w'' + (\beta + \frac{\alpha'}{\alpha})w' + \alpha\gamma w = 0$$

# Discrete Riccati

Discrete Riccati equation is nothing but a homographic mapping (two point linearisable mapping):

$$x_{n+1} = \frac{a_n x_n + b_n}{c_n x_n + d_n}$$

Linearisation through the discrete equivalent of the Cole-Hopf transformation:
Substitute

$$x_n = \frac{P_n}{Q_n}$$

Write the equation for $P_n$ in the form

$$P_n = Q_{n+1} - \frac{d_n}{c_n} Q_n$$

Use this expression in the remaining equation we finally obtain

$$Q_{n+2} - (d_{n+1} + \frac{a_n c_{n+1}}{c_n}) Q_{n+1} + (\frac{a_n d_n c_{n+1}}{c_n} - d_n c_{n+1}) Q_n = 0$$

The basic idea of the singularity confinement is to consider the properties of solution close to some singularities.

If at some stage of iteration $\dfrac{\partial x_{n+1}}{\partial x_{n-1}} = 0$

Lose of one degrees of freedom - the initial memory is not there.

(Recall the talk of B.Grammaticos)

We recall the Gambier equation:

$$y' = -y^2 + by + c$$
$$x' = ax^2 + nxy + \sigma$$

$a = a(t), b = b(t), c = c(t), \sigma = constant$ The singularity analysis shows that the Gambier system can not be integrable unless the coefficient of $xy$ in the Gambier equation is an integer $n$. In addition, depending on the value of $n$ one can find constraints on the $a, c, \sigma$ which are necessary for integrability.

Generic discrete Gambier equation is given by:

$$y_{n+1} = \frac{\alpha y_n + \beta}{\gamma y_n + \delta}$$

$$x_{n+1} = \frac{a y_n x_n + b x_n + c y_n + d}{f y_n x_n + g x_n + h y_n + k}$$

where $\alpha, \ldots, \delta$ and $a, \ldots, k$ are all functions of independent discrete variable $n$.

Now consider the special case

Loss of one degrees of freedom
There is no way for $x_n$ to leave the singularities untill

i.e. $y_n$ becomes again $y_n \to \infty$ or $y_n = \dfrac{-a\sigma}{d}$
This happens after $N$ steps.

Periodic singularity appear-Fixed singularity.

Start with $\dfrac{-a\sigma}{d}$ reach $\infty$ after $N$ steps

Start with $\infty$ reach $\dfrac{-a\sigma}{d}$ after $N'$ steps.

Thus confinement of singularity when parameter satisfy certain conditions:

Let the iteration step is denoted by $N$
Conditions on the parameters for first two iteration

$$N = 1,\ \frac{-a\sigma}{d} = -1$$
$$N = 2,\ \frac{-a\sigma}{d} = \frac{c+1}{b+1}$$
with
$$N' = 1,\ \frac{-a\sigma}{d} = b_{n-1}$$
$$N' = 2,\ \frac{-a\sigma}{d} = \frac{b_{n-1}b_{n-2} + c_{n-1}}{b_{n-1}+1}$$

# Algebraic entropy

- Arnold:
  The notion of complexity for mapping in the plane is the number of intersection points of a fixed curve with the image of a second curve obtained under the iterations of the mapping at hand.

  While complexity grows exponentially with the iteration for generic mapping, it can be shown to grow only polynomially for a large class of integrable mapping.
- Veselov:
  Integrability has an essential correlation with the weak growth of certain characteristics
- Viallet et al introduced algebraic entropy, which is a global index of the complexity of the rational mapping

  There exists a link between the dynamical complexity of a mapping and the degree of its iterates

# Bézout's theorem:

Consider two curves in $\mathbb{P}^2$ defined by polynomials of degree $m$ and $n$ respectively. If they have no common components, their intersection consists of $mn$ points. The $mn$ points are distinct, provided that the curves are not tangent to each other at any of their intersection points.

In the generic case, the two curves are not tangent at any point, and there are precisely $mn$ intersection points.

In order to interpret the number of intersection points as exactly $mn$ in general, the points must be counted with "multiplicities".

$$x_{n+1} = f(x_{n-1}, x_n)$$

$$x_0 = p, \text{ finite.}$$

$$x_1 = \frac{q}{r},$$

Assume $deg(p) = 0, \ deg(q) = 1, \ deg(r) = 1$
$f$ is homogeneous i.e.

$$f(\lambda x, \lambda y) = \lambda^n f(x, y)$$

$$deg(\lambda^n) = n$$

The notion of algebraic entropy for the rational mapping was introduced by Viallet and Heitarinta as an integrability detector.

Let $\{d_n\}$ be the degree of the map at each iteration.
Algebraic entropy is defined as

$$\epsilon = \lim_{n\to\infty} \frac{1}{n} log(d_n)$$

If $\epsilon \neq 0$ the sequence $d_n$ **grows exponentially**

If $\epsilon = 0$ the sequence $d_n$ **grows only polynomial in** $n$

Linear degree growth $\rightarrow$ linearizable mappings

For integrable second order mappings (for QRT type) degree growth should be quadratic. (See Favre for more results.)

## Projective mapping

Singularity appearing in projective systems a confined in one step

$$x_{n+1} = \frac{\alpha x_n + \beta}{\gamma x_n + \delta}$$

when $x_n = \frac{-\delta}{r} \to x_{n+1} \to \infty$ but $x_{n+2}, x_{n+3}, \ldots$ are finite.
Where as algebraic growth analysis gives If

$$x_0 = f \text{ some finite and } x_1 = \frac{q}{r}$$

Replace $q = \lambda q$ and $r = \lambda r$ in the above expression then we get

$$\lambda(\alpha q + \beta r),$$

$$, \ldots ,$$

$$\lambda(....)$$

at each iteration $deg(\lambda) = 1$ at each iteration.

Thus

$$\{d_n\} = \{1, 1, \dots\}$$

Therefore degree growth is 1, degree is constant $d_n = 1$

Another linearizable map

$$x_{n+1} = a x_{n-1} \left( \frac{x_n - a}{x_n - 1} \right)$$

Singularity appear $x_n = 1, \; x_n = a$. Then singularity pattern $\{1, \infty, a\}$

Singularity is confined.

Degree growth $0, 1, 2, 3, 4, \dots$
Linear growth.

# Non-integrable mapping

$$x_{n+1} = a x_{n-1} \left( x_n + \frac{1}{x_n} \right)$$

Singularity is confined $\{i, 0, \infty, -i\}$ but not integrable.

The degree growth $0, 1, 2, 4, 8, 14, 24, 40, 66, 108, \ldots$

$$d_{n+1} - 2d_n + d_{n-1} = 0$$

Asymptotic ratio of two consecutive $x_n$

$$\frac{1+\sqrt{5}}{2}$$

“golden” ratio -same as Fibonacci series
Exponential growth- Nonintegrable.

## Gambier degree growth

$$y_n = \frac{1}{y_{n-1}} + a$$

$$x_{n+1} = \frac{\frac{x_n y_n}{d} + c^2}{x_n + d y_n}$$

Substitute $x_0 = r$ and $x_1 = \frac{p}{q}$ in the expression

The degree growth is

$$0, 1, 2, 3, 4, 5, \ldots$$

The degree growth is linear for any $a, c, d$

Singularity confinement is not satisfied in this case but for certain condition on $a, c, d$ singularity is confined.

For iteration $N = 1$ the constraints for singularity confinement is

$$c(a + c) + 1 = 0, \ d^2 - 1 = 0$$

and the degree growth in this case is $0, 1, 1, 1, \ldots$

Therefore the degree growth is saturated

Saturation of degree clearly associated with singularity confinement.

For iteration $N = 2$ the constraints for singularity confinement is

$$(ac + c)(a + c) + c = 0, \ d^4 - 1 = 0$$

and the degree growth is $0, 1, 2, 2, 2, \ldots$ and singularity confinement is satisfied

Therefore the degree growth is saturated.

## Third-kind linearizable

$$\frac{1}{x_{n+1}+x_n}+\frac{1}{x_n+x_{n-1}}=\frac{1}{x_n}+a$$

The degree growth $0, 1, 3, 5, 7, 9, \ldots$

Linear growth

Remark:

1. Degree growth is in step 2, where as degree growth in generic Gambier system is in step 1
2. In this case Gambier system when the singularities are confined the degree growth is saturated.

Third kind do not exhibit degree growth saturation.

## Favre definition of singularity

Singularity confinement examine only singularities which appear at *a* finite distance

Favre:

From algebraic point of view, one should also examine singularities associated with an infinite value of the QRT invariant.

i,e. The relation between $x$ and $x_{n-1}$ is such that the invariant vanish.

# Example

Third-kind:

$$x_{n+1}x_{n-1} = {x_n}^2 + 1$$

Invariant

$$K = \frac{x_n}{x_{n+1}} + \frac{x_{n+1}}{x_n} + \frac{1}{x_n x_{n+1}}$$

$K \to \infty$ if either $x_1 \to 0$ or $x_1 \to \infty$

If $x_1 = \frac{p}{q}$, then $p = 0$ or $q = 0$ (point at infinity in $\mathbb{P}^2$)

Iterating the mapping forwards, overall homogeneous degree of the iterates growth linear with a step of 2

If $p = 0,\ x_1 = 0$

Singularity pattern:

$$\ldots, \Omega^3, \Omega^2, \Omega, x_0, x_1 = 0, \frac{1}{x_0}, \Omega, \Omega^2, \Omega^3, \ldots$$

If $q = 0$, then $x_1 = \infty$

Singularity pattern:

$$\ldots, \Omega^3, \Omega^2, \Omega, \frac{1}{x_0}, x_1 = 0, x_0, \Omega, \Omega^2, \Omega^3, \ldots$$

The mapping of the third kind has singularities (at infinity) which extends to infinity in both directions but more and more singular.

On the other hand

Gambier type mapping

$$(x_{n+1} + x_n)(x_n + x_{n-1}) = a(x_n^2 - 1)$$

Invariant:

$$K = x_{n+1} + x_n + \frac{1 - a x_n x_{n+1}}{x_n + x_{n+1}}$$

$$x_1 = \frac{p}{q}$$

$K \to \infty$ if $q = 0$
or $x_0 + x_1 \propto q x_0 + p = 0$
Iterate the denominator for $x_n,\ n > 1$ is exactly $q(qx_0 + p)$
$n < -1,\ qx_0 + p$
when $x_1 \to \infty$
Singularity pattern has the form

$$\ldots, \Omega, \Omega, \Omega, -x_0, x_0, x_1, \Omega, \Omega, \Omega, \ldots$$

## Recall:

Gambier:

$$\ldots, \Omega, \Omega, \Omega, \text{finite}, \Omega, \Omega, \Omega, \ldots$$

Infinity in both directions and remain as singular as started (not becoming more singular.)

Third-kind:

$$\ldots, \Omega^3, \Omega^2, \Omega, \text{finite}, \Omega, \Omega^2, \Omega^3, \ldots$$

infinity in both direction and become more singular in each direction.

Singularities of Gambier mapping, make to disappear by a suitable transformation.

This not possible for third kind mapping

Result:
The linearizable mapping of Gambier type can be considered as having confined singularities at infinity.

This is not the case for third-kind mapping.

## Projective mapping

We already seen that projective mapping can be linearized to second-order linear equation **Three point linearisable mapping**

$$x_{n+1}x_nx_{n-1} + f_nx_nx_{n-1} + g_nx_{n-1} + 1 = 0$$

$f$ and $g$ are free functions of $n$ Linearise through a Cole-Hopf

transformation $x_n = u_{n+1}/u_n$

$$u_{n+2} + f_nu_{n+1} + g_nu_n + u_{n-1} = 0$$

The generic Gambier mapping is the derivative of the discrete Riccati equation

We start from the expression

$$\frac{a_n x_n x_{n+1} + b_n x_n + c_n x_{n-1} + d_n}{e_n x_n x_{n+1} + f_n x_n + g_n x_{n-1} + h_n} = k$$

where $a_n, \ldots, h_n$ are free functions of $n$ and $k$ is a constant.

Eliminate $k$ the above expression and its up shift. The resulting second-order mapping is the Gambier equation.
Two initial conditions are given say $x_n$ and $x_{n-1}$ at some given $n$ compute $k$ and then linearize the equation through a simple Cole-Hopf transformation.

# Third-kind mapping

$$\frac{\alpha x_{n+1} + \beta x_n + \gamma x_{n-1} + \delta}{\epsilon x_{n+1} + \zeta x_n + \eta x_{n-1} + \theta} = K, \qquad (*)$$

where $\alpha, \ldots, \theta$ are all functions of $n$ with $K$ a constant.

A nonlinear mapping

$$f(x_{n+1}, x_n, x_{n-1}; n) = M \qquad (**)$$

where $f$ is globally polynomial of degree two in all the $x'$s but not more then linear separately in each of $x_{n-1}$ and $x_{n+1}$.

If L.H.S of equation $(*)$ and $(**)$ is same as that of its upshift, we get an equation relating $x_{n-1}, x_n, x_{n+1}$ and $x_{n+2}$.
Then demand that both of them are same
We get conditions for $\alpha, \ldots, \theta$.

The integration method is quite similar to continuous case.

Given $M$, starting with $x_{n-1}, x_n$ at some $n$, one gets $x_{n+1}$ from equation $(**)$.

Implementing $(*)$, this fixes the value of $K$.

Now, one can integrate the linear equation $(*)$ for all $n$.

This implies, $f$ computed at any $n$ has constant values, which is just $M$, so $(**)$ is satisfied.

## Constructing explicit solutions of linearizable QRT mapping:

-Autonomous form

-Non autonomous form using SC/Algebraic entropy

-Invariant $K_{n+1} = K_n$ gives conditions on the parameters

-Assume solution of the form

$$x_n = p_n \phi_n + q_n + \frac{r_n}{\phi_n}$$

-Substitute $k_n$ find $\phi_{n+1} = f_n \phi_n$ with recurrence relation with $p_n$, $q_n$ and $r_n$.
Using $x_n$ and linear equation

$$a_n x_{n+1} + b_n x_n + c_n x_{n-1} + d_n = 0$$

Check for consistency that fixes all the unknowns .
Thus the solution is obtained.

# Linearizable QRT mappings

## Solutions in terms of trigonometric functions:

For QRT mappings the invariant should be **without quartic and cubic**

$$\gamma(x_{n+1}^2 + x_n^2) + \epsilon x_{n+1}x_n + \zeta(x_{n+1} + x_n) + \mu = 0$$

Integration (in terms of a hyperbolic cosine function)

$$x_n = p\phi_n + q + r/\phi_n$$

$\phi$ is an exponential function

$$\phi_{n+1} = \lambda\phi_n \qquad \text{with } \lambda \text{ given by} \quad \gamma\lambda^2 + \epsilon\lambda + \gamma = 0$$

and

$$q = -\frac{\zeta}{\epsilon + 2\gamma}, \qquad pr = \frac{\lambda}{(\lambda - 1)^2}\frac{(\zeta^2 - \mu(\epsilon + 2\gamma)}{(\epsilon^2 - 4\gamma^2)}$$

Solutions in terms of a hyperbolic cosine function.

## Explicit solution of linearizable, non-autonomous, equations of the third-kind

Equations of alternate $dP_{\mathrm{I}}$ type:

$$\frac{(z_{n+1}+z_n)}{(x_{n+1}+x_n)} + \frac{(z_n+z_{n-1})}{(x_n+x_{n-1})} = \frac{2z_n}{x_n} - \frac{1}{x_n^2}$$

where $z_n$ is a free function (This is typical for linearizable mapping)
Linearisation: introduce auxiliary function $u_n$

$$x_n = \frac{1}{u_{n+1}-u_n}$$

and ($k^2$ is an integration constant)

$$u_{n+1} = \frac{k^2+z_n u_n}{z_n+u_n}$$

Solution

$$u_n = k\frac{\phi_n - 1}{\phi_n + 1}$$

$\phi$ satisfies recursion relation

$$\phi_{n+1} = \frac{z_n + k}{z_n - k}\phi_n$$

the solution of which can be expressed in terms of infinite product.

Solution for $x$

$$x_n = \frac{1}{4k^2}\left((z_n + k)\phi_n + 2z_n + \frac{z_n - k}{\phi_n}\right)$$

Linear equation satisfied by $x_n$

$$\frac{x_{n+1}+x_n}{z_{n+1}+z_n}+\frac{x_n+x_{n-1}}{z_n+z_{n-1}}=\frac{1-2z_nx_n}{k^2-z_n^2}$$

Meaning of the constant $k$. Solve for $k^2$ and eliminate $x_{n+1}$ using alternate d-$\mathrm{P_I}$

$$k^2=\frac{(x_nz_{n-1}-x_{n-1}z_n)^2+(z_{n-1}+z_n)(x_{n-1}+x_n)}{(x_n+x_{n-1})^2}$$

**Important Remark**: $k^2$ is non-autonomous extension of the QRT

invariant.The meaning of the invariant is that eliminate $k$ between the above expression and its upshift we get alternate d-$\mathrm{P_I}$
We have presented a closed form solution explicitly for third kind mapping. This not the case for projective and Gambier mappings, where we have linearized the mapping but no explicit solution was given

# “Higher” examples of third-kind linearisable do exist

“Higher”: mappings belonging to the same family as d-Ps associated to the affine Weyl groups $E_8^{(1)}$ and $E_7^{(1)}$ in the Sakai classification

A first example

$$\frac{(x_{n+1}z_{n+1}z_n + x_n)(z_n z_{n-1} x_{n-1} + x_n)}{(x_{n+1} + z_{n+1}z_n x_n)(z_n z_{n-1} x_n + x_{n-1})} = z_{n+1}z_{n-1}\frac{x_n^2 + a x_n z_n + b z_n^2}{x_n^2 z_n^2 - a x_n z_n + b}$$

QRT invariant

$$z_n z_{n-1}(z_n z_{n-1} x_n + x_{n-1})(z_n z_{n-1} x_{n-1} + x_n) K_n = \Big((x_n z_n (z_{n-1}^2 + 1) + x_{n-1} z_{n-1}(z_n^2 + 1) + a(z_n^2 z_{n-1}^2 - 1)/2\Big)^2 + (b - a^2/4)(z_n^2 z_{n-1}^2 - 1)^2$$

Mapping is obtained by $K_n - K_{n+1} = 0$

Solution of the mapping

$$x_n = (k - (z_n + 1/z_n)^2)\phi_n + \frac{a(z_n - 1/z_n)}{k-4} \frac{a^2 + b(k-4)}{k(k-4)^2 \phi_n}$$

with $(k = 2 - h - 1/h)$

$$\phi_n = -\frac{z_{n-1} + h/z_{n-1}}{h z_n + 1/z_n} \phi_{n-1}$$

Linear equation:

$$(k-(z_n+\frac{1}{z_n})^2)\left(\frac{x_{n+1}z_{n+1}z_n+x_n}{{z_{n+1}}^2{z_n}^2-1}+\frac{z_nz_{n-1}x_{n-1}+x_n}{{z_n}^2{z_{n-1}}^2-1}\right) + \left(k-2-\frac{2}{{z_n}^2}\right)x_n-a\left(z_n+\frac{1}{z_n}\right)=0.$$

There are many more examples exist.

Conclusions:

We have analyzed three kinds of linearizable systems:

- Projective
- Gambier
- Third-kind

we have presented linearization, singularities, degree growth of systems belongs to these families

Moreover, we gave explicit integration of both autonomous and non-autonomous mapping

**THANK YOU!**