On the Dynamics of the Heisenberg Anisotropic Spin Chain

M. Lakshmanan

Centre for Nonlinear Dynamics
Bharathidasan University
Tiruchirappalli – 620024
India

NMI Workshop on 'Discrete Integrable Systems'
Indian Institute of Science, Bangalore

10 June 2014

Objectives

To give a brief overview of the Nonlinear Dynamics underlying the Heisenberg anisotropic spin chain on both integrability and nonintegrability aspects

Plan

Motivation

"The synchronization of STNOs raises complex problems that are new in spintronics and is related to the general field of dynamics of nonlinear systems"

J. Grollier, V. Cros and A. Fert (2006)

L.L.G - Equation

- Magnetic moment $\longrightarrow$ Spin: $\vec{S}$
- Spin in an external magnetic field
  $\Longrightarrow$ Precession

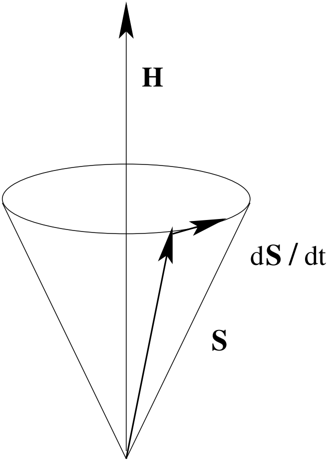
$$\frac{d\vec{S}}{dt} = -\vec{S} \times \vec{H},$$

$$\vec{S} = (S^x, S^y, S^z),\ \ \vec{S}^2 = 1,\ \ \vec{H} = (h^x, h^y, h^z)$$

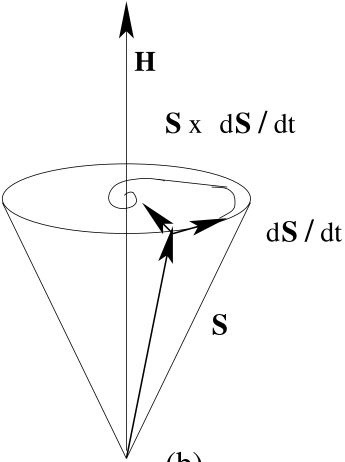
- Spin in the presence of damping

$$\frac{d\vec{S}}{dt} = -\vec{S} \times \vec{H} + \lambda\vec{S} \times \frac{d\vec{S}}{dt} = -\vec{S} \times \left(\vec{H} - \lambda\frac{d\vec{S}}{dt}\right).$$

L.L.G - Equation



(a)



(b)

L.L.G - Equation

- **Lattice of spins:**

$$\begin{aligned}\frac{d\vec{S}_i}{dt} &= -\vec{S}_i \times \left(\vec{H} + AS_i^x\vec{i} + BS_i^y\vec{j} + CS_i^z\vec{k} + \vec{S}_{i+1} + \vec{S}_{i-1} + \cdots\right) \\ &\quad + \text{damping}, \qquad\qquad i = 1, 2, \cdots, N \\ &= -\vec{S}_i \times \left(\vec{H}_{\text{eff}} - \lambda\frac{d\vec{S}_i}{dt}\right),\end{aligned}$$

$$\vec{S}_i = (S_i^x, S_i^y, S_i^z),\ (S_i^x)^2 + (S_i^y)^2 + (S_i^z)^2 = 1$$

$$\vec{H}_{\text{eff}} = \vec{H}_{\text{exchange}} + \vec{H}_{\text{anisotropy}} + \vec{H}_{\text{demag}} + \vec{H}_{\text{applied}} \cdots$$

- Landau & Lifshitz (1935)
- Gilbert (1955)

Continuum limit

$$\frac{\partial \vec{S}(\vec{r},t)}{\partial t} = -\Big[\vec{S} \times (\nabla^2 \vec{S} + AS^x \vec{i} + BS^y \vec{j} + CS^z \vec{k} + \vec{H}_{\rm dip} + \vec{H}_{\rm demag} + \vec{H}) - \lambda \vec{S} \times \frac{\partial \vec{S}}{\partial t}\Big]$$

$$\vec{S} = (S^x, S^y, S^z), \ (S^x)^2 + (S^y)^2 + (S^z)^2 = 1$$

or

$$\frac{\partial \vec{S}}{\partial t} = -\left[\vec{S} \times \vec{H}_{\rm eff} - \lambda \vec{S} \times \frac{\partial \vec{S}}{\partial t}\right] = -\vec{S} \times \left(\vec{H}_{\rm eff} - \lambda \frac{\partial \vec{S}}{\partial t}\right),$$

- Spin waves
- Elliptic function waves
- Solitons
- Vortices

$$\vec{H}_{\rm eff} = \vec{H}_{\rm exchange} + \vec{H}_{\rm anisotropy} + \vec{H}_{\rm demag} + \vec{H}_{\rm applied} \cdots$$

(M. Lakshmanan, *Phys. Lett. A*, **61** 53-54 (1977); *Phil. Trans. R. Soc. A*, **369**, 1280-1300 (2011))

Continuum limit

(i) Isotropic spin system : (1+1) dimensions (Lakshmanan 1977; Takhtajan 1977)

$$\vec{S}_t = \vec{S} \times \vec{S}_{xx}, \quad \vec{S} = (S^x, S^y, S^z), \quad \vec{S}^2 = 1$$

$$\begin{aligned} L &= i\lambda S \\ B &= \lambda S S_x + 2i\lambda^2 S, \quad S = \begin{pmatrix} S^z & S^- \\ S^+ & -S^z \end{pmatrix} \equiv \vec{S} \cdot \vec{\sigma} \end{aligned}$$

so that

$$L_t = [L, B] \quad \Longleftrightarrow \quad S_t = [S, S_{xx}]$$

- Geometrical / Gauge equivalence to nonlinear Schrödinger equation :

$$iq_t + q_{xx} + 2|q|^2 q = 0,$$

$\Longrightarrow$

Continuum limit

$$q = \frac{1}{2}\kappa \; exp\left[i\int_x^{+\infty} \tau \; dx'\right],$$
$$\kappa^2 = \vec{S}_x \cdot \vec{S}_x, \quad \kappa^2\tau = \vec{S}\cdot(\vec{S}_x \times \vec{S}_{xx})$$

(ii) Single site anisotropy (Borovick 1980; Nakamura & Sasada 1982)

$$\vec{S}_t = \vec{S} \times \vec{S}_{xx} + A\vec{S} \times \vec{n}, \quad \vec{n} = (0,0,1)$$

(iii) Biaxial anisotropy (Sklyanin 1979)

$$\vec{S}_t = \vec{S} \times \vec{S}_{xx} + \vec{S} \times J\vec{S}, \quad J\vec{S} = \sum J_\alpha S_\alpha \vec{n}_\alpha$$

Continuum limit

(iv) (2+1) dimensional Ishimori spin system (Ishimori 1984)

$$\vec{S}_t = \vec{S} \times (\vec{S}_{xx} + \alpha^2 \vec{S}_{yy} + u_x \vec{S}_y + u_y \vec{S}_x,$$

$$(u)xx - \alpha^2 u_{yy} = -2\alpha^2 \vec{S}.(\vec{S}_x \times \vec{S}_y), \alpha^2 = 1$$

$\Rightarrow$ Geometrical & Gauge equivalent to (2+1) dimensional Davey-Stewartson equation.

(v) Isotropic spin systems with damping (Lakshmanan & Daniel 1982)

$$\vec{S}_t = \vec{S} \times \vec{S}_{xx} + \lambda[\vec{S}_{xx} - (\vec{S}.\vec{S}_{xx})\vec{S}].$$

$\Rightarrow$ Geometrically equivalent to damped NLS equation :

$$iq_t + q_{xx} + 2|q|^2 q = i\lambda[q_{xx} - 2q \int_{-\infty}^{x} (qq^*_{x'} - q^* q_{x'})dx'],$$

Hamiltonian Formulation

(vi) Hamiltonian formulation (Lakshmanan, Ruijgrok & Thompson 1976)

- Undamped spin chain (cubic lattice) Hamiltonian

$$H = -\sum_{\{n\}}(AS_n^x S_{n+1}^x + BS_n^y S_{n+1}^y + CS_n^z S_{n+1}^z) - D\sum_n (S_n^z)^2 - \vec{\mathcal{H}}\cdot\sum_n \vec{S}_n$$

- Quantum Heisenberg equation of motion :

$$\frac{d\vec{S}_i}{dt} = [\vec{S}_i, H]$$

and then take the $s \to \infty$ limit to obtain the semiclassical/classical dynamics.

- Or start with the spin Poisson brackets

$$\{S_i^\alpha, S_j^\beta\}_{PB} = \delta_{ij}\epsilon_{\alpha\beta\gamma}S_j^\gamma, \quad \alpha,\ \beta,\ \gamma = 1, 2, 3,$$

so that

Hamiltonian Formulation

$\Longrightarrow$

$$\{\mathcal{A},\mathcal{B}\}_{PB} = \sum_{\alpha,\beta,\gamma}\sum_{i=1}^{N}\epsilon_{\alpha\beta\gamma}\frac{\partial\mathcal{A}}{\partial S_i^\alpha}\frac{\partial\mathcal{B}}{\partial S_i^\beta}S_i^\gamma,$$

$\Longrightarrow$

- Then the equation of motion

$$\begin{aligned}\frac{d\vec{S}_i}{dt} &= \{\vec{S}_i, H\}\\ &= \vec{S}_i\times[A(S_{i+1}^x+S_{i-1}^x)\vec{i}+B(S_{i+1}^y+S_{i-1}^y)\vec{j}\\ &+ C(S_{i+1}^z+S_{i-1}^z)\vec{k}+2DS_i^z\vec{k}]+\vec{S}_i\times\vec{\mathcal{H}},\quad i=1,2,...,N,\end{aligned}$$

Hamiltonian Formulation

- In the continuum case,

$$\begin{aligned}\vec{S}_{\vec{i}}(t) &= \vec{S}(\vec{r},t), \qquad \vec{r}=(x,y,z),\\ \vec{S}_{\vec{i}\pm 1} &= \vec{S}(\vec{r},t) \pm \vec{a}.\vec{\nabla}\vec{S} + \frac{a^2}{2}\nabla^2\vec{S} + higher\ orders,\end{aligned}$$

  and then take the $a \to 0$ limit.

- Or define the Poisson brackets :

$$\left\{S^\alpha(\vec{r},t), S^\beta(\vec{r}^{'},t^{'})\right\} = \in_{\alpha\beta\gamma} S^\gamma \delta(\vec{r}-\vec{r}^{'}, t-t^{'}),$$

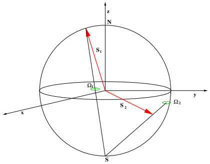
$\Rightarrow$

$$\frac{\partial \vec{S}(\vec{r},t)}{\partial t} = \frac{\delta \mathcal{H}}{\delta \vec{S}}, \quad \mathcal{H} = \int_{-\infty}^{\infty} H dx$$

(vi) Stereographic projection

Defining $\omega \in \mathbb{C}$,

$$\omega = \frac{S^x + iS^y}{(1 + S^z)}$$



$$S^x = \frac{\omega + \omega^*}{1 + \omega\omega^*}, \qquad S^y = \frac{1}{i}\frac{(\omega - \omega^*)}{(1 + \omega\omega^*)}, \qquad S^z = \frac{1 - \omega\omega^*}{1 + \omega\omega^*}$$

$\Longrightarrow$ (M. Lakshmanan & K. Nakamura, Phys. Rev. Lett. (1984))

$$i(1 - i\lambda)\omega_t + \nabla^2\omega - \frac{2\omega^*(\nabla\omega)^2}{(1 + \omega\omega^*)} + \frac{A}{2}\frac{(1 - \omega^2)(\omega + \omega^*)}{(1 + \omega\omega^*)}$$
$$+ \frac{B}{2}\frac{(1 + \omega^2)(\omega - \omega^*)}{(1 + \omega\omega^*)} - C\left(\frac{1 - \omega\omega^*}{1 + \omega\omega^*}\right)\omega$$
$$+ \frac{1}{2}(H^x - ij^x)(1 - \omega^2) + \frac{1}{2}(j^y + iH^y)(1 + \omega^2) - (H^z - ij^z)\omega = 0.$$

Stereographic projection

⇒

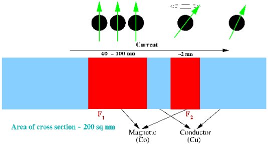
$$\Omega_n = \frac{S_n^x + iS_n^y}{1 + S_n^z},$$

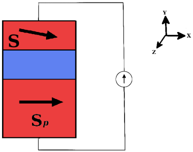
we get

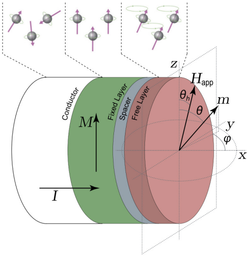
$$\frac{d\Omega_n}{dt} = iC\Omega_n\Big(\frac{1-|\Omega_{n+1}|^2}{1+|\Omega_{n+1}|^2} + \frac{1-|\Omega_{n-1}|^2}{1+|\Omega_{n-1}|^2}\Big)$$
$$-i\frac{A}{2}\Omega_n^2\Big(\frac{\Omega_{n+1}+\bar{\Omega}_{n+1}}{1+|\Omega_{n+1}|^2} + \frac{\Omega_{n-1}+\bar{\Omega}_{n-1}}{1+|\Omega_{n-1}|^2}\Big)$$
$$+i\frac{B}{2}\Omega_n^2\Big(\frac{\Omega_{n+1}-\bar{\Omega}_{n+1}}{1+|\Omega_{n+1}|^2} + \frac{\Omega_{n-1}-\bar{\Omega}_{n-1}}{1+|\Omega_{n-1}|^2}\Big)$$
$$+2iD\frac{1-|\Omega_n|^2}{1+|\Omega_n|^2}\Omega_n + i\frac{H_x}{2}(1+\Omega_n^2),$$

Extended L-L-G Equation: Spin torque effect

$\Longrightarrow$ Spin current, Slonczewski, J. Magn. Magn. Mater. (1996), Berger, Phys. Rev.B (1996)







Extended L-L-G Equation: Spin torque effect

Then

$$\frac{\partial \vec{S}}{\partial t} = -\vec{S} \times \left[\vec{H}_{\text{eff}} - \lambda \frac{\partial \vec{S}}{\partial t} + \vec{S} \times \vec{j} \right]$$

Then

$$\frac{dE}{dt} = \left[-\lambda |\vec{S}_t|^2 + \left(\vec{S}_t \times \vec{S} \right) \cdot \vec{j} \right]$$

$\Longrightarrow$Energy is not necessarily decreasing along trajectories

$\Longrightarrow$Novel features may arise

- Slonczewski's form of $\vec{j}$:

$$\vec{j} = \frac{a.\vec{S}_p}{f(P)(3 + \vec{S} \cdot \vec{S}_p)}; \qquad f(P) = \frac{(1 + P^3)}{4P^{\frac{3}{2}}}.$$

- a, strength of spin current
- f(P), polarization factor
- P, degree of polarization of pinned layer(F_1)

L-L-G Equation with spin current: Switching, Bifurcations and Chaos

- **Homogeneous magnetization**

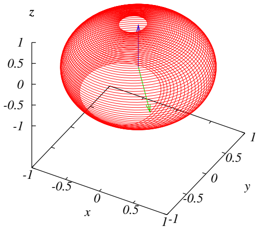
$$(1-i\lambda)\omega_t = -(j^z + iH^z)\omega + i[A\omega - B(1-\omega^2)]$$
$$\frac{-i}{(1+|\omega|^2)}\Big[N_3(1-|\omega|^2)\omega - N_1(1-\omega^2-|\omega|^2)\omega$$
$$-\frac{N_2}{2}(1+\omega^2-|\omega|^2)\omega - \frac{(N_1-N_2)\omega}{2}\Big]$$

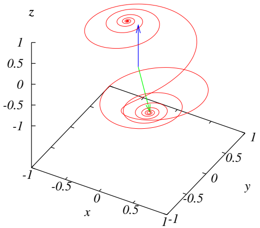
- **Isotropic sphere:**

$$\begin{aligned}(1-i\lambda)\omega_t &= -(j^z+iH^z)\omega \\ \implies \omega(t) &= \omega(0)\exp\left[\frac{-(j^z-\lambda H^z)}{(1+|\lambda|^2)}t\right] \\ &\quad \exp\left[\frac{-i(\lambda j^z+H^z)}{(1+|\lambda|^2)}t\right]\end{aligned}$$

L-L-G Equation with spin current: Switching, Bifurcations and Chaos

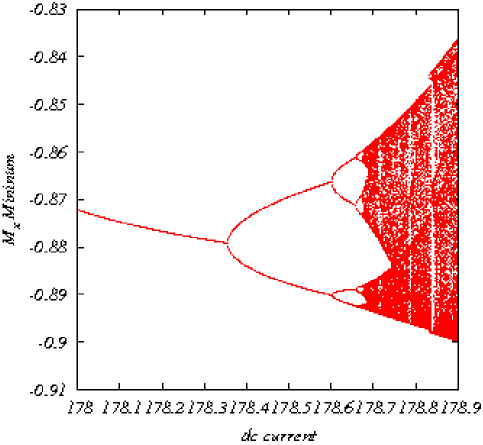
- **(a) Magnetic field alone present.**
- **(b) Current alone present: Faster switching**
  $\Longrightarrow$ Development of MRAMS
  S. Murugesh & M. Lakshmanan, *Chaos, Solitons & Fractals*, **41** 2773 (2009)





As j is varied (d.c+a.c)

Z. Yang, S. Zhang & Y. C. Lai, Phys. Rev. Lett. **99** (2007)
S. Murugesh & M. Lakshmanan, *Chaos*, **19** 043111 (2009)



Anisotropy + External magnetic field

$$H = -\sum_{\{n\}}^{N}(AS_n^x S_{n+1}^x + BS_n^y S_{n+1}^y + CS_n^z S_{n+1}^z) - D\sum_n (S_n^z)^2 - \vec{\mathcal{H}}\cdot\sum_n \vec{S}_n,$$

$$(S_n^x)^2 + (S_n^y)^2 + (S_n^z)^2 = 1, \;\; n = 1, 2, ..., N.$$

$$\begin{aligned} \frac{d\vec{S}_n}{dt} &= \vec{S}_n \times [A(S_{n+1}^x + S_{n-1}^x)\vec{i} + B(S_{n+1}^y + S_{n-1}^y)\vec{j} \\ &+ C(S_{n+1}^z + S_{n-1}^z)\vec{k} + 2DS_n^z\vec{k}] + \vec{S}_n \times \vec{\mathcal{H}}, \quad n = 1, 2, ..., N. \end{aligned}$$

$$\begin{aligned} \frac{dS_n^x}{dt} &= CS_n^y(S_{n+1}^z + S_{n-1}^z) - BS_n^z(S_{n+1}^y + S_{n-1}^y) - 2DS_n^y S_n^z, \\ \frac{dS_n^y}{dt} &= AS_n^z(S_{n+1}^x + S_{n-1}^x) - CS_n^x(S_{n+1}^z + S_{n-1}^z) + 2DS_n^x S_n^z + \mathcal{H}S_n^z, \\ \frac{dS_n^z}{dt} &= BS_n^x(S_{n+1}^y + S_{n-1}^y) - AS_n^y(S_{n+1}^x + S_{n-1}^x) - \mathcal{H}S_n^y, n = 1, 2, ..., N \end{aligned}$$

(a) XY - Integrable Map

- Let us restrict the spin motion to (x, y) space only, i.e. $S_i^z = 0$.

⇒ Then dynamics becomes statics ! (Thompson, Ross, Thompson & Lakshmanan, 1985)

⇒ Discrete XY-Spin system

$$\begin{aligned} B(S_{i+1}^y + S_{i-1}^y) - AS_i^y(S_{i+1}^x + S_{i-1}^x) - HS_i^y &= 0 \\ (S_i^x) + (S_i^y)^2 &= 1 \end{aligned}$$

- When the magnetif field $\vec{H} = 0$

$$\begin{aligned} B(S_{i+1}^y + S_{i-1}^y) - AS_i^y(S_{i+1}^x + S_{i-1}^x) &= 0 \\ (S_i^x)^+(S_i^y)^2 &= 1 \end{aligned}$$

⇒ Hamiltonian

$$H = -\sum_n (AS_i^x S_{i+1}^x + BS_i^y S_{i+1}^y)$$

(a) XY - Integrable Map

- Exact solution : (Roberts & Thompson 1988; Lakshmanan & Saxena 2008)

$$
\begin{aligned}
S_n^x &= sn(pn+\delta, k) \\
S_n^y &= cn(pn+\delta, k)
\end{aligned}
$$

where $dn(p,k) = \frac{B}{A}$, k : *arbitrary*

$$H = E = -N\frac{B}{A}\frac{\sqrt{B^2 - A^2 + A^2k^2}}{k} + N\sqrt{A^2 - B^2}\frac{Z(p,k)}{k}$$

XY - Integrable Map

- Domain wall structure
- When $k \to 1$:

$$
\begin{aligned}
S_n^x &= \tanh(pn + \delta), \\
S_n^y &= \sqrt{1-\gamma^2}\,\mathrm{sech}(pn + \delta), \\
S_n^z &= \gamma\ \mathrm{sech}(pn + \delta)
\end{aligned}
$$

System admits a first integarl W: (Granovkii & Zhedanov, 1986)

$$W = B\ S_n^x\ S_{n+1}^x + A\ S_n^y S_{n+1}^y$$

Proof:

$$(S_{n+1}^x - S_{n-1}^x) \times [B\ S_n^x(S_{n+1}^y + S_{n-1}^y) - A\ S_n^y(S_{n+1}^x + S_{n-1}^x)] = 0$$
$$(S_{n+1}^y - S_{n-1}^y) \times [B\ S_n^x(S_{n+1}^y + S_{n-1}^y) - A\ S_n^y(S_{n+1}^x + S_{n-1}^x)] = 0$$

Use $(S_n^x)^2 + (S_n^y)^2 = 1 \Rightarrow$

$$[B(S_n^x\ S_{n+1}^x - S_{n-1}^x S_n^x) + A(S_n^y\ S_{n+1}^y - S_{n-1}^y S_n^y)] \times (S_{n+1}^x + S_{n-1}^x + S_{n+1}^y + S_n^y$$

$\Rightarrow$

$$BS_n^x S_{n+1}^x + AS_n^y S_{n+1}^y = BS_{n-1}^x S_n^x + AS_{n-1}^y S_n^y$$

$\Rightarrow$ W: Integral

XY - Integrable Map

- Complete integrability stands proved.
- Another form of the map: $(\vec{S}_n = (\cos\theta_n, \sin\theta_n))$

$$B\ tan(\theta_n) = A\ tan\frac{\theta_{n+1} + \theta_{n-1}}{2}$$

⇒ With $x_n = \tan\theta_n$

⇒ Integrable QRT map

$$x_{n+1} = \frac{2\lambda x_n - x_{n-1}(1 - \lambda^2 x_n^2)}{1 - \lambda^2 x_n^2 + x_{n-1}(2\lambda x_n)}, \quad \lambda = \frac{A}{B}$$

(Quispel, Roberts & Thompson, Physica D 1989)

XYZ - Integrable Map

(b) **XYZ Integable map:**

- Next consider the static case of the anisotrpic Heisenberg spin chain (with $\vec{H} = 0$)

$$C\ S_n^y(S_{n+1}^z + S_{n-1}^z) - B\ S_n^z(S_{n+1}^y + S_{n-1}^z) = 0$$
$$A\ S_n^z(S_{n+1}^x + S_{n-1}^x) - C\ S_n^x(S_{n+1}^z + S_{n-1}^z) = 0$$
$$B\ S_n^x(S_{n+1}^y + S_{n-1}^y) - A\ S_n^y(S_{n+1}^x + S_{n-1}^x) = 0$$

and also

$$(S_n^x)^2 + (S_n^y)^2 + (S_n^z)^2 = 1$$

- Or equivalently

$$H = -\sum_n \vec{S}_n.\hat{J}\vec{S}_{n+1},$$

$$\hat{J}\vec{S}_n = A\ S_n^x \vec{i} + B\ S_n^y \vec{j} + C\ S_n^z \vec{k}$$

XYZ - Integrable Map

- Static equation of motion:

$$\vec{S}_n \times (\hat{J}\vec{S}_{n+1} + \hat{J}\vec{S}_{n-1}) = 0$$

Or

$$\begin{aligned} \vec{S}_{n+1} + \vec{S}_{n-1} &= \lambda_n \hat{J}_n^{-1} \; S_n, \\ \lambda_n &= \frac{2(\vec{S}_n \; .J^{-1}\vec{S}_{n+1})}{S_n \; .J^{-2} S_n} \end{aligned}$$

(Granovskii & Zhedanov 1987; Veselov 1987)

- First integral:
- As before multiplying scalarly by $(\vec{S}_{n+1} - \vec{S}_{n-1})$: $W_1 = \vec{S}_n \; . \; J^{-1}\vec{S}_{n+1}$
- Second integarl: From the defining equation:

$$\vec{S}_n \times (\hat{J}\vec{S}_{n+1}) = -\vec{S}_n \times (\hat{J}\vec{S}_{n-1})$$

XYZ - Integrable Map

- Squaring and rearranging:

$$(\vec{S}_n\times\vec{A}_{n+1}).(\vec{S}_n\times\vec{A}_{n+1}) = (\vec{S}_n\times\vec{A}_{n-1}).(\vec{S}_n\times\vec{A}_{n-1}) \quad (\vec{A}_n = \hat{J}\vec{S}_n)$$

$$\Rightarrow \quad \vec{S}_n.(\vec{A}_{n+1}\times\vec{S}\times\vec{A}_{n+1}) = \vec{S}_n.(\vec{A}_{n-1}\times\vec{S}_n\times\vec{A}_{n-1})$$

$$\Rightarrow$$

$$\vec{S}_n.[(\vec{A}_{n+1}.\vec{S}_n)\vec{A}_{n+1} - |\vec{A}_{n+1}|^2\vec{S}_n] = \vec{S}_n[(A_{n-1}.\vec{S}_n)(\vec{A}_{n-1}) - |\vec{A}_{n-1}|^2 S_n]$$

$$(\vec{A}_{n+1}.\vec{S}_n)^2 - |\vec{A}_{n+1}|^2 = (\vec{A}_{n-1}.\vec{S}_n)^2 - |\vec{A}_{n-1}|^2 = (\vec{S}_{n-1}.\vec{A}_n)^2 - |\vec{A}_{n-1}|^2$$

$$\Rightarrow$$

$$|\vec{A}_{n+1}|^2 + |\vec{A}_n|^2 - (\vec{A}_{n+1}.\vec{S}_n)^2 = |\vec{A}_n|^2 + |\vec{A}_{n-1}|^2 - (\vec{A}_n.\vec{S}_{n-1})^2$$

$$\Rightarrow \quad W_2 = |\hat{J}\vec{S}_{n+1}|^2 + |\hat{J}\vec{S}|^2 - (\hat{J}\vec{S}_{n+1}.\vec{S}_n)^2$$

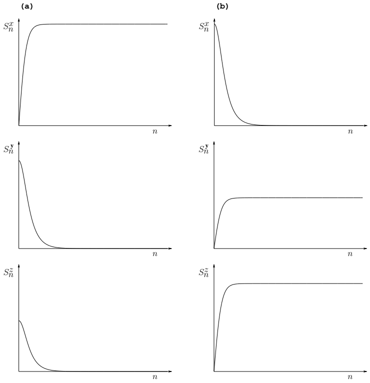
XYZ - Integrable Map

- **Some exact solutions** (Lakshmanan & Saxena, Physica D 2008)

$$S_n^x = \sqrt{1-\gamma^2 k'^2}\ \mathrm{sn}(pn+\delta,k),$$
$$S_n^y = \sqrt{1-\gamma^2}\ \mathrm{cn}(pn+\delta,k),$$
$$S_n^z = \gamma\ \mathrm{dn}(pn+\delta,k),$$

$$k^2 = \frac{A^2-B^2}{A^2-C^2}, \quad \mathrm{dn}(p,k) = \frac{B}{A}.$$

- XYY model : $k \to 1 \quad (A \neq B, C)$



$$S_n^x = \tanh(pn+\delta), \quad S_n^y = \sqrt{1-\gamma^2}\ \mathrm{sech}(pn+\delta) \quad S_n^z = \gamma\ \mathrm{sech}(pn+\delta),$$

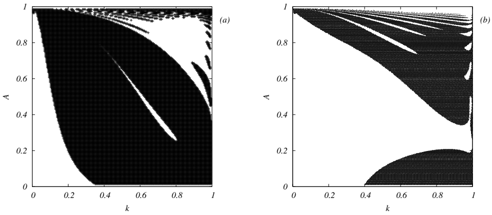
Total Energy

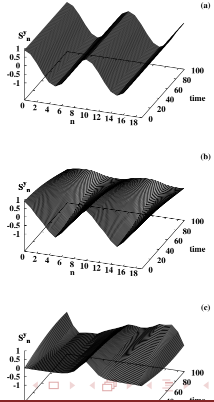
$$E = -N\frac{BC}{A} - N\frac{\sqrt{A^2-C^2}}{(A^2-B^2)}Z(p,k)\left[-(A^2-B^2)+\gamma^2C^2\frac{(B^2-C^2)}{(A^2-C^2)}\right].$$

- Linear Stability

$$\mathbf{S}_n^p = \mathbf{S}_n^0 + \delta\mathbf{S}_n$$

- Floquet theory





Ishimori Integrable Spin Chain

- Anisotropic spin chain : In general not integrable
- But a modified version - Ishimori spin chain - is integrable.
- Isotropic spin chain $(A = B = C)$ (Ishimori, Prog. Theor. Phys. 1984)

$$\frac{d\vec{S}_n}{dt} = \vec{S}_n \times (\vec{S}_{n+1} + \vec{S}_{(n-1)}).$$

- Ishimori spin chain

$$\frac{d\vec{S}_n}{dt} = \frac{2}{(1 + \vec{S}_n \cdot \vec{S}_{n+1})}(\vec{S}_n \times \vec{S}_{n+1}) + \frac{2}{(1 + \vec{S}_n \cdot \vec{S}_{n-1})}(\vec{S}_n \times \vec{S}_{n-1}).$$

Ishimori Integrable Spin Chain

- Lax pair

$$L_n = \frac{(\lambda + \lambda^{-1})}{2}\mathrm{I} + \frac{(\lambda - \lambda^{-1})}{2} S_n$$

$$M_n = i\left(1 - \frac{\lambda^2 + \lambda^{-2}}{2}\right)\frac{S_n + S_{n-1}}{1 + \vec{S}_n \cdot \vec{S}_{n+1}} - i\frac{(\lambda^2 - \lambda^{-2})}{2} \cdot \frac{(\mathrm{I} + S_{n-1}S_n)}{(1 + \vec{S}_n \cdot \vec{S}_{n-1})}$$

$$S_n = \vec{\sigma} \cdot \vec{S}_n, \ \mathrm{I}: \text{ unit matrix.}$$

- Hamiltonian

$$H = -2\sum_n \log(1 + \vec{S}_n \cdot \vec{S}_{n+1})$$

Ishimori Integrable Spin Chain

- 1- soliton solution

$$
\begin{aligned}
S_n^z &= 1 - \frac{\sinh^2 2P}{\cosh 2P - \cos 2k} \cdot \mathrm{sech}\xi_n \cdot \mathrm{sech}\xi_{n+1}, \\
S_n^x + iS_n^y &= \frac{\sinh 2P}{\cosh 2P - \cos 2k} \cdot \mathrm{sech}\xi_{n+1} \times \\
&\quad \times (\cosh 2P - e^{-2ik} + \sinh 2P \tanh \xi_n) e^{i\eta n}, \\
\xi_n &= 2Pn + 2\sinh 2P \sin(2kt + \xi_0) \\
\eta_n &= 2k_n + 2(1 - \cosh 2P \cos 2k)t + \eta^0, \\
&\quad (P, k, \xi_0, \eta^0 : \text{constant}).
\end{aligned}
$$

Ishimori Integrable Spin Chain

- Gauge equivalence to discrete NLS

$$
\begin{aligned}
i\frac{dQ_n}{dt} &= Q_{n+1} + Q_{n-1} - 2Q_n + \mid Q_m \mid^2 (Q_{n+1} + Q_{(n+1)}) \\
&\quad \text{(Ablowitz-Ladik)} \\
L_n &= \begin{bmatrix} \lambda & \frac{1}{\lambda} Q_n \\ -\lambda Q_n^* & \frac{1}{\lambda} \end{bmatrix} \\
M_n &= i \begin{bmatrix} 1 - \lambda^2 - Q_n Q_{n-1}^* & -Q_n + Q_{n-1} \cdot \lambda^{-2} \\ -Q_n^* + Q_{n-1}^* \lambda^2 & -1 + \lambda^{-2} + Q_m^* Q_{n-1} \end{bmatrix} \\
\frac{dL_n}{dt} &= M_{n+1} L_n - L_n M_{n+1}
\end{aligned}
$$

Ishimori Integrable Spin Chain

- Integrable mapping (Quispel, Roberts & Thompson, 1988)
- With the assumption

$$S_n(t) = (\cos\phi_n \cos\omega t, \cos\phi_n \sin\omega t, \sin\phi_n)$$
$$x_n = \tan\frac{1}{2}\phi_n$$

$\Rightarrow$

$$x_{n+1} = \frac{[2x_n^3 + \omega x_n^2 + 2x_n - \omega + x_{n-1}(x_n^4 + \omega x_n^3 - \omega x_n - 1)]}{[-x_n^4 - \omega x_n^3 + \omega x_n + 1 - x_{n-1}(\omega x_n^4 - -2x_n^3 - \omega x_n^2 - 2x_n)]}$$

$\Rightarrow$ One among the 18 parameter integrable QRT map.

Anisotropic spin chain - Nonintegrable case

- Equation of motion

$$\frac{dS_n^x}{dt} = CS_n^y(S_{n+1}^z + S_{n-1}^z) - BS_n^z(S_{n+1}^y + S_{n-1}^y) - 2DS_n^yS_n^z,$$

$$\frac{dS_n^y}{dt} = AS_n^z(S_{n+1}^x + S_{n-1}^x) - CS_n^x(S_{n+1}^z + S_{n-1}^z) + 2DS_n^xS_n^z + \mathcal{H}S_n^z,$$

$$\frac{dS_n^z}{dt} = BS_n^x(S_{n+1}^y + S_{n-1}^y) - AS_n^y(S_{n+1}^x + S_{n-1}^x) - \mathcal{H}S_n^y.$$

$$S_x^2 + S_y^2 + S_z^2 = 1$$

(Lakshmanan & Saxena, Physica D 2006)

Parametrization of the unit spin vector

E.g. : Lamé polynomials :

$$\frac{d^2\psi(u)}{du^2} + [E - n(n+1)k^2\mathrm{sn}^2(u,k)]\psi(u) = 0$$

$n = 1$:

$$\psi_{11} \propto \mathrm{sn}(u,k),\ \psi_{12} \propto \mathrm{cn}(u,k),\ \psi_{13} \propto \mathrm{dn}(u,k),$$

$n = 2$:

$$\psi_{21} \propto \mathrm{sn}(u,k)\mathrm{cn}(u,k),\ \psi_{22} \propto \mathrm{cn}(u,k)\mathrm{dn}(u,k),\ \psi_{23} \propto \mathrm{sn}(u,k)\mathrm{dn}(u,k)$$

Other parametrizations :

$$S_n^x = \mathrm{cn}(u, k_1), \quad S_n^y = \mathrm{sn}(u, k_1)\mathrm{cn}(v, k_2), \quad S_n^z = \mathrm{sn}(u, k_1)\mathrm{sn}(v, k_2),$$

$$S_n^x = \frac{\alpha \mathrm{cn}(u, k_1)}{1 - \gamma \mathrm{sn}(u, k_1)\mathrm{sn}(v, k_2)}, \quad S_n^y = \frac{\alpha \mathrm{sn}(u, k_1)\mathrm{sn}(v, k_2)}{1 - \gamma \mathrm{sn}(u, k_1)\mathrm{sn}(v, k_2)},$$

$$S_n^z = \frac{\mathrm{sn}(u, k_1)\mathrm{sn}(v, k_2) - \gamma}{1 - \gamma \mathrm{sn}(u, k_1)\mathrm{sn}(v, k_2)}, \quad \alpha = \sqrt{1 - \gamma^2},$$

Moving solutions : $(D = 0, \quad \vec{H} = 0)$

- With $u = (pn - \omega t + \delta)$

$\Rightarrow$

$$S_n^x = \alpha \, \mathrm{sn}(pn - \omega t + \delta, k), \qquad S_n^y = \beta \, \mathrm{cn}(pn - \omega t + \delta, k),$$
$$S_n^z = \gamma \, \mathrm{dn}(pn - \omega t + \delta, k),$$

$\Rightarrow$

$$-\omega\alpha = \frac{2\beta\gamma[C\mathrm{dn}(p,k) - B\mathrm{cn}(p,k)]}{1 - k^2\mathrm{sn}^2(u,k)\mathrm{sn}^2(p,k)},$$

$$\omega\beta = \frac{2\alpha\gamma\mathrm{dn}(p,k)[A\mathrm{cn}(p,k) - C]}{1 - k^2\mathrm{sn}^2(u,k)\mathrm{sn}^2(p,k)},$$

$$\omega\gamma k^2 = \frac{2\alpha\beta\mathrm{cn}(p,k)[B - A\mathrm{dn}(p,k)]}{1 - k^2\mathrm{sn}^2(u,k)\mathrm{sn}^2(p,k)}, \qquad u = pn - \omega t + \delta.$$

Four possible choices :

(i) $p = 4K(k)$:

$$\omega = 2\gamma\sqrt{(B-C)(A-C)}, \quad k^2 = \frac{1-\gamma^2}{\gamma^2}\frac{(B-A)}{(A-C)}, \quad (B > A > C),$$

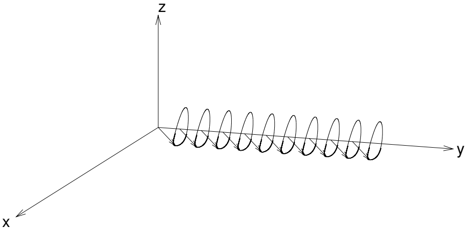
$$S_n^x = \sqrt{1-\gamma^2 k'^2}\,\mathrm{sn}(4Kn-\omega t+\delta, k) = -\sqrt{1-\gamma^2 k'^2}\,\mathrm{sn}(\omega t+\delta, k),$$

$$S_n^y = \sqrt{1-\gamma^2}\,\mathrm{cn}(4Kn-\omega t+\delta, k) = \sqrt{1-\gamma^2}\,\mathrm{cn}(\omega t+\delta, k),$$

$$S_n^z = \gamma\,\mathrm{dn}(4Kn-\omega t+\delta, k) = \gamma\,\mathrm{dn}(\omega t+\delta, k).$$

Moving solutions : $(D = 0, \quad \vec{H} = 0)$

$$\begin{aligned} E &= -\sum_{n=1}^{N}[AS_n^x(S_{n+1}^x + S_{n-1}^x) + BS_n^y(S_{n+1}^y + S_{n-1}^y) + CS_n^z(S_{n+1}^z + S_{n-}^z \\ &= -2N(B\beta^2 + C\gamma^2) = -N[B + (C - B)\gamma^2] \\ &= -2N[B - (B - C)\gamma^2], \quad (B > C, \quad 0 \leq \gamma \leq 1) \end{aligned}$$



Bohr-Sommerfield quantization:

$$\oint p_i dq_i = \left(n_i + \frac{1}{2}\right) h, \qquad n_i = 0, 1, 2, ..., \; i = 1, 2, ..., N$$

$$p_n = S_n^z, \quad q_n = \arctan\left(\frac{S_n^y}{S_n^x}\right), \quad n = 1, 2, ..., N.$$

$$\frac{4}{\gamma}\sqrt{\frac{1-\gamma^2 k'^2}{1-\gamma^2}}\left[\Pi\left(\frac{-\gamma^2 k^2}{(1-\gamma^2)}, k\right) - (1-\gamma^2)K(k)\right] = \left(n_i + \frac{1}{2}\right) h,$$
$$n_i = 0, 1, 2, ..., \quad i = 1, 2, \ldots, N.$$

(ii) $p = 2K(k)$:

$$\omega = 2\gamma\sqrt{(A+C)(B+C)}, \quad k^2 = \frac{1-\gamma^2}{\gamma^2}\left(\frac{B-A}{A+C}\right), \quad p = 2K(k).$$

The corresponding spatially alternating time periodic solutions are

$$S_n^x = \sqrt{1-\gamma^2 k'^2}\,\mathrm{sn}(2Kn-\omega t+\delta, k) = (-1)^{n+1}\sqrt{1-\gamma^2 k'^2}\,\mathrm{sn}(\omega t+\delta, k),$$

$$S_n^y = \sqrt{1-\gamma^2}\,\mathrm{cn}(2Kn-\omega t+\delta, k) = (-1)^n\sqrt{1-\gamma^2}\,\mathrm{cn}(\omega t+\delta, k),$$

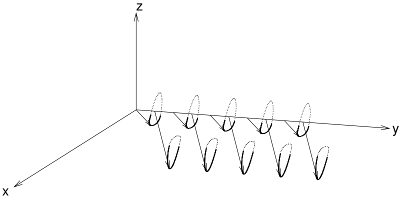
$$S_n^z = \gamma\,\mathrm{dn}(2Kn-\omega t+\delta, k) = \gamma\,\mathrm{dn}(\omega t+\delta, k).$$

$$E = N[B-(B+C)\gamma^2].$$

(ii) $p = 2K(k)$:

$$\oint p_{1,i} dq_{1,i} = \left(n_{1,i} + \frac{1}{2}\right) h, \quad n_{1,i} = 0, 1, 2, \ldots, \quad i = 1, 2, \ldots \frac{N}{2}$$

$$\oint p_{2,i} dq_{2,i} = \left(n_{2,i} + \frac{1}{2}\right) h, \quad n_{2,i} = 0, 1, 2, \ldots, \quad i = 1, 2, \ldots \frac{N}{2}$$



(iii) $k = 0$: Linear / Nonlinear magnon solutions

(iv) $\omega = 0$: Static solutions (already discussed).

(b) Moving solutions ($D \neq 0, \vec{H} = 0$)
All the above solutions exist with modified dispersion relations.

(c) Constant external magnetic field

$$\frac{dS_n^x}{dt} = C[S_n^y(S_{n+1}^z + S_{n-1}^z) - S_n^z(S_{n+1}^y + S_{n-1}^y)],$$

$$\frac{dS_n^y}{dt} = AS_n^z(S_{n+1}^x + S_{n-1}^x) - CS_n^x(S_{n+1}^z + S_{n-1}^z) + H_x S_n^z,$$

$$\frac{dS_n^z}{dt} = CS_n^x(S_{n+1}^y + S_{n-1}^y) - AS_n^y(S_{n+1}^x + S_{n-1}^x) - H_x S_n^y.$$

$$S_n^x = \mathrm{sn}(pn + \delta, k),$$

$$S_n^y = \sin(\omega t + \gamma)\mathrm{cn}(pn + \delta, k),$$

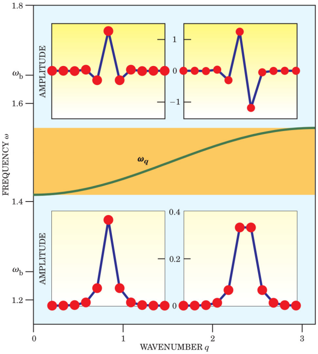
$$S_n^z = \cos(\omega t + \gamma)\mathrm{cn}(pn + \delta, k).$$

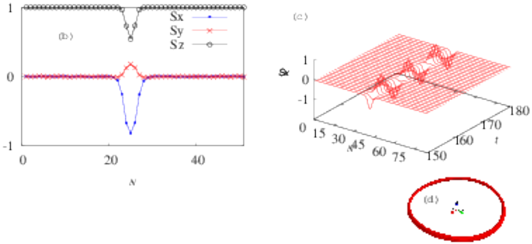
$$
\begin{aligned}
E &= -NC\,\mathrm{cn}(p,k) - \frac{NZ(p,k)}{k^2\,\mathrm{sn}(p,k)}[-A + C\,\mathrm{dn}(p,k)] - H_x \sum_n \mathrm{sn}(pn+\delta,k) \\
&= -N\frac{C}{A}\frac{\sqrt{C^2 - A^2 + k^2A^2}}{k} + N\frac{\sqrt{A^2 - C^2}}{k} Z(p,k) + H_x \sum_n \mathrm{sn}(pn+\delta,k).
\end{aligned}
$$

(Pierls-Nabarro potential barrier present)

Internal localized modes

Introduction :





(Zolotaryuk, Flach & Fleurov, Phys. Rev. B 2001)

(a) One-spin excitation :

(Lakshmanan, Subash & Saxena, Phys. Lett. A 2014)

$$\vec{S}_n = \ldots, (1,0,0), (1,0,0), (S_i^x(t), S_i^y(t), S_i^z(t)), (1,0,0), (1,0,0), \ldots$$

$$\begin{aligned} \frac{dS_0^x}{dt} &= -2DS_0^y S_0^z, \\ \frac{dS_0^y}{dt} &= (2A+H)S_0^z + 2DS_0^x S_0^z, \\ \frac{dS_0^z}{dt} &= -(2A+H)S_0^y. \end{aligned}$$

$D = 0$

$$S_0^x = \sqrt{1-a^2}, \quad S_0^y = a\sin(\Omega t+\delta), \quad S_0^z = a\cos(\Omega t+\delta), \quad \Omega = (2A+H),$$

(a) One-spin excitation :

$D \neq 0$

$$S_0^y = -\frac{1}{2A+H}\frac{dS_0^z}{dt},$$

$$S_0^x = -\frac{1}{2A+H}\left[\frac{1}{S_0^z}\left(\frac{d^2S_0^z}{dt^2}\right) + (2A+H)^2\right].$$

Substituting Eqs. (2) - (2) into Eq. (2) we find

$$\frac{dS_0^x}{dt} \equiv -\frac{1}{2D(2A+H)}\left[-\frac{1}{(S_0^z)^2}\left(\frac{dS_0^z}{dt}\right)\left(\frac{d^2S_0^z}{dt^2}\right) + \frac{1}{S_0^z}\frac{d^3S_0^z}{dt^3}\right]$$
$$= \frac{2D}{(2A+H)}S_0^z\left(\frac{dS_0^z}{dt}\right),$$

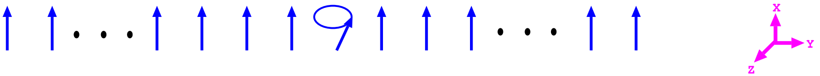
(a) One-spin excitation :

$$S_0^x = \frac{Da^2}{2A+H}\mathrm{cn}^2(\hat{\Omega}t+\delta,k)+d,$$

$$S_0^y = \frac{a\hat{\Omega}}{2A+H}[\mathrm{sn}(\hat{\Omega}t+\delta,k)\mathrm{dn}(\hat{\Omega}t+\delta,k)],$$

$$S_0^z = a\ \mathrm{cn}(\hat{\Omega}t+\delta,k),$$

$$E = 2d(2A+H) = 2\sqrt{1-a^2}(2A+H) - 2Da^2 + \{vacuum\ state\ energy - A\}.$$



(b) Two-spin excitations :

$$\vec{S}_n = ..., (1,0,0), (1,0,0), (S_i^x, S_i^y, S_i^z), (S_{i+1}^x, S_{i+1}^y, S_{i+1}^z), (1,0,0), (1,0,0)$$

$$= ..., (1,0,0), (1,0,0), (S_0^x, S_0^y, S_0^z), (S_1^x, S_1^y, S_1^z), (1,0,0), (1,0,0), ...$$

Equivalently one can also choose

$$\vec{S}_n = ..., (1,0,0), (1,0,0), (S_{i-1}^x, S_{i-1}^y, S_{i-1}^z), (S_i^x, S_i^y, S_i^z), (1,0,0), (1,0,0)$$

$$\frac{dS_0^x}{dt} = CS_0^y S_1^z - BS_0^z S_1^y - 2DS_0^y S_0^z,$$

$$\frac{dS_0^y}{dt} = AS_0^z(S_1^x + 1) - CS_0^x S_1^z + 2DS_0^x S_0^z + HS_0^z,$$

$$\frac{dS_0^z}{dt} = BS_0^x S_1^y - AS_0^y(1 + S_1^x) - HS_0^y,$$

(b) Two-spin excitations :

$$\frac{dS_1^x}{dt} = CS_0^z S_1^y - BS_1^z S_0^y - 2DS_1^y S_1^z,$$

$$\frac{dS_1^y}{dt} = AS_1^z (S_0^x + 1) - CS_1^x S_0^z + 2DS_1^x S_1^z + HS_1^z,$$

$$\frac{dS_1^z}{dt} = BS_1^x S_0^y - AS_1^y (1 + S_0^x) - HS_1^y.$$

$$\vec{S}_0(t) = \vec{S}_1(t).$$

$$\frac{dS_0^x}{dt} = (C - B - 2D) S_0^y S_0^z,$$

$$\frac{dS_0^y}{dt} = (A + H) S_0^z + (A - C + 2D) S_0^x S_0^z,$$

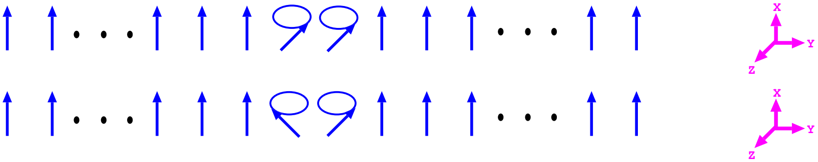
$$\frac{dS_0^z}{dt} = (B - A) S_0^x S_0^y - (A + H) S_0^y.$$

(b) Two-spin excitations :

$$S_0^x(t) = \frac{\Gamma c\ \mathrm{sn}^2(\Omega t + \delta) - b}{1 - \Gamma \mathrm{sn}^2(\Omega t + \delta)}, \quad \Gamma < 1,$$

$$S_0^y(t) = \sqrt{\frac{2(b-c)(A - C + 2D)(a + b - (a+c)\Gamma \mathrm{sn}^2(\Omega t)}{(C - B - 2D)(1 - \Gamma \mathrm{sn}^2(\Omega t))^2}},$$

$$\vec{S}_1(t) = (S_0^x, -S_0^y, -S_0^z),$$



(c) Three-spin excitations :

$$\vec{S}_n(t) = ..., (1,0,0), (1,0,0), (S^x_{-1}, S^y_{-1}, S^z_{-1}), (S^x_0, S^y_0, S^z_0), (S^x_1, S^y_1, S^z_1), (1$$

$$\frac{dS^x_{-1}}{dt} = CS^y_{-1}S^z_0 - BS^z_{-1}S^y_0 - 2DS^y_{-1}S^x_{-1},$$

$$\frac{dS^y_{-1}}{dt} = AS^z_{-1}(1 + S^x_0) - CS^x_{-1}S^z_0 + 2DS^x_{-1}S^z_{-1} + HS^z_{-1},$$

$$\frac{dS^z_{-1}}{dt} = BS^x_{-1}S^y_0 - AS^y_{-1}(1 + S^x_0) - HS^y_{-1},$$

$$\frac{dS^x_0}{dt} = CS^y_0(S^z_1 + S^z_{-1}) - BS^z_0(S^y_1 + S^y_{-1}) - 2DS^y_0 S^x_0,$$

$$\frac{dS^y_0}{dt} = AS^z_0(S^x_1 + S^y_{-1}) - CS^x_0(S^z_1 + S^z_{-1}) + 2DS^x_0 S^z_0 + HS^z_0,$$

$$\frac{dS^z_0}{dt} = BS^x_0(S^y_1 + S^y_{-1}) - AS^y_0(S^x_1 + S^x_{-1}) - HS^y_0,$$

(c) Three-spin excitations :

$$\frac{dS_1^x}{dt} = CS_1^y S_0^z - BS_1^z S_0^y - 2DS_1^y S_1^x,$$
$$\frac{dS_1^y}{dt} = AS_1^z(1 + S_0^x) - CS_1^x S_0^z + 2DS_1^x S_1^z + HS_1^z,$$
$$\frac{dS_1^z}{dt} = BS_1^x S_0^y - AS_1^y(S_0^x + 1) - HS_1^y.$$

$$\vec{S}_0(t) = (1, 0, 0),$$
$$\frac{dS_{-1}^x}{dt} = -2DS_{-1}^y S_{-1}^z,$$
$$\frac{dS_{-1}^y}{dt} = (2A + H)S_{-1}^z + 2DS_{-1}^x S_{-1}^z,$$
$$\frac{dS_{-1}^z}{dt} = -(2A + H)S_{-1}^y,$$
$$0 = 0$$

(c) Three-spin excitations :

$$0 = -C(S_1^z + S_{-1}^z),$$

$$0 = B(S_1^y + S_{-1}^y),$$

$$\frac{dS_1^x}{dt} = -2DS_1^y S_1^x,$$

$$\frac{dS_1^y}{dt} = (2A + H)S_1^z + 2DS_1^x S_1^z,$$

$$\frac{dS_1^z}{dt} = -(2A + H)S_1^z.$$

$$S_1^z = -S_{-1}^z,$$

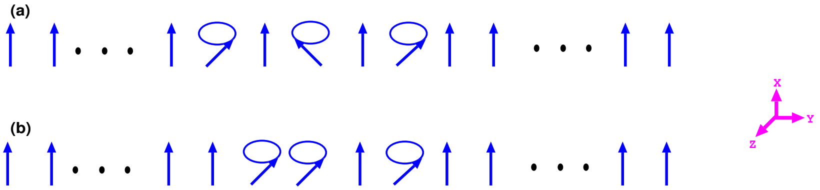
$$S_1^y = -S_{-1}^y.$$

$$\hat{\vec{S}}_1(t) = (S_{-1}^x, -S_{-1}^y, -S_{-1}^z)$$

(c) Three-spin excitations :

(a) $\vec{S}_n(t) = ..., (1,0,0), (S^x_{-1}, S^y_{-1}, S^z_{-1}), (1,0,0), (S^x_{-1}, -S^y_{-1}, -S^z_{-1}), (1,$

(b) $\vec{S}_n(t) = ..., (1,0,0), (S^x_{0}, S^y_{0}, S^z_{0}), (S^x_{0}, S^y_{0}, S^z_{0}), (1,0,0), (S^x_{-1}, S^y_{-1}, S^z_{-}$



- Linear stability.
- Semiclassical quantization.

Merits and Demerits of STNO

Merits

- Nanoscale source of microwaves
- Resistant to radiation damage
- Can be tuned over large frequency range

Demerits

- Very low output power (theory $\sim$ nW, experiment $\sim$ pW)

Common Driven Magnetic Field : Synchronization of STNOs

- Current coupling (W. H. Rippard, M. R. Pufall, S. Kaka, T. J. Silva, S. E. Russek & J. A. Katine(2005))
- Current coupling with delay (J. Grollier, V. Cros & A. Fert (2006))
- Common external noise (K. Nakada, S. Yakata & T. Kimura (2012))
- Common external periodically driven magnetic field

Consider an array of two STNOs in the presence of a common applied magnetic field

$$\vec{H}_{app} = (h_{dc} + h_{ac}\cos\omega t, 0, 0)$$

(Subash, Chandrasekar & Lakshmanan, Europhys. Lett. 2013)

Common Driven Magnetic Field : Synchronization of STNOs

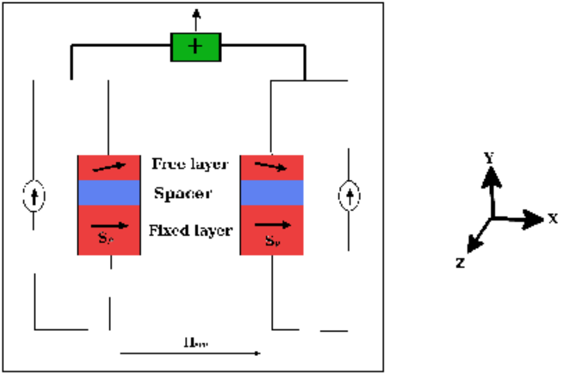


Figure: Schematic model of an array of two STNOs placed in a common driven magnetic field

Common Driven Magnetic Field : Synchronization of STNOs

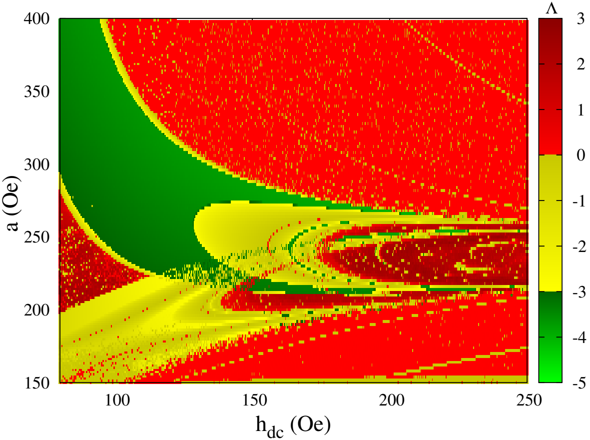
- The LLGS equation of the spins of the two STNOs

$$
\begin{aligned}
\frac{d\vec{S_1}}{dt} &= -\gamma\vec{S_1}\times\vec{H}_{1eff} + \lambda\vec{S_1}\times\frac{d\vec{S_1}}{dt} - \gamma a\vec{S_1}\times(\vec{S_1}\times\hat{S}_p) \\
\frac{d\vec{S_2}}{dt} &= -\gamma\vec{S_2}\times\vec{H}_{2eff} + \lambda\vec{S_2}\times\frac{d\vec{S_2}}{dt} - \gamma a\vec{S_2}\times(\vec{S_2}\times\hat{S}_p)
\end{aligned}
$$

$$\vec{S_1}^{\,2} = S_{1x}^2 + S_{1y}^2 + S_{1z}^2 = 1\ ,\ \vec{S_2}^{\,2} = S_{2x}^2 + S_{2y}^2 + S_{2z}^2 = 1$$
$$\hat{S}_p = \hat{i}\ ,\ \vec{H}_{eff}^{1,2} = H_{app}\hat{i} + \kappa S_{1x,2x}\hat{i} - 4\pi M_s S_{1z,2z}\hat{k}$$

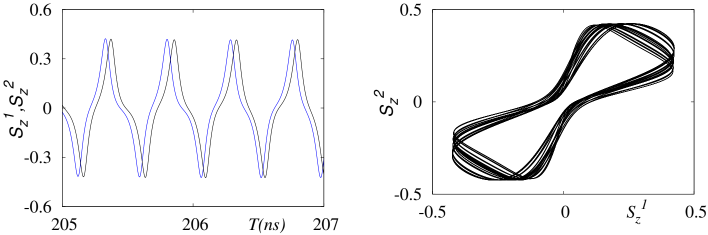
- H_{app}, applied magnetic field along easy axis
- κ, anisotropy field
- Demagnetization field perpendicular to the layer($z - axis$)

Regions of Chaos in a Single STNO



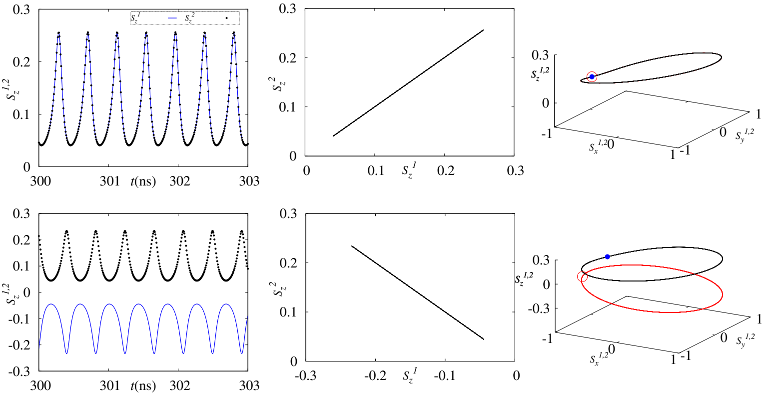
Periodic and Chaotic region for a single STNO in the a-h_{dc} space for an oscillating external magnetic field of strength h_{ac} = 10 Oe of frequency ω = 15 GHz and anisotropic field κ = 45.0 Oe along the inplane axis

Desynchronized oscillation of two similar STNO



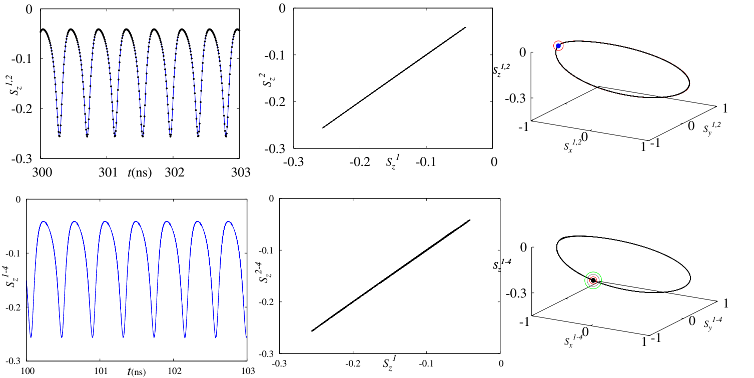
Desynchronized oscillation of two similar STNOs having same anisotropy field $\kappa = 45.0$ Oe placed in the oscillating external magnetic field of strength $h_{ac} = 10$ Oe of frequency $\omega = 15\ ns^{-1}$, $h_{dc} = 400$ Oe and $a = 220$ Oe

Synchronization of two similar STNO



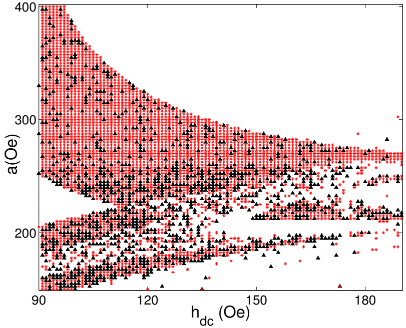
The time series and phase space plot of an array of two similar STNOs having same anisotropy field $\kappa = 45.0$ Oe placed in the oscillating external magnetic field of strength $h_{ac} = 10$ Oe of frequency $\omega = 15\ ns^{-1}$, $h_{dc} = 500$ Oe and $a = 220$ Oe exhibiting inphase and antiphase($a = 221$ Oe) locked synchronization

Synchronization of different STNO



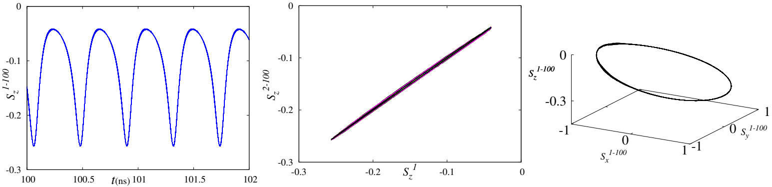
Synchronization of an array of two different($1^{s}t$ row) STNOs with anisotropy fields $\kappa_1 = 45.0$ Oe and $\kappa_2 = 46.0$ Oe placed in the oscillating external magnetic field, $h_{dc} = 130$ Oe and $a = 300$ Oe. Synchronization dynamics of an array of four different STNOs($2^{n}d$ row) having anisotropy fields $\kappa_1 = 45$, $\kappa_2 = 46$, $\kappa_3 = 47$ and $\kappa_4 = 48$ Oe, $h_{dc} = 130$ Oe and $a = 300$ Oe

h_{dc} Vs a parameter space for two similar STNO



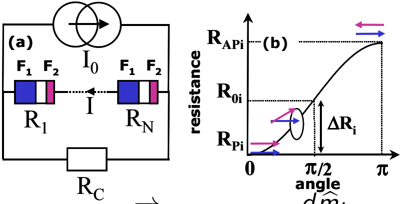
The inphase and antiphase synchronization regions of two similar STNOs having same anisotropy field $\kappa = 45.0$ Oe placed in the oscillating external magnetic field of strength $h_{ac} = 10$ Oe and all the parameters remains same

Synchronization of an array of STNOs



The time series and phase space plots of an array of 100 nonidentical STNOs showing the in-phase synchronization for external magnetic field of strength $h_{dc} = 130$ Oe, external current $a = 300$ Oe and anisotropy strength κ_i, i = 1,2,...100 distributed randomly between 45 to 55 Oe.

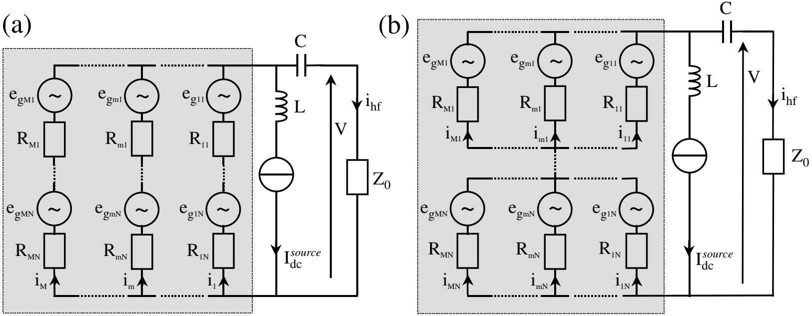
Current coupling : array of STNOs



$$\frac{d\widehat{m}_j}{dt} = -\gamma_0 \widehat{m}_j \times \vec{H}_{eff} + \alpha \widehat{m}_j \times \frac{d\widehat{m}_j}{dt}$$
$$+ \gamma_0 J[1 + \sum_{i=1}^{N} \beta_{\Delta Ri} \, cos(\theta_i(t)] \, \widehat{m}_j \times (\widehat{m}_j \times \widehat{M})$$

J. Grollier, V. Cros, and A. Fert, *Phys. Rev. B* , **73** 060409(R) (2006)
J. Turtle et al : Gluing Bifurcations in coupled STNOs, J. Appl. Phys. **113** 114901 (2013)

Hybrid arrays of STNOs (N x M)



B. Georges, J. Grollier, V. Cros, and A. Fert, *Appl. Phys. Lett.*, **92** 232504 (2008)

Coupled phase oscillators

- The synchronization of coupled STNOs via external ac field $\Rightarrow$ the energy injected from the external ac field H_{ac} to the ith STNO

$$E_i = -\mu_0 M_S V_0 \int \mathbf{H_{ac}.dm_i} \qquad (3)$$

- m_i - orbit of the small amplitude in-plane oscillation, μ_0 - vacuum permeability, V_0 - volume of the free layer.
- Phase dynamics of the $i_t h$ STNO

$$\dot{\theta}_i = \omega_i - \frac{\sigma}{N} \sum_{j=1}^{N} \sin(\theta_i - \theta_j + \alpha), i = 1, 2, \ldots, N, \qquad (4)$$

- α is the phase shift.

Jian-qing Xu and Guojun Jin, *J. Appl. Phys.*, **111** 066101 (2012)

Coupled phase oscillators

- The phase of the oscillator (i, η) in the hybrid array

$$\dot{\theta}_i^{(\eta)} = \omega_i^{(\eta)} - \sum_{\eta'=1}^{N'} \frac{\sigma_{\eta\eta'}}{N_{\eta'}} \sum_{j=1}^{N_{\eta'}=N} \sin(\theta_i^{(\eta)} - \theta_j^{(\eta')} + \alpha_{\eta\eta'}) + \zeta_i^{(\eta)}(t),$$

$$i = 1, 2, \ldots, N_{\eta'} = N, \quad (5)$$

- η parallel branches have each N STNOs connected in series.
- $\sigma_{\eta\eta'}$ is the strength of the coupling between the STNO in η' and those in η.
- $\zeta_i^{(1,2)}$ are independent Gaussian white noises

N- Coupled Oscillators- Kuromoto Model:

- $\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N} sin(\theta_j - \theta_i), \; i = 1, 2, ...N$
- Consider frequency distribution as a unimodal function $g(\omega) = g(-\omega)$.
- Global Coupling $\Longrightarrow$ Mean field approximation.
- Define the complex order parameter

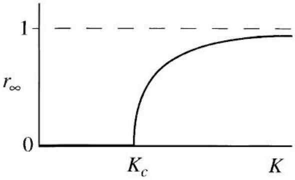
$$re^{i\psi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}$$

$\Longrightarrow$ $r(t)$: A measure of phase coherence

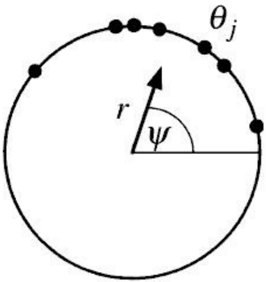
$\psi(t)$: Average phase (Strogatz, Physica D 2000)

N- Coupled Oscillators- Kuromoto Model:

- $r = \sqrt{1 - \frac{K_c}{K}}$ for Lorenzian distribution $g(\omega) = \frac{r}{\pi(\gamma^2+\omega^2)}$
- Second order phase transition:
- $\dot{\theta}_i = \omega_i + Kr\sin(\psi - \theta_i), \; i = 1, 2, 3, ..., N$



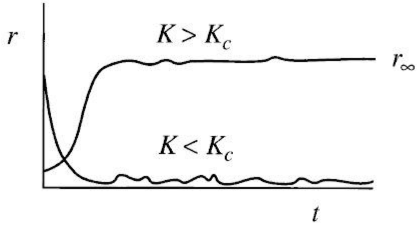
N- Coupled Oscillators- Kuromoto Model:



- $r = 1$: Synchrony
  $0 < r < 1$- Partial synchronization
  $r = 0$ - Desynchronization(phase drift)

N- Coupled Oscillators- Kuromoto Model:

- Kuromoto: For $r = constant$, the threshold condition for synchrony is $K \geq K_c$, $K_c = \frac{2}{\pi g(0)}$



Synchronization

- Synchronization of fireflies

Synchronization

- Delay coupling

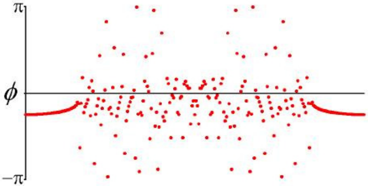
$$\frac{d\phi_i(t)}{dt} = \omega_0 + K \sum_j \sin\left[\phi_j(t-\tau) - \phi_i(t)\right]$$

- Nonlocal coupling

$$\frac{d\phi(t)}{dt} = \omega - \int_{-\pi}^{\pi} G(x-x^{'}) \sin\left[\phi(x,t) - \phi(x^{'},t) + \alpha\right] dx^{'}$$

$$G(y) = \frac{k}{2} exp(-K|y|)$$

- Chimera states: Simultaneous existence of coherent and incoherent states.



Coupled Phase Oscillators: $N \to \infty$

- Kuromoto model of coupled phase oscillators:

$$\frac{d\theta_i}{dt} = \omega_i + \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_j(t) - \theta_i(t))$$

- In the $N \to \infty$ limit, the state of the oscillator system can be described by a continuous distribution function $f(\omega, \theta, t)$

Coupled Phase Oscillators: $N \to \infty$

- Continuity equation

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial t}(vf) = 0$$

$\Longrightarrow$

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \theta}\left[\omega + \frac{K}{2i}\, re^{-i\theta} - r^* e^{i\theta}\right] f \, = 0$$

where

$$r(t) = \int_0^{2\pi} d\theta \int_{-\infty}^{\infty} d\omega \; e^{i\theta} f(\theta, \omega, t) \quad [|r(t)| \leq 1]$$

Ott - Antonsen Ansatz:

- Expanding $f(\theta, \omega, t)$ as a Fourier series in θ

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \theta}\left[\omega + \frac{K}{2i}\left(re^{-i\theta} - r^* e^{i\theta}\right)\right] f = 0$$

where

$$r(t) = \int_0^{2\pi} d\theta \int_{-\infty}^{\infty} d\omega \; e^{i\theta} f(\theta, \omega, t) \quad (|r(t)| \leq 1)$$

Ott - Antonsen Ansatz:

- Consider a restricted class of $f_n(\omega, t)$: (Ott & Antonsen, Chaos 2009)

$$f_n(\omega, t) = [\alpha(\omega, t)]^n \, , |\alpha(\omega, t)| \leq 1$$

- Then

$$\begin{aligned} r &= \int_0^{2\pi} d\theta \int_{-\infty}^{\infty} d\omega \;\; f \;\; e^{i\theta} \\ &= \int_{-\infty}^{\infty} d\omega \int_0^{2\pi} d\theta \left(e^{i\theta} + \sum_{n=1}^{\infty} \alpha^n e^{i(n+1)\theta} + \sum_{n=1}^{\infty} (\alpha^*)^n e^{-i(n-1)\theta} \right) \\ &= \int_{-\infty}^{\infty} d\omega \;\; g(\omega) \;\; \alpha^*(\omega, \; t) \end{aligned}$$

Ott - Antonsen Ansatz:

- Then the continuity equation becomes

$$\sum_{n=1}^{\infty}\left[n\alpha^{n-1}\frac{\partial\alpha}{\partial t}\quad e^{in\theta}+c.c\right]$$
$$+[\frac{-K}{2}\left(r\ e^{-i\theta}+r^{*}\ e^{i\theta}\right)$$
$$\left(1+\sum_{n=1}^{\infty}\alpha^{n}\ e^{in\theta}+c.c\right)+\omega\left(\sum_{n=1}^{\infty}in\alpha^{n}\ e^{in\theta}+c.c\right)+$$
$$\frac{K}{2i}\sum_{n=1}^{\infty}\left(n\ r\alpha^{n}\ e^{i(n-1)\theta}+c.c\right)-\frac{K}{2i}\sum_{n=1}^{\infty}\left(r^{*}\ n\ \alpha^{*n}\ e^{i(n+1)\theta}+c.c\right)]$$

Macroscopic Equations:

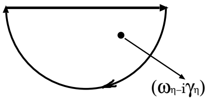
- Equating coefficients of powers of $e^{in\theta}$ and c.c:

$$\frac{\partial \alpha}{\partial t} + \frac{K}{2}(r\ \alpha^2 - r^*) + i\omega\ \alpha = 0$$

- But $r(t) = \int_{-\infty}^{\infty} d\omega\ g(\omega)\ \ \alpha^*(\omega, t)$
- Let

$$\begin{aligned} \textit{Lorentzian} \quad g(\omega) &= (\frac{\Delta}{\pi})\frac{1}{[(\omega-\omega_0)^2+\Delta^2]} \\ &= \frac{1}{\pi}\left[\frac{1}{(\omega-\omega_0-i\Delta)} - \frac{1}{(\omega-\omega_0+i\Delta)}\right] \end{aligned}$$

Macroscopic Equations:



- Then

$$r^* = \int_{-\infty}^{\infty} d\omega \; \alpha(\omega, t) \; \frac{\Delta}{2\pi} \; \frac{1}{[(\omega - \omega_0)^2 + \Delta^2]}$$
$$= \frac{-1}{2\pi i} \oint_c d\omega \; \frac{\alpha(\omega, t)}{(\omega - \omega_0 + i\Delta)}$$
$$= \alpha(\omega_0 - i\Delta, \; t)$$

- By changing the variables $(\theta, \omega) \longrightarrow \left[\theta - \omega_0(t), \; \frac{\omega - \omega_0}{\Delta}\right]$
- we can set $\omega_0 = 0$, $\Delta = 1$

Macroscopic Equations:

- $r(t) = \alpha^*(-i,\ t)$

  $\Longrightarrow \frac{dr}{dt} + \frac{K}{2}(r^* r^2 - r) + r = 0$

  with $r(t) = \rho(t)\ e^{i\phi}$

  $\Longrightarrow \frac{d\rho}{dt} + \frac{K}{2}(\rho^2 - 1)\rho + \rho = 0$

$$\frac{d\rho}{dt} + \left(1 - \frac{1}{2}K\right)\rho + \frac{K}{2}\rho^3 = 0$$

$$\dot{\phi}_t = 0$$

Synchronized/ desynchronized states

- $$\frac{\rho(t)}{R} = \left[1 + \left\{\left(\frac{R}{\rho(0)}\right) - 1\right\}\ e^{(1-\frac{1}{2}K)t}\right]^{\frac{-1}{2}}$$

- where $R = (1 - \frac{2}{K})^{\frac{1}{2}}$

- For $K < K_c = 2, \quad r \longrightarrow 0 \ \ as \ \ t \rightarrow \infty$

- For $K > 2, \quad r \rightarrow (1 - \frac{2}{K})^{\frac{1}{2}}$

Synchronized/ Desynchronized states

- Linear Stability
- $\rho = \rho_0 + \xi(t) \Longrightarrow \frac{d\xi}{dt} + \left(1 - \frac{1}{2}K\right)\xi = 0$
- $\xi = c' \; e^{-(1-\frac{K}{2})t}$
- (i) $\rho = 0$: Stable for $K < 2$, Unstable for $K > 2$.
- (ii) $\rho = \rho_c = \left(1 - \frac{2}{K}\right)^{\frac{1}{2}}$: $\frac{d\xi}{dt} + \left(1 - \frac{K}{2}\right)\xi + 3\rho_e^2 \;\; \xi = 0$

$\Longrightarrow \frac{d\xi}{dt} + (K-2)\xi = 0$

$\Longrightarrow \xi = c \; e^{-(K-2)t}$

$\Longrightarrow \rho_e$ is stable for $K > 2$.

Conclusion

- The topic of dynamics of Heisenberg spin chain is an unending source of nonlinear dynamical systems - integrable (maps, odes, pdes, etc) and nonintegrable.
- What has been understood so far is only a tiny part of the general system.
- Many more exciting problems remain to be tackled.