Pairwise spanners

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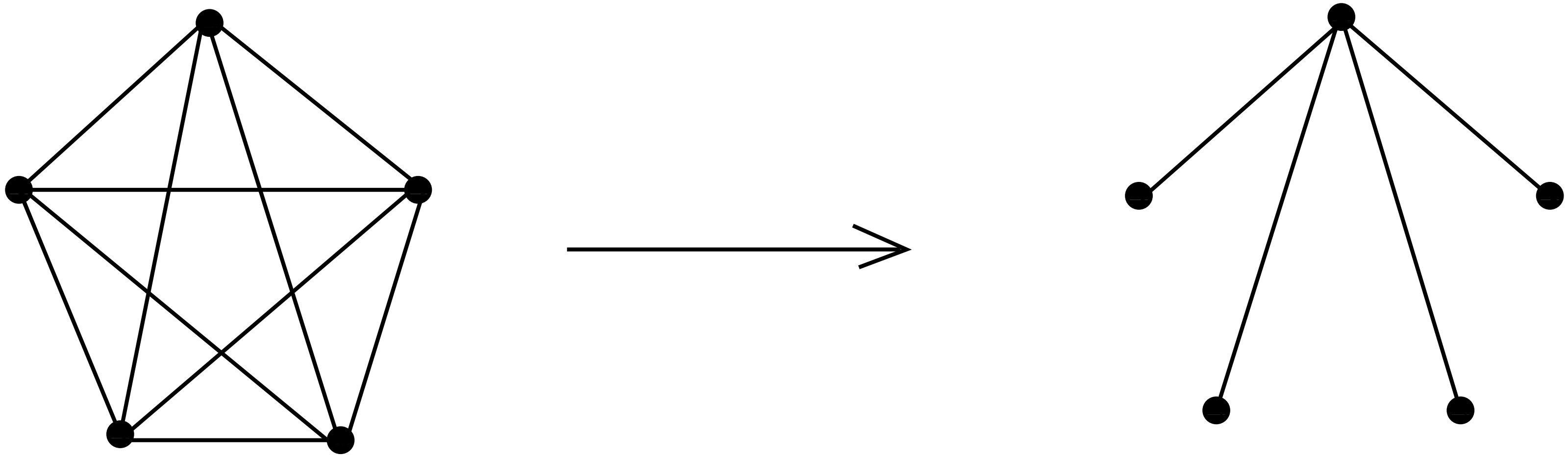
- We want a sparse subgraph that well-approximates every u-v distance, where $(u, v) \in V \times V$.
  - such a subgraph is called a *spanner* of G [PS89].

An example



- On the left is G with $\binom{n}{2}$ edges while the subgraph H on the right has only $n-1$ edges.
  - for all (u, v): $d_H(u, v) \leq d_G(u, v) + 1$.

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- For all pairs of vertices (u, v):
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    - for every integer $k \geq 1$, G admits a $(2k-1, 0)$-spanner of size $O(n^{1+1/k})$ [ADDJS93].

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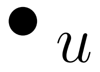

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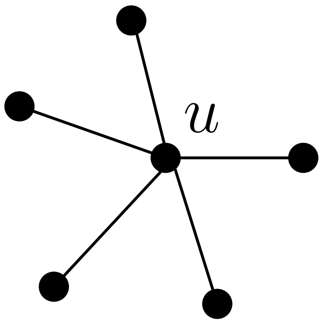

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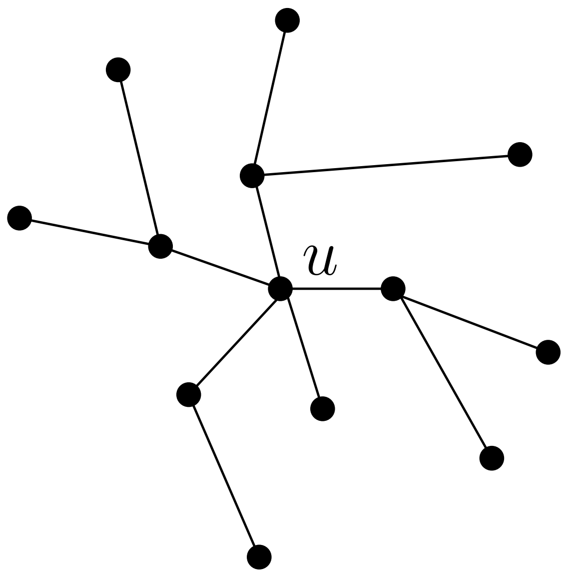

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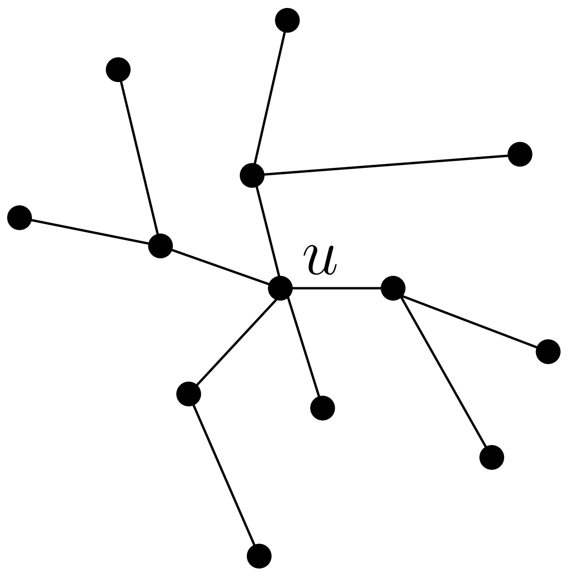

- $radius(C_i) \leq k-1$ since the scaling factor is $> n^{1/k}$.

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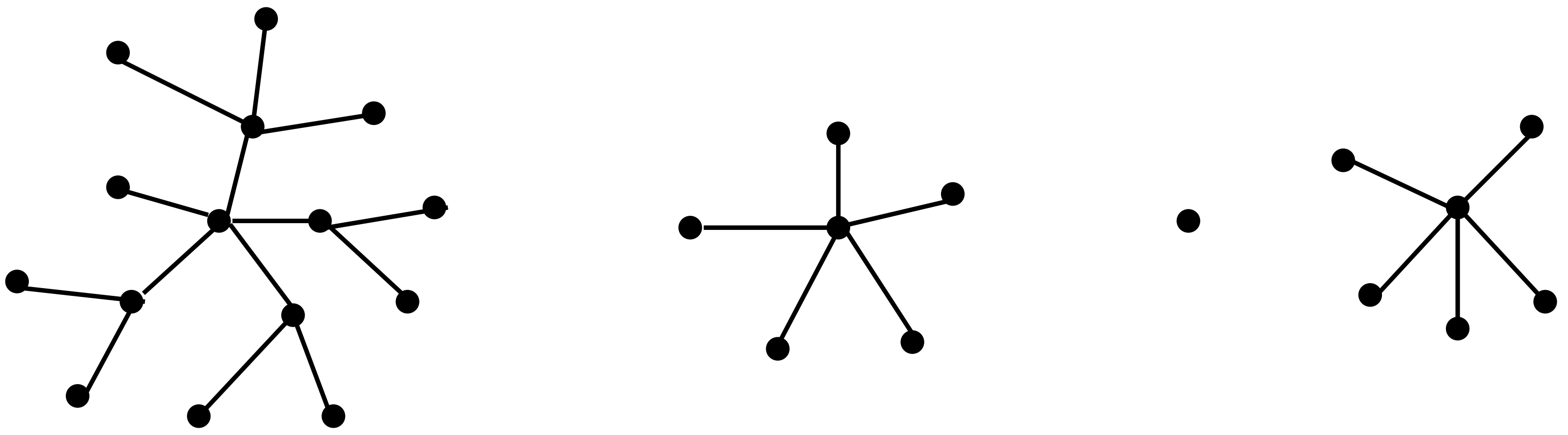

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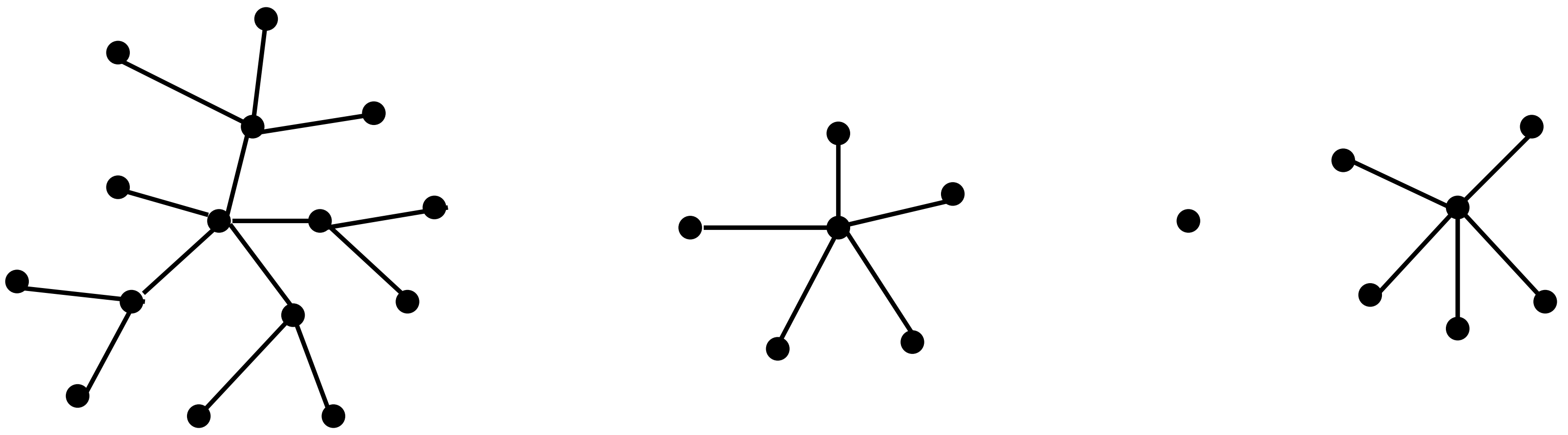

- Initialize $H =$ forest defined by the clusters.

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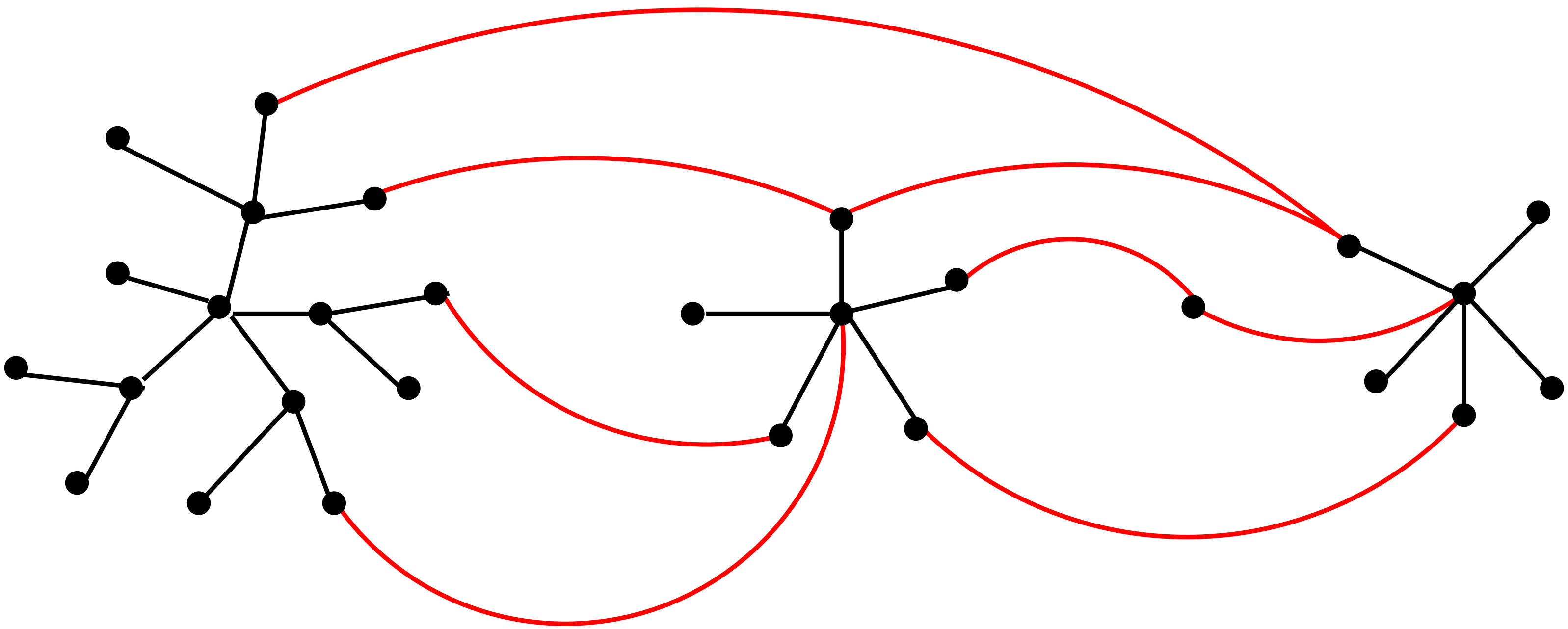

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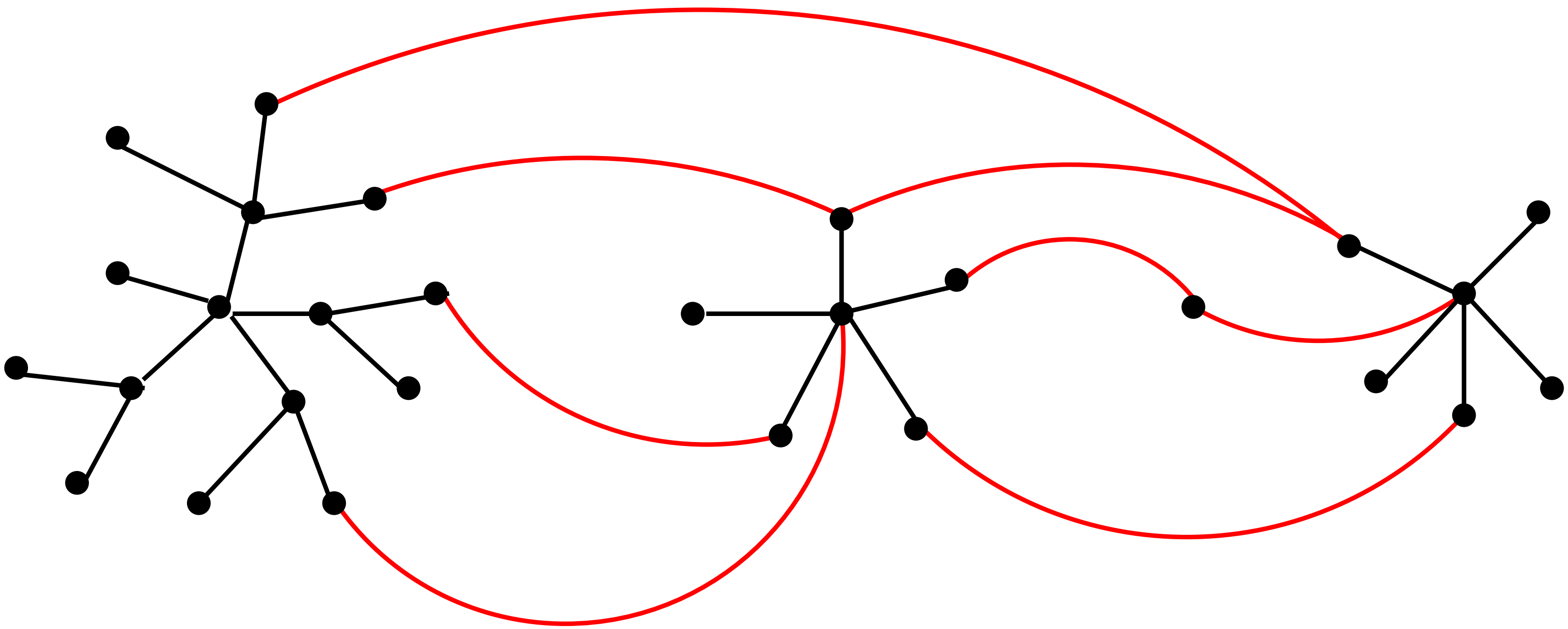

  - total number of red edges $\leq n^{1/k}(\sum_i |C_i|) = n^{1+1/k}$.

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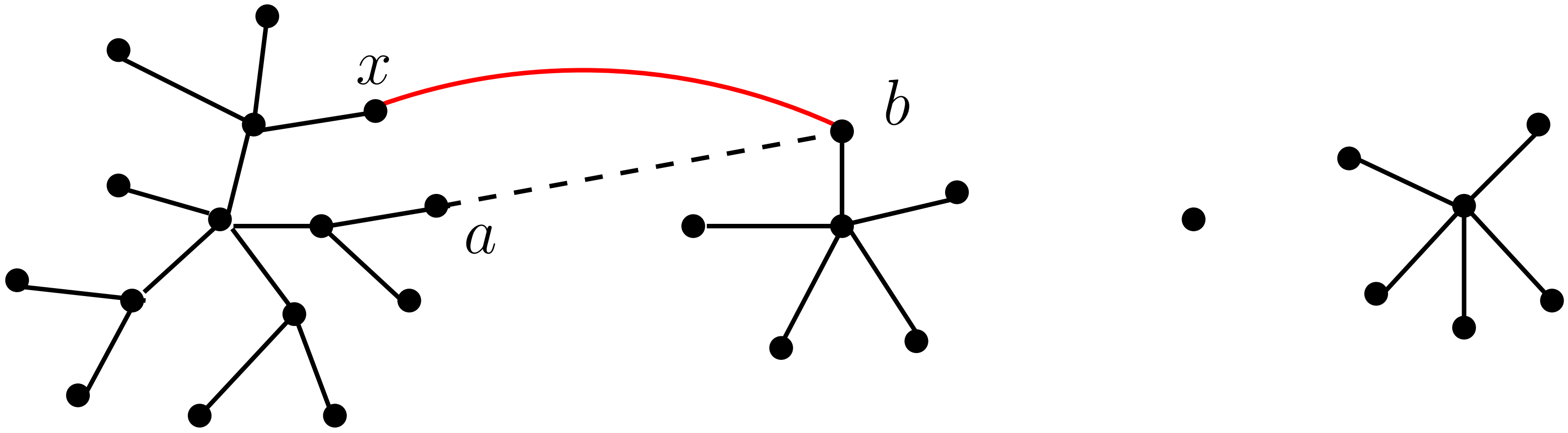

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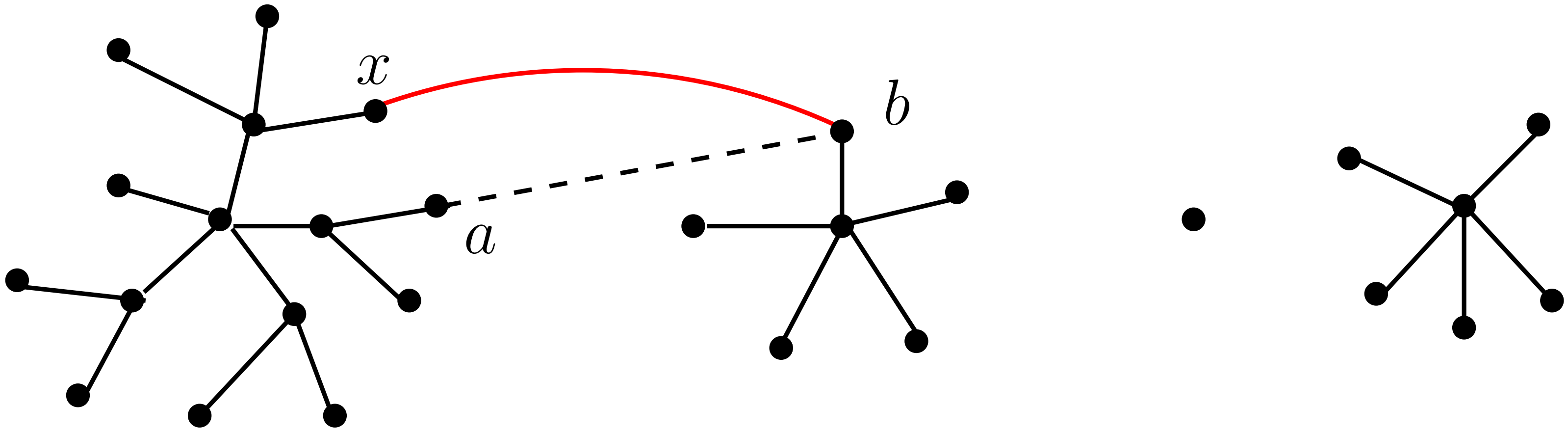

  - diameter of each cluster is $\leq 2k-2$, so $\delta_H(a, b) \leq 2k-1$.

Spanners with additive stretch

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- Such an H is called a *purely additive* spanner.

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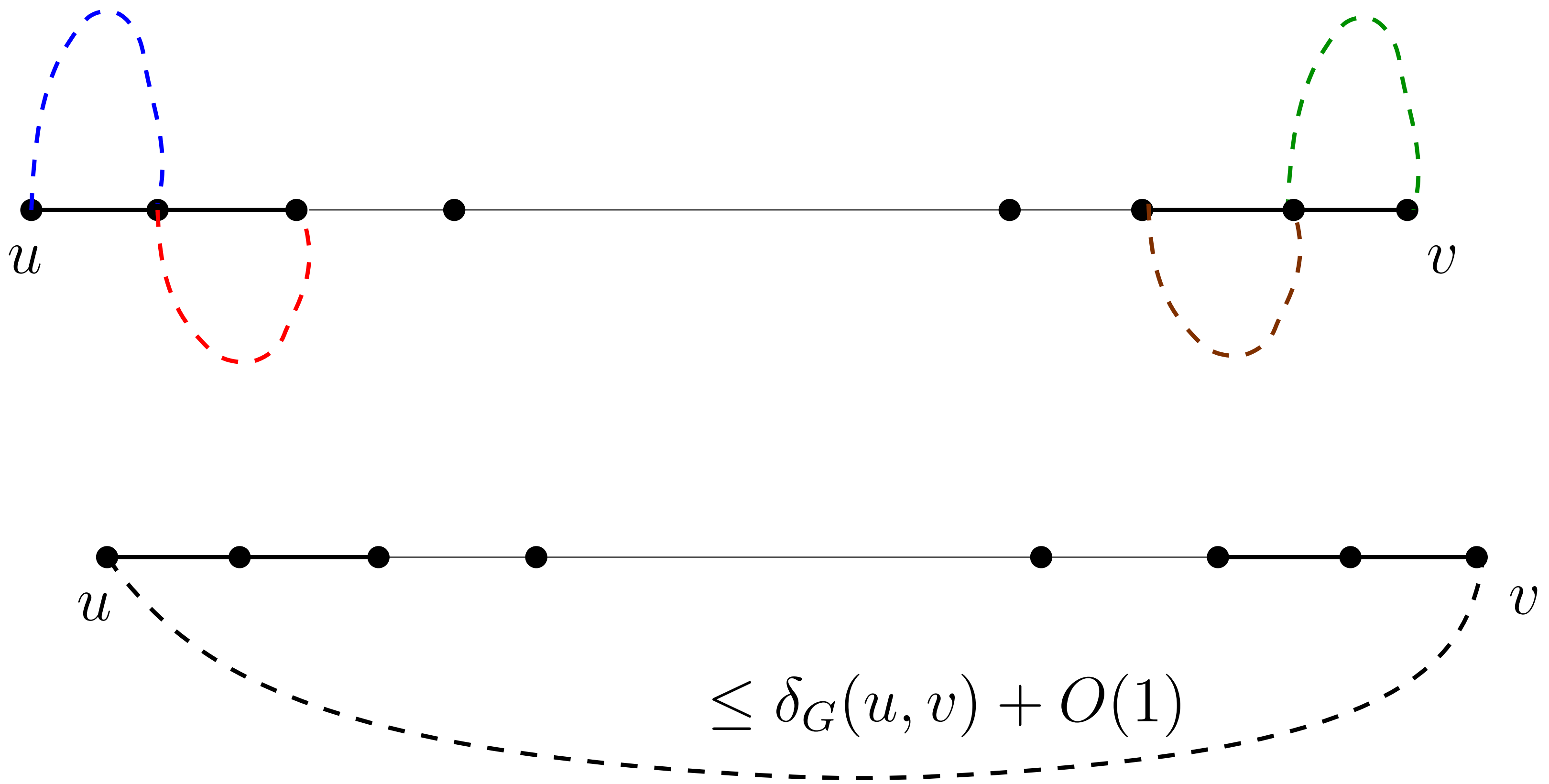

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  - no other results for purely additive spanners are known.
    - size $\tilde{O}(n^{1+\delta})$ vs additive stretch $\tilde{O}(n^{\frac{1-3\delta}{2}})$ [C13]

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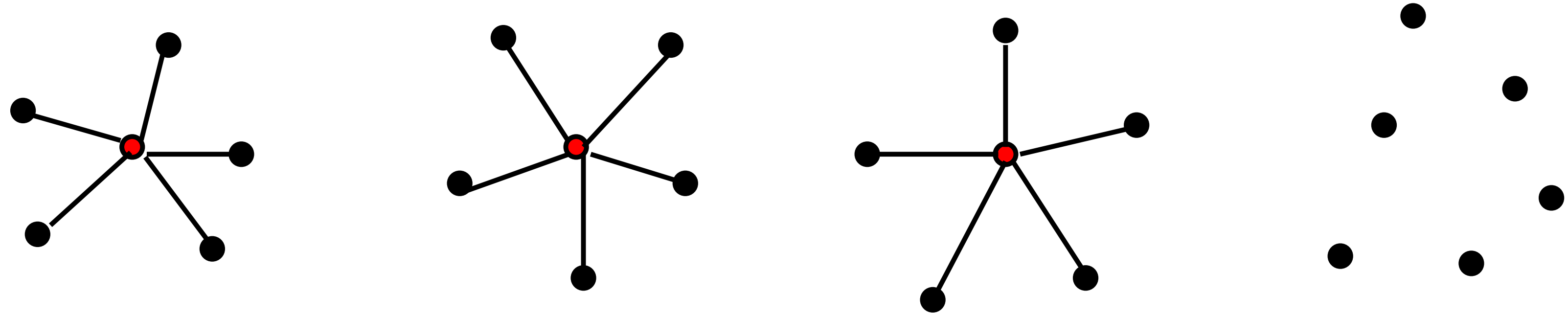

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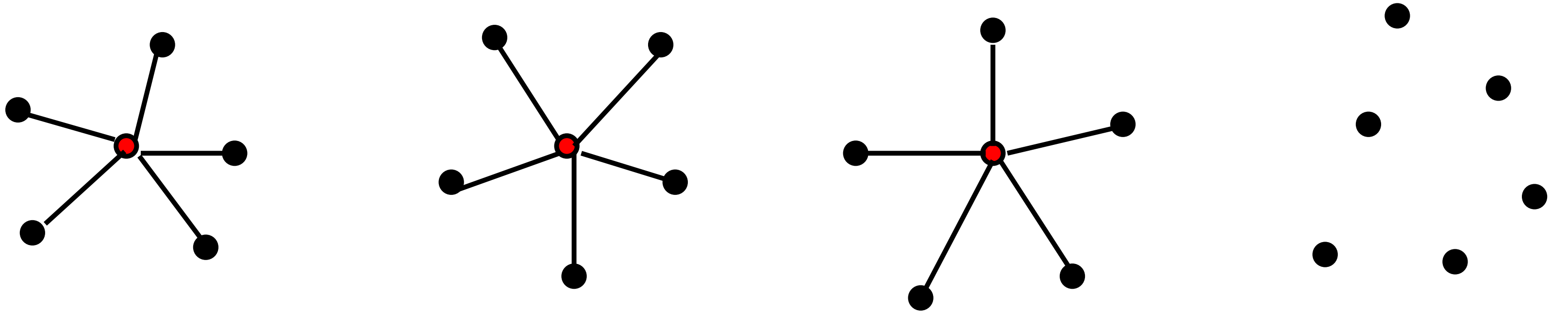

- there will be at most n/h special (high degree) vertices

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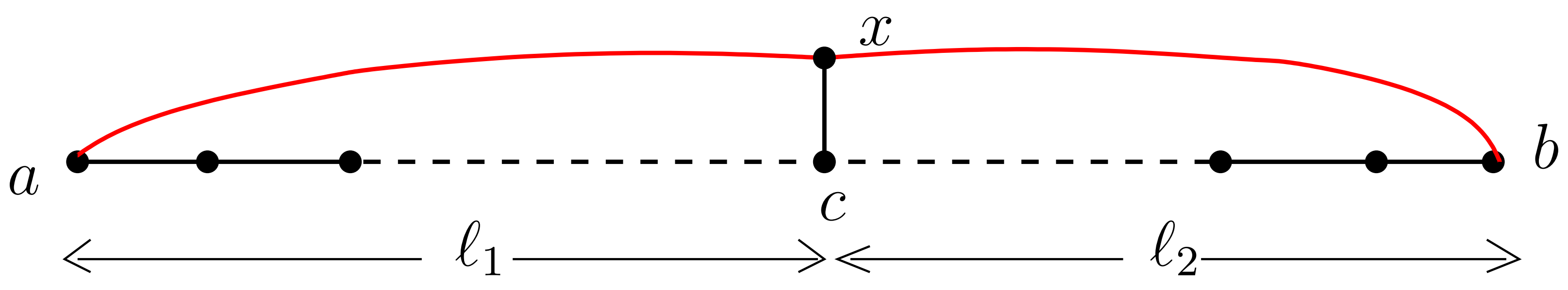

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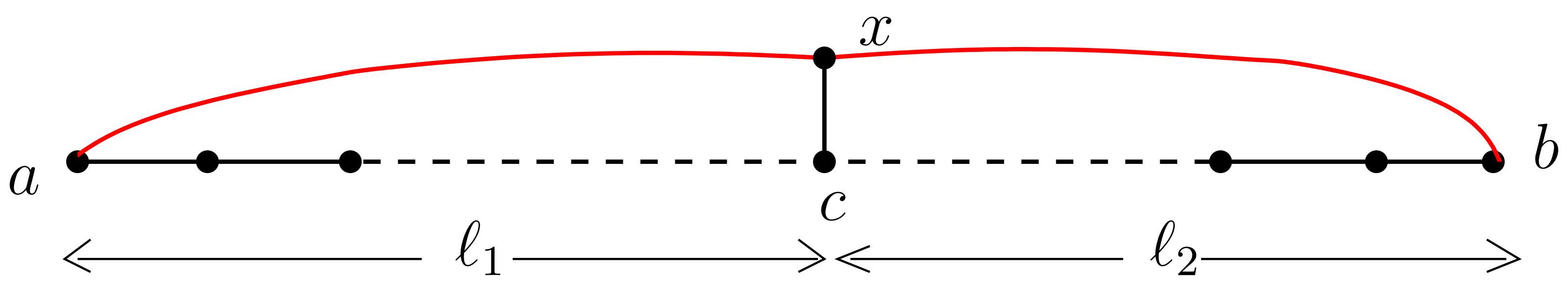

- $\delta_H(a, b) \leq \delta_H(a, x) + \delta_H(x, b) \leq \ell_1 + 1 + \ell_2 + 1$.

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- As before, we have low degree vertices (those with degree $\leq h \approx (n|S|)^{1/4}$) and high degree vertices.

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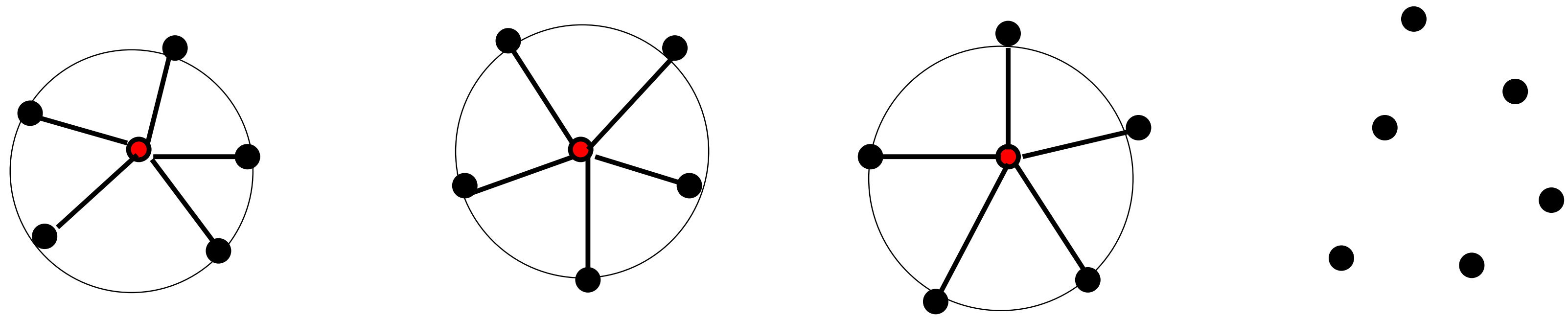

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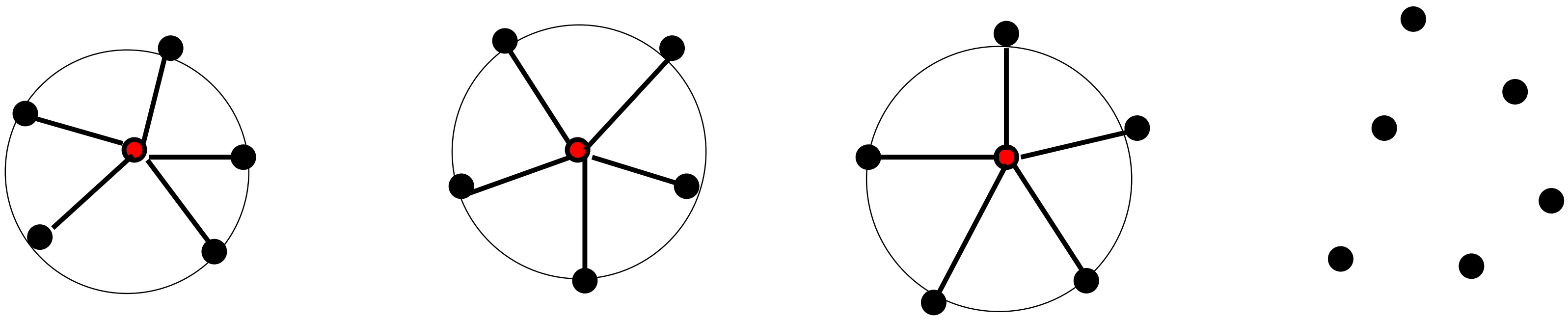

- H is initialized to the forest defined by clusters along with all edges incident on unclustered vertices.

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- Sample special high degree vertices w.p. $\approx 3/\tau$ to get very special vertices.

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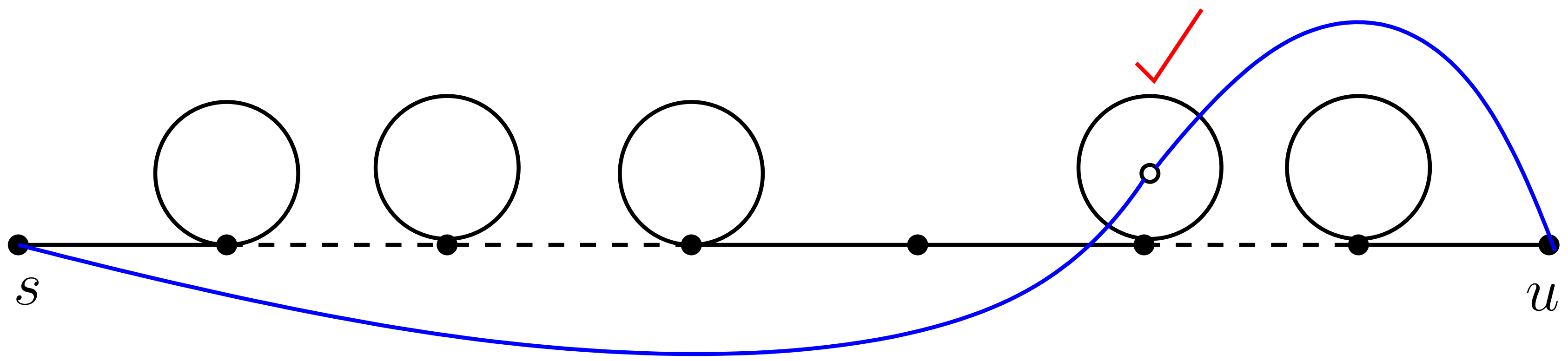

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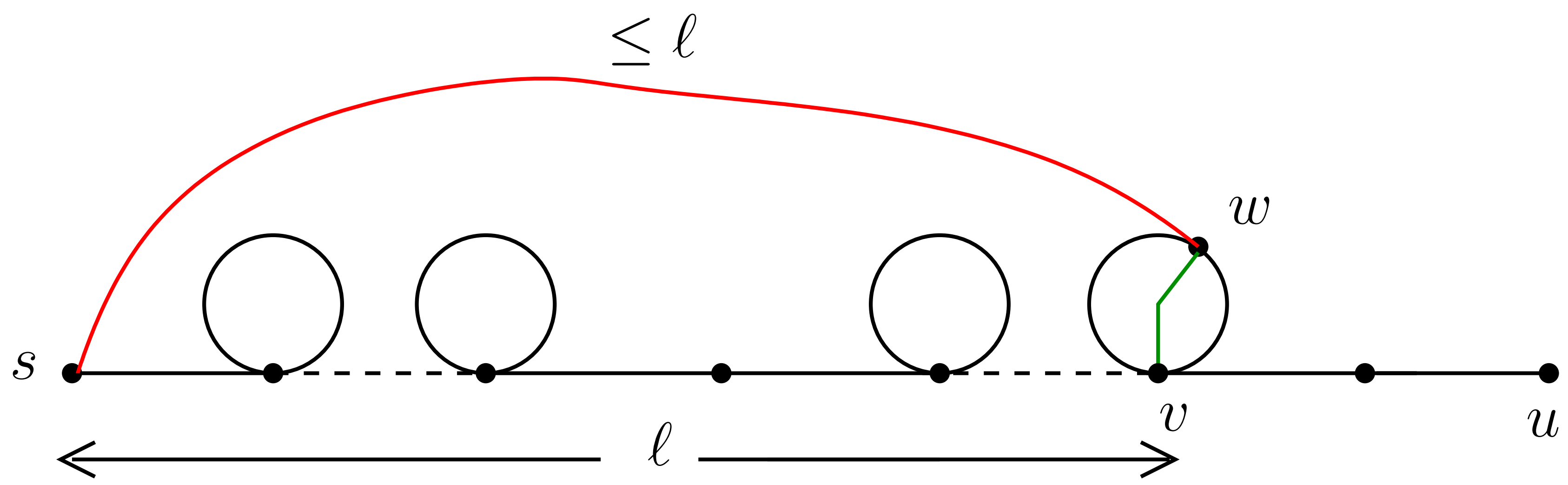

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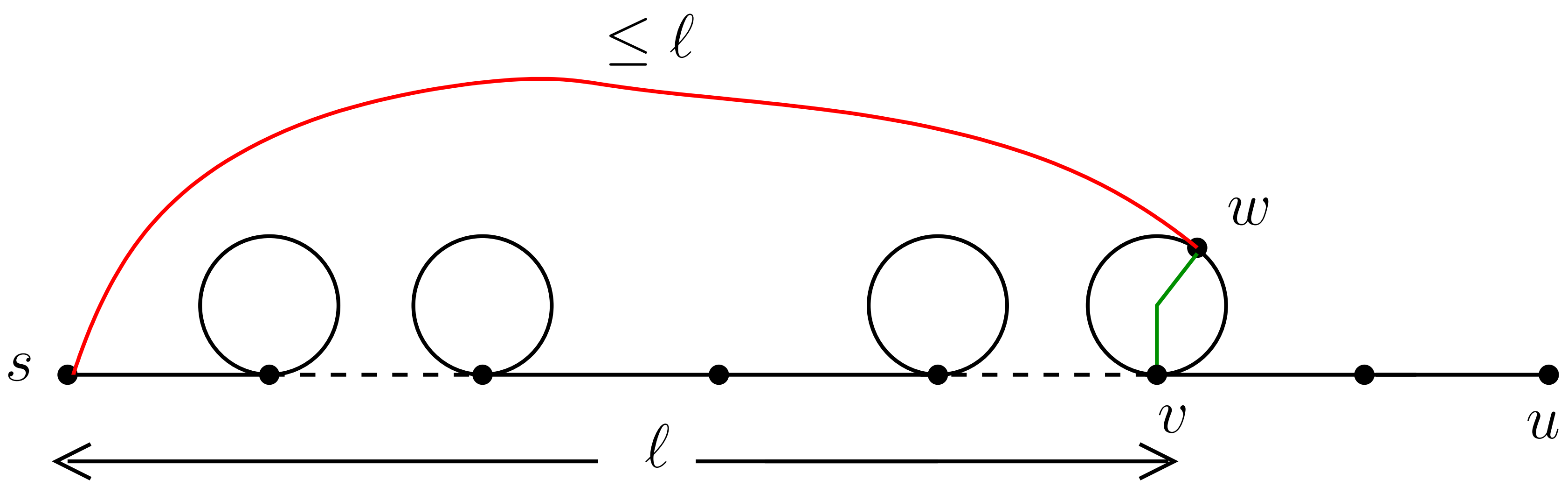

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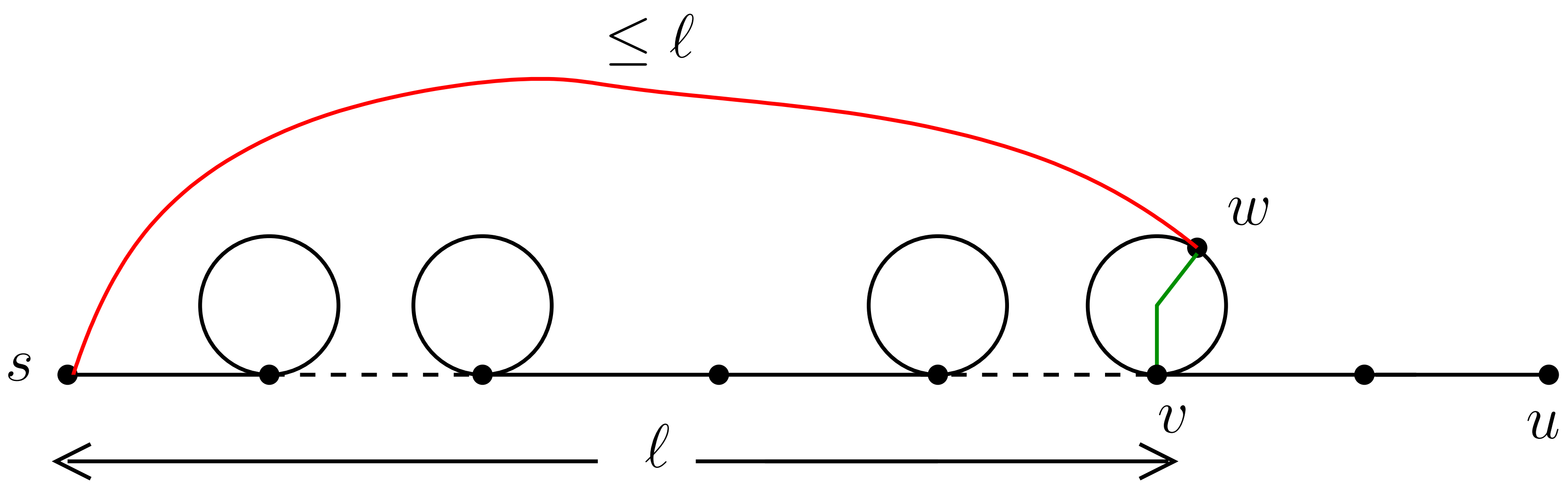

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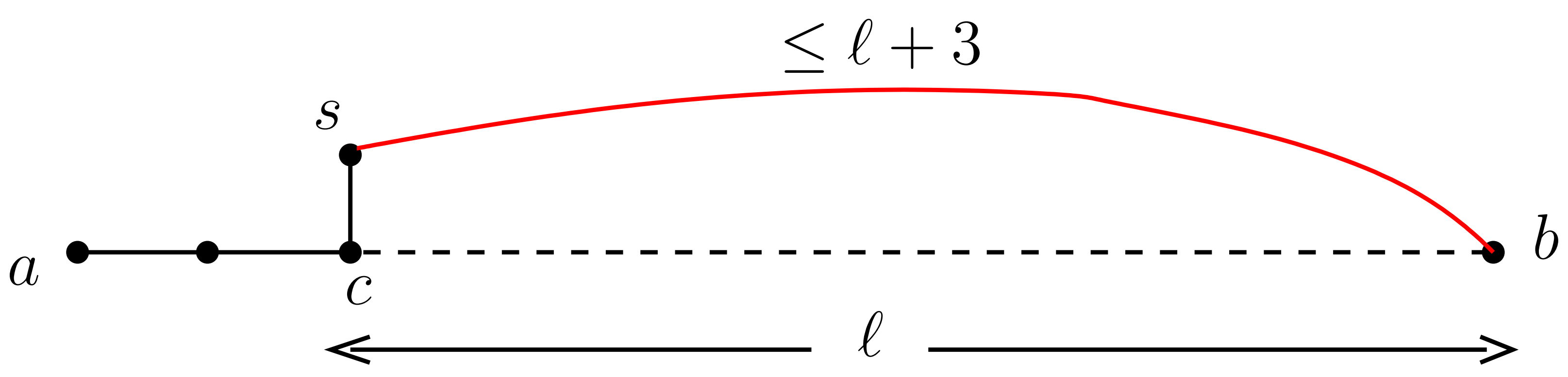

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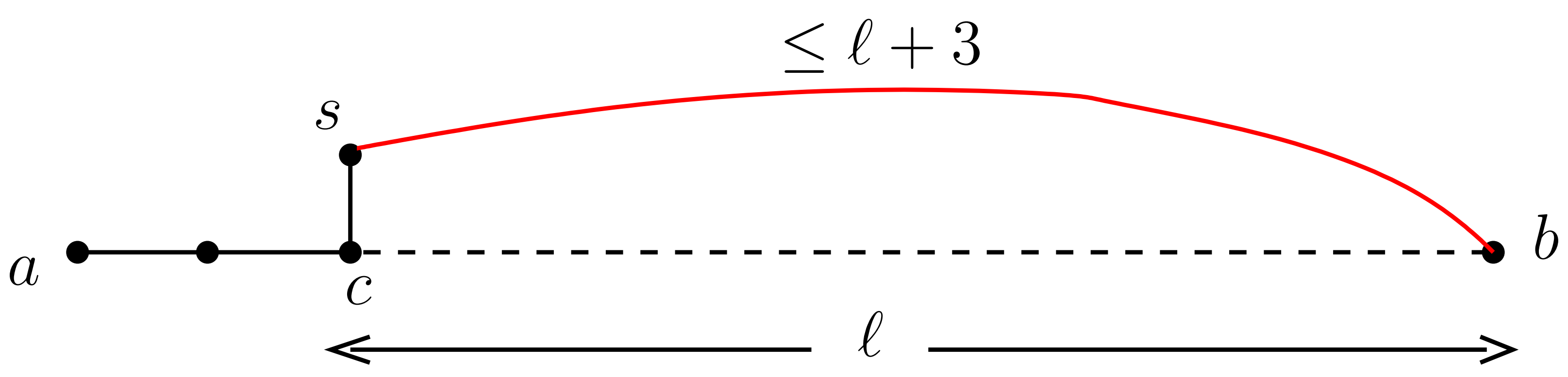

  - so $\delta_H(a,b) \leq \delta_H(a,s) + \delta_H(s,b) \leq \delta_G(a,c) + 1 + \ell + 3$
    $= \delta_G(a,b) + 4$.

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- As before, some high degree vertices are unclustered and the rest are clustered around special vertices.

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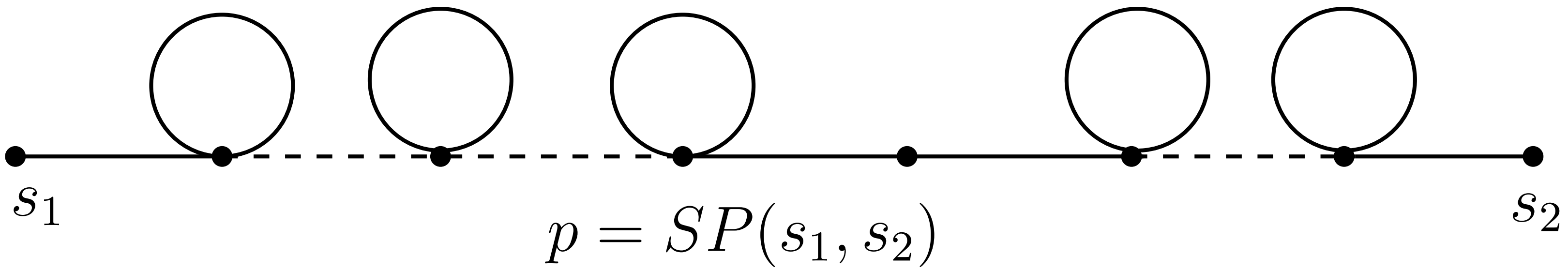

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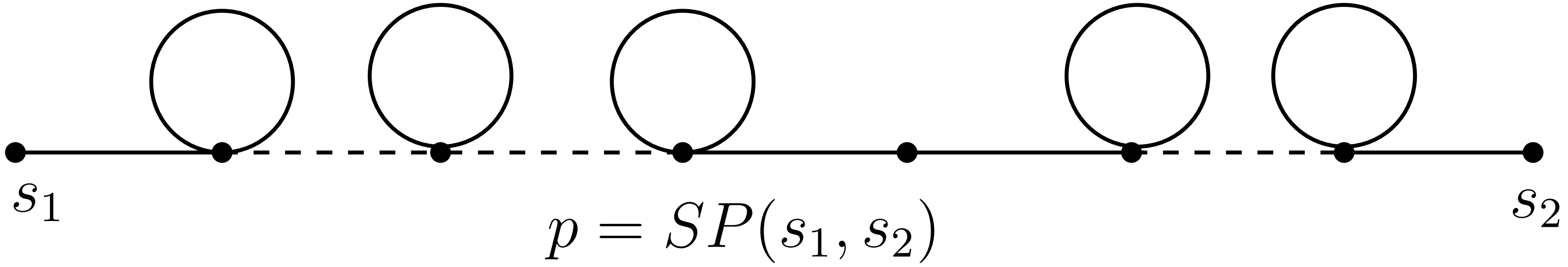

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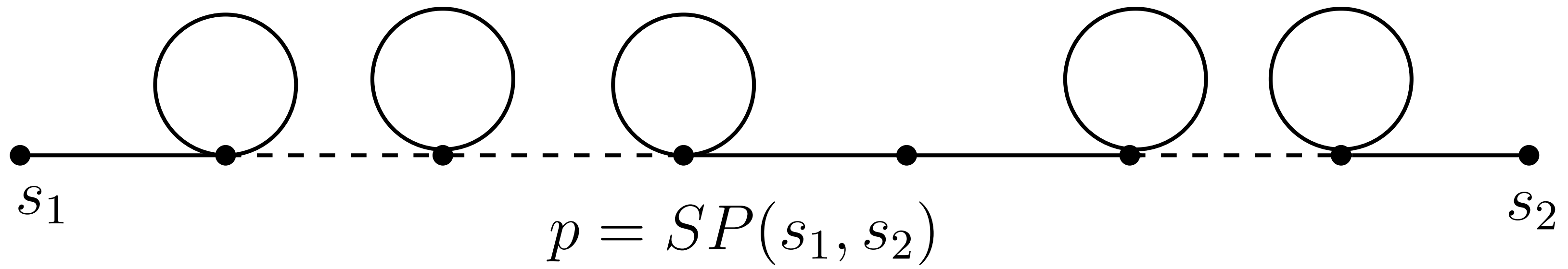

- p is one of two types:
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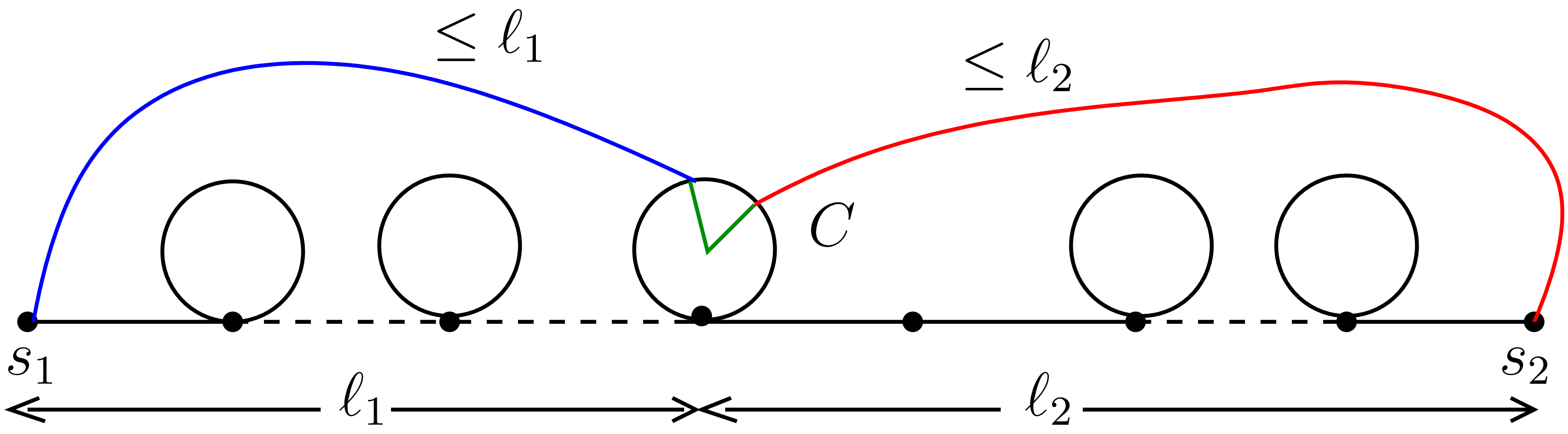

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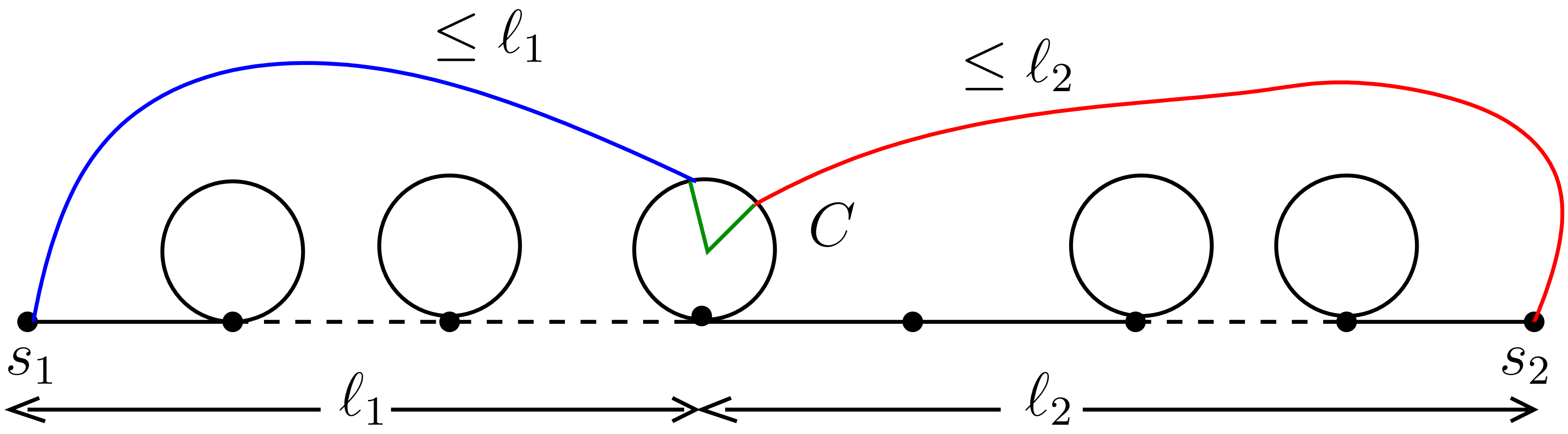

- so we already have $\delta_H(s_1, s_2) \leq \delta_G(s_1, s_2) + 2$ in this case

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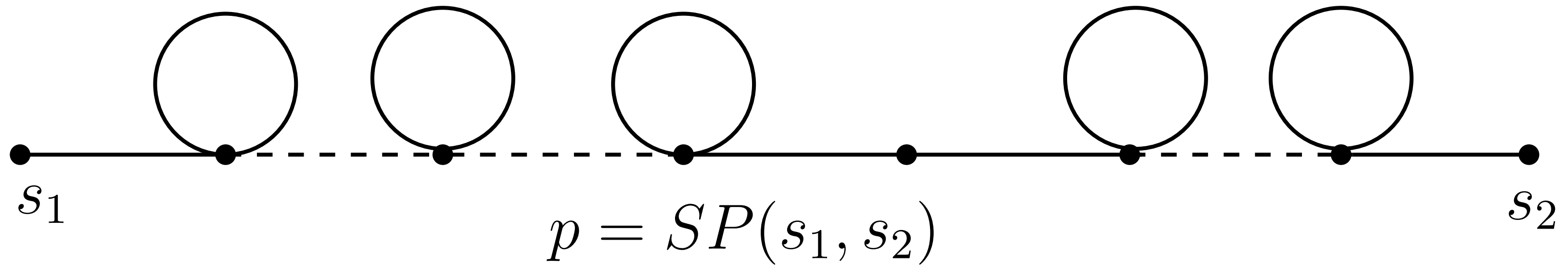

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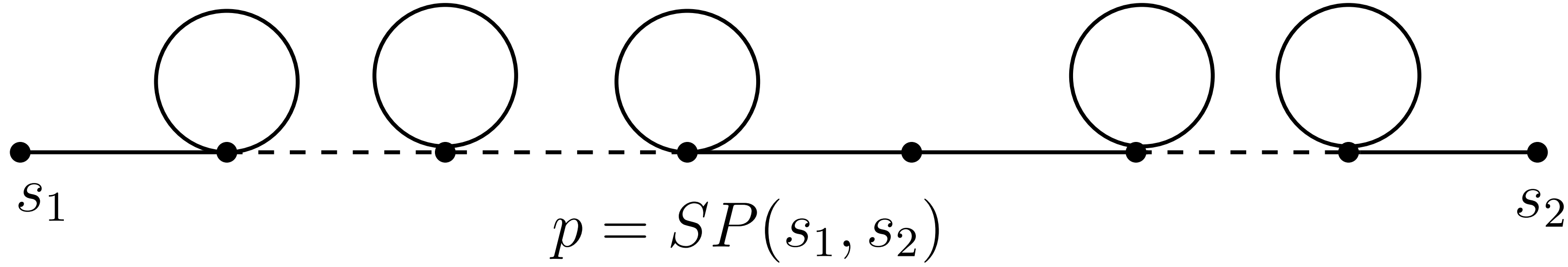

- So $\delta_H(s_1, C)$ or $\delta_H(s_2, C)$ has strictly decreased and we *charge* such (s, C) pairs to pay for the edges added.

The size of H

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- The number of edges incident on unclustered vertices is $O(nh)$, which is again $O(n\sqrt{|S|})$.

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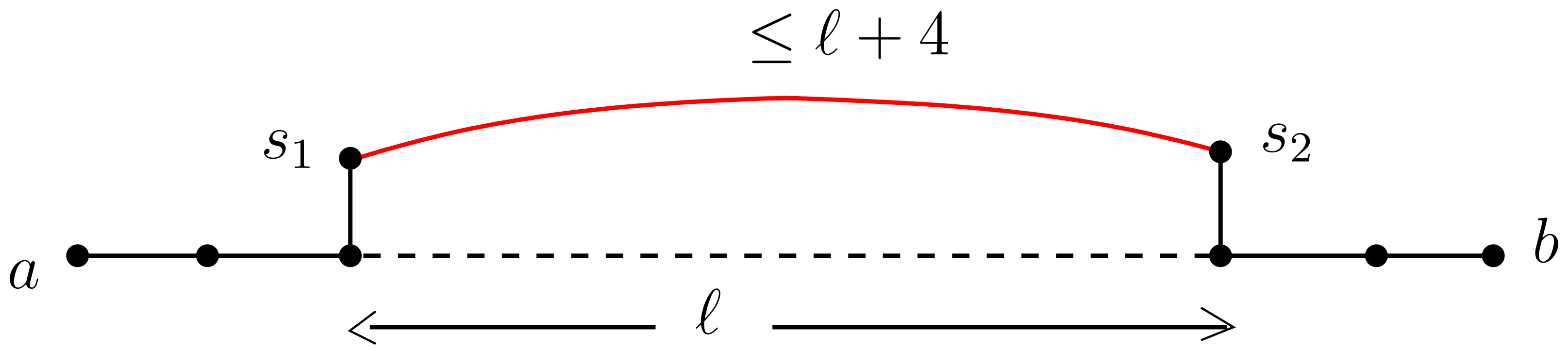

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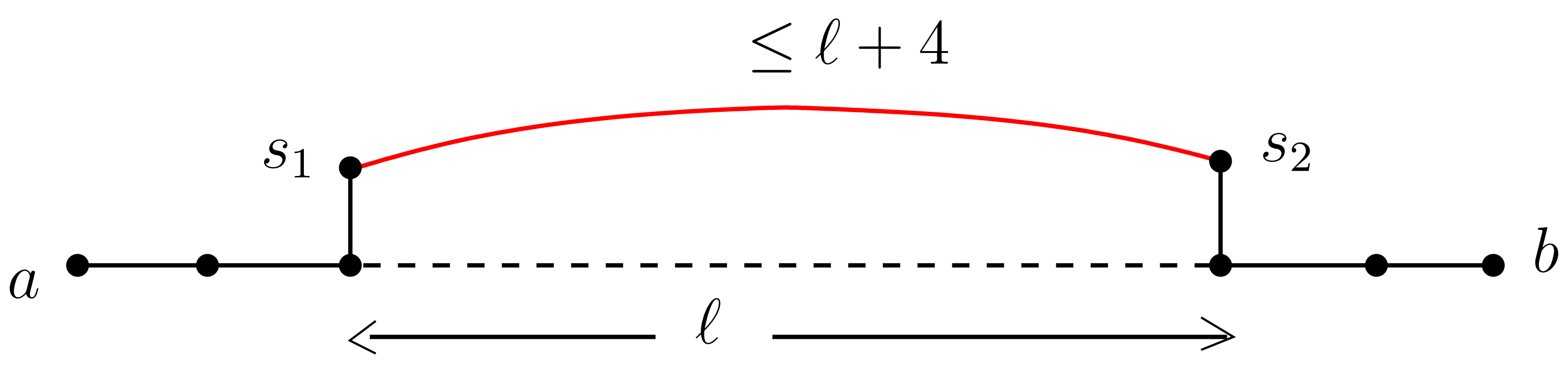

  - so $\delta_H(a,b) \leq \delta_H(a,s_1) + \delta_H(s_1,s_2) + \delta_H(s_2,b)$
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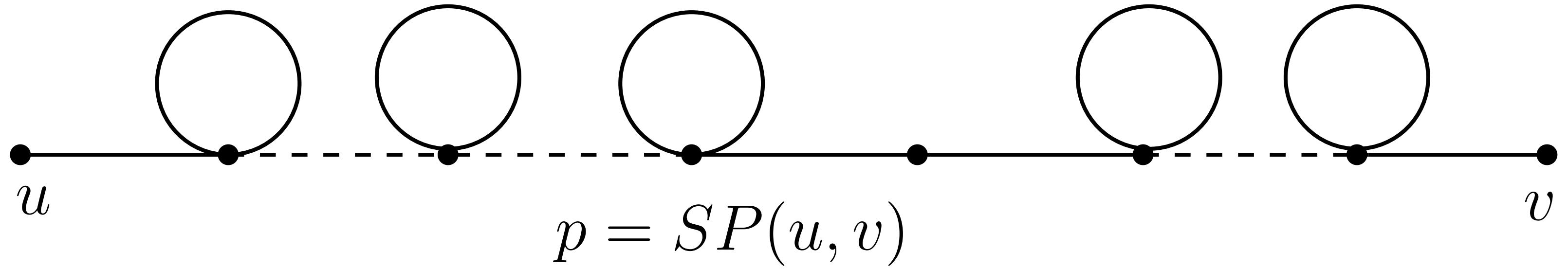

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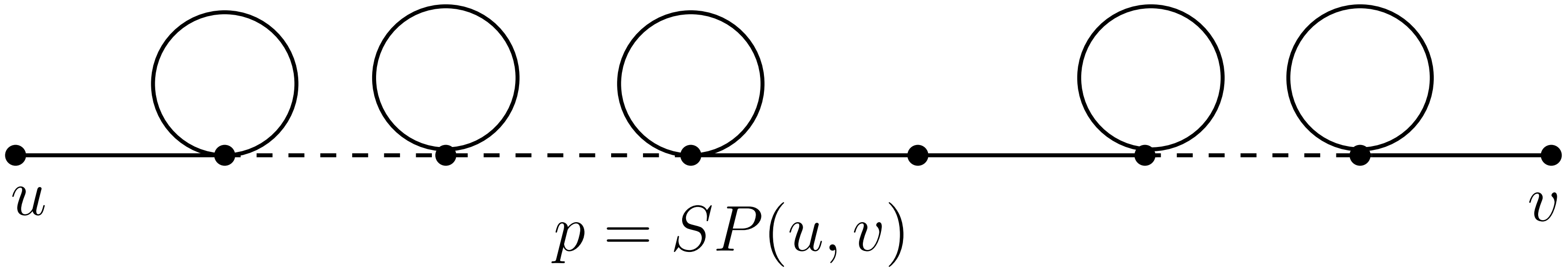

- Sample each special vertex with probability $\approx 3/\tau$ $(\tau = n/|\mathcal{P}|^{2/3})$ to get very special vertices

A $\mathcal{P}$-spanner with additive stretch 2

- Add to H a BFS tree rooted at each very special vertex.

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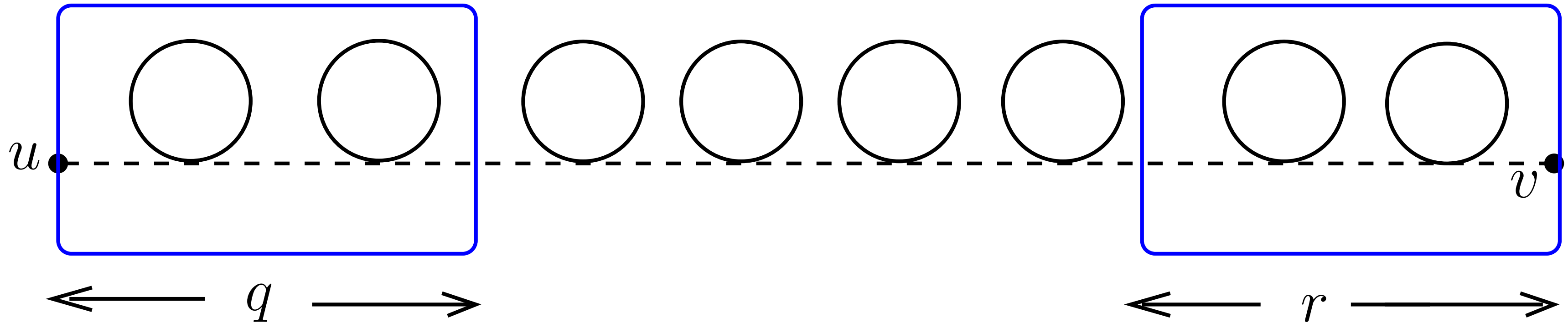

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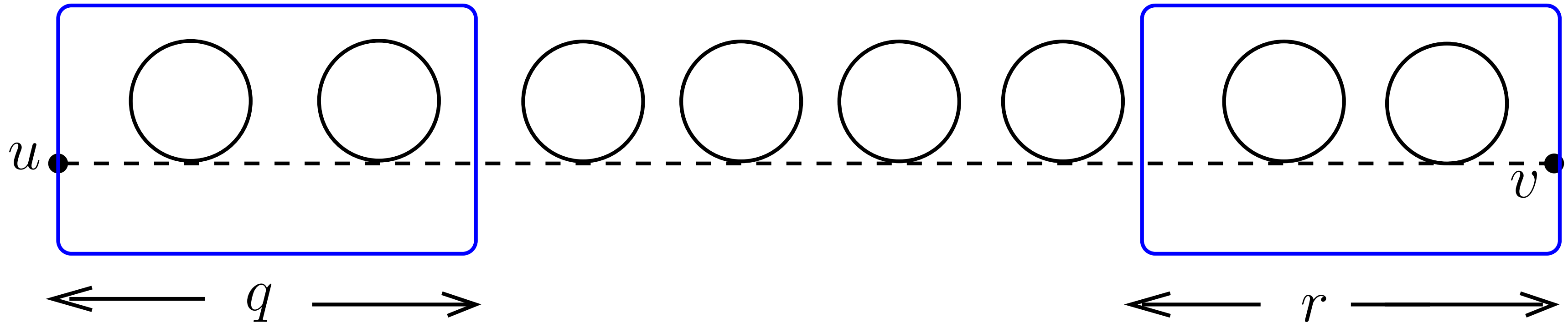

- Add to H all missing edges in the *prefix* q and *suffix* r of each such path.

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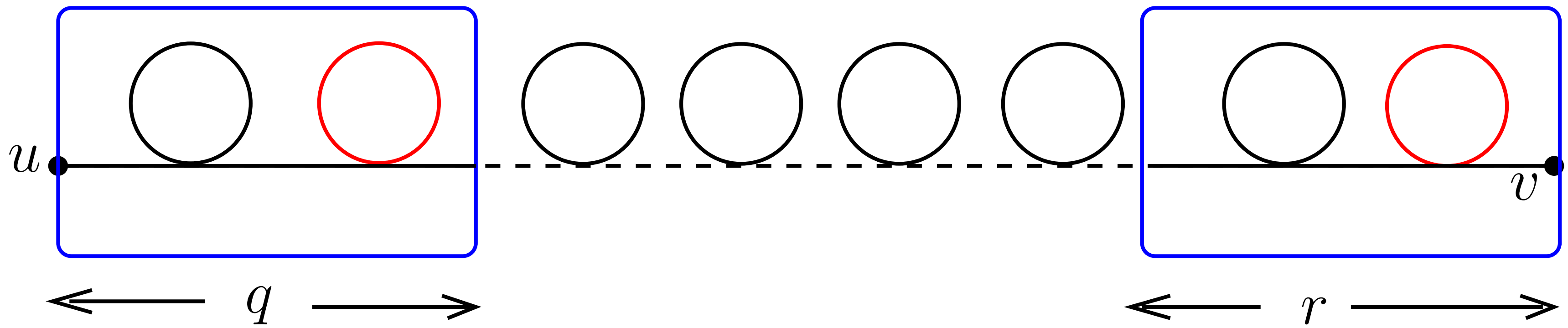

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- We buy at most one $SP(x, y)$ for $x \in C_1$ and $y \in C_2$, where C_1 and C_2 are clusters centered at selected special vertices.

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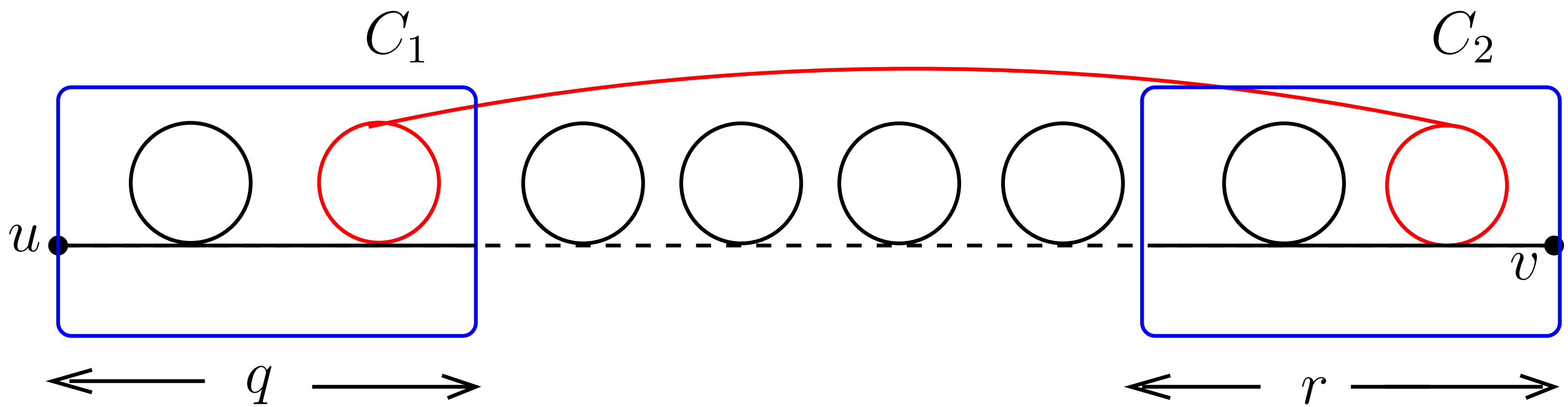

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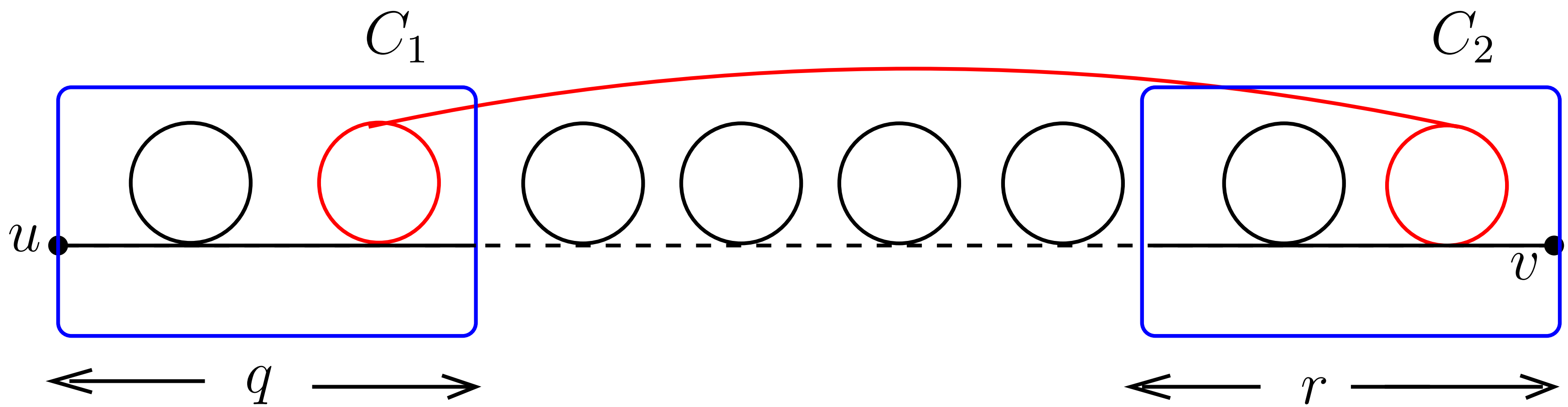

- This ensures $\delta_H(u, v) \leq \delta_G(u, v) + 4$ in this case.

A $\mathcal{P}$-spanner with additive stretch 6

- Run the clustering step with parameter $h \approx |\mathcal{P}|^{1/4}$ and we get the starting subgraph H.

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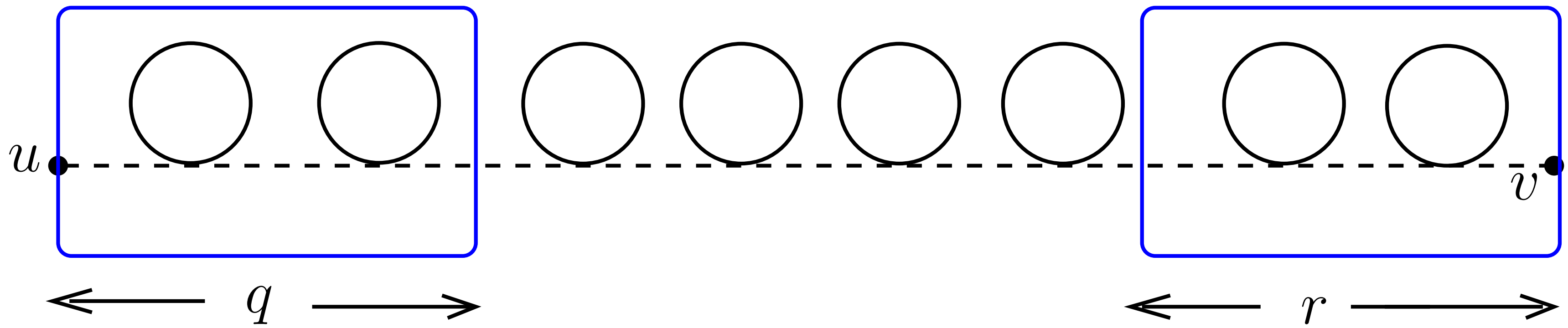

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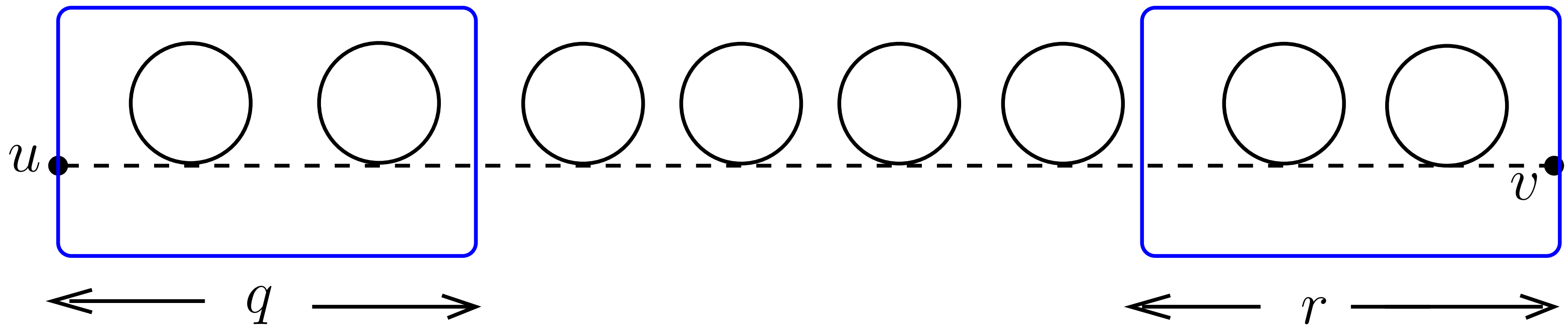

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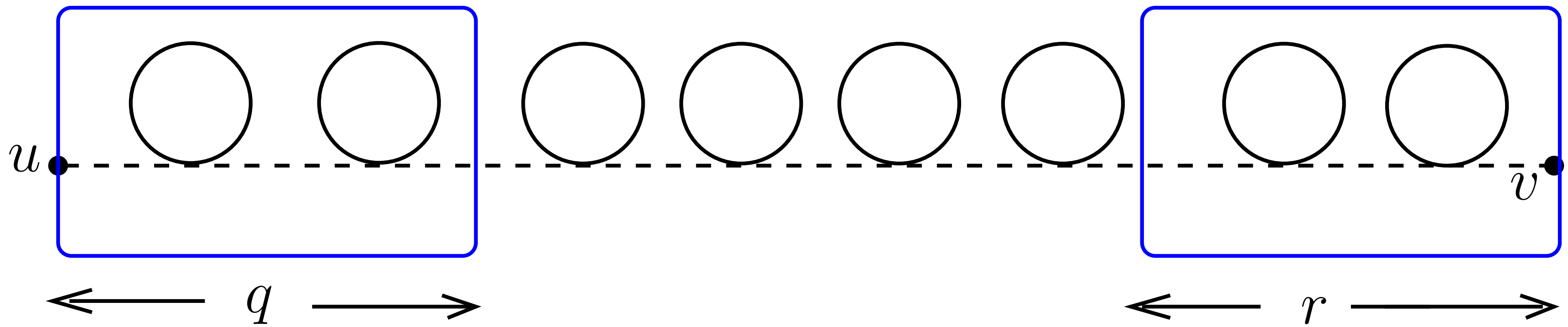

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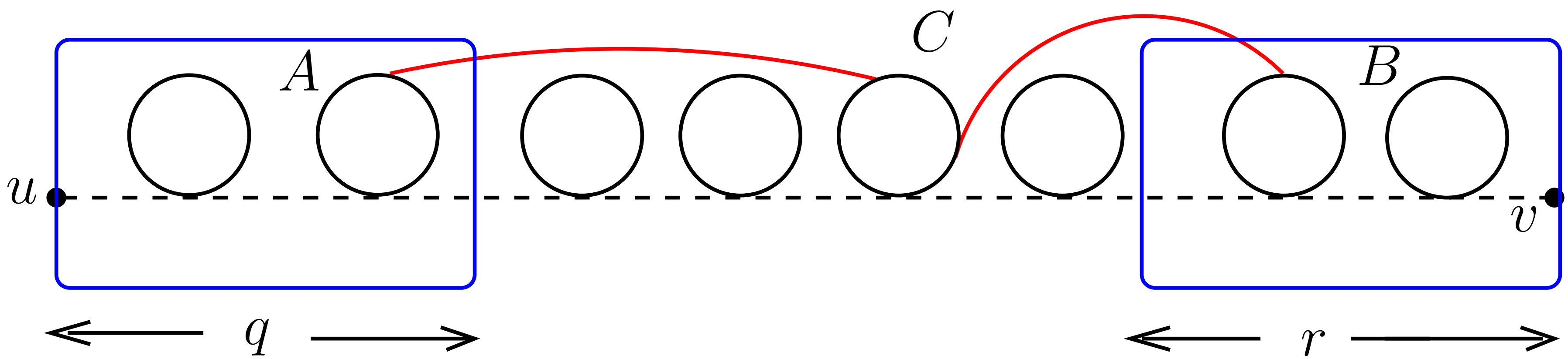

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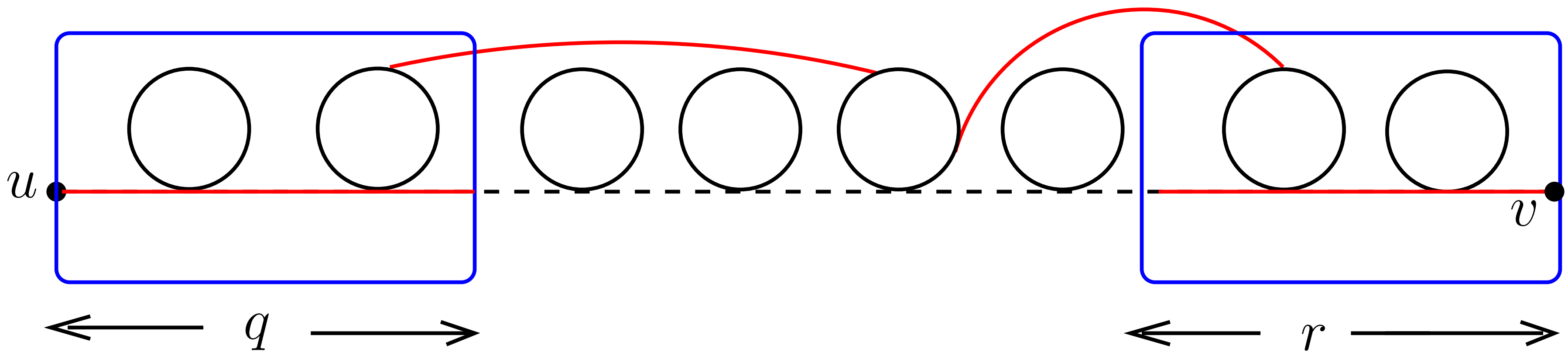

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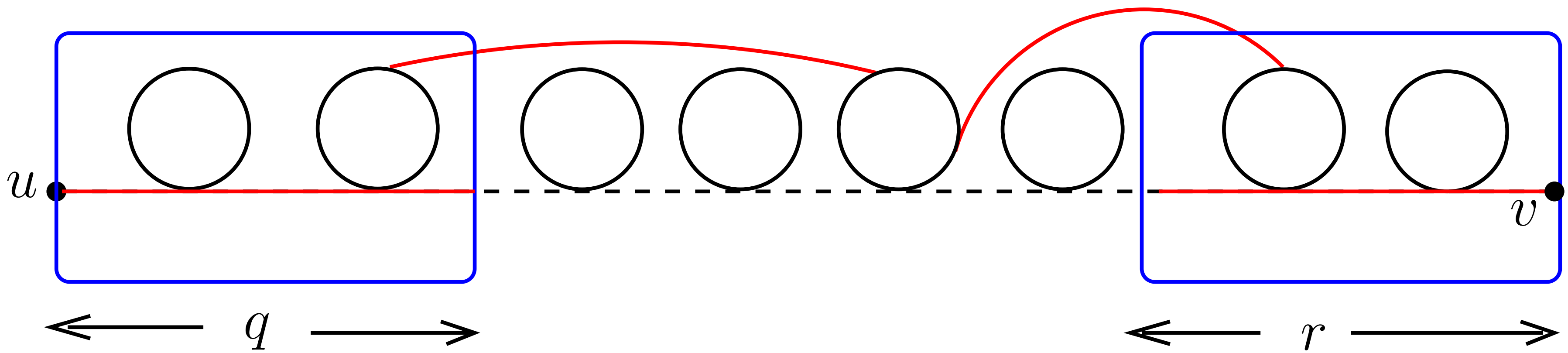

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  - this would imply an all-pairs of size $o(n^{4/3})$ and $O(1)$ additive stretch

Thank you!