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Analytic and number theoretic detectors of integrability

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Overview

PART 1: Introduction to Nevanlinna theory

(a) General theory

(b) Applications to differential equations

PART 2: Detecting integrability in discrete systems

(a) Singularity confinement

(b) Measures of complexity in discrete systems

i. Growth of meromorphic solutions (Nevanlinna theory)

ii. Diophantine integrability

iii. Algebraic entropy

PART 1: Introduction to Nevanlinna theory

(a) General theory

- Polynomial: $P(z) = a_0 + a_1 z + \cdots + a_d z^d$.
  If $a_d \neq 0$, then the degree of P is $\deg(P) = d$.

- Rational functions:
$$R(z) = \frac{P(z)}{Q(z)},$$
  where P and Q are polynomials.
  If P and Q have no common factors, then the degree of R is
$$\deg(R) = \max\{\deg(P), \deg(P)\}.$$

Entire and meromorphic functions

- A complex function f is called entire if it is differentiable at all $z \in \mathbb{C}$. This is equivalent to saying that there is a power series $\sum_{n=0}^{\infty} a_n z^n$, with infinite radius of convergence, such that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

for all $z \in \mathbb{C}$.

- Examples of entire functions: constant functions, polynomials, e^z, $\sin z$, $\cos z$.
- A function is said to be meromorphic (on $\mathbb{C}$) if it is analytic everywhere except at poles.
- Equivalently, a function is meromorphic if and only if it can be written as $g(z)/h(z)$, where g and h are entire.
- Examples of meromorphic functions: entire fns, rational fns, $\tan z$, $\wp(z)$.
- Examples of non-meromorphic functions:

$$\sqrt{z}, \quad \log z, \quad \exp(1/z).$$

Jensen's formula

- Suppose f is analytic and nowhere vanishing in the disc $D = \{z : |z| \leq r\}$.
- Then $\log f(z)$ is analytic in D.
- Cauchy's integral formula for $\log f$ gives

$$\log f(0) = \frac{1}{2\pi i} \int_{|z|=r} \frac{\log f(z)}{z} dz.$$

- On taking the real part we have

$$\log |f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log \left| f(re^{i\theta}) \right| \, d\theta.$$

Jensen's formula

- Now let f be any meromorphic function.
- For simplicity we assume that $f(0) \neq 0$ or ∞.
- Then f has finitely many zeros $a_1, \ldots, a_m$ and poles $b_1, \ldots, b_n$ in D.
- Then the function

$$g(z) := \frac{\prod B(a_j, z)}{\prod B(b_k, z)} f(z),$$

where

$$B(a, z) = \frac{r^2 - \bar{a}z}{r(z - a)},$$

has no zeros or poles in D.

- From $\log|g(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log\left|g(re^{i\theta})\right| \, d\theta$, we obtain Jensen's formula:

$$\log|f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log\left|f(re^{i\theta})\right| \, d\theta + \sum \log \frac{r}{|b_k|} - \sum \log \frac{r}{|a_j|}.$$

A symmetric form of Jensen's formula

- For any $x > 0$, define $\log^+ x := \max(\log x, 0)$.
- Then $\log x = \log^+ x - \log^+(x^{-1})$.
- Jensen's formula can now be written as

$$\frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f(re^{i\theta})\right| \, d\theta + \sum \log \frac{r}{|b_k|}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|\frac{1}{f(re^{i\theta})}\right| \, d\theta + \sum \log \frac{r}{|a_j|} + \log|f(0)|.$$

- Define the *proximity function* to be $m(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta.$
- The *enumerative function* is $N(r, f) := \int_0^r \frac{n(t, f)}{t} dt$,
  where $n(r, f)$ is the number of poles of f (counting multiplicities) in $|z| \leq r$.

The Nevanlinna characteristic

- The *proximity function* is $m(r,f) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta$, where $\log^+ x := \max(\log x, 0)$.
- The *enumerative function* is $N(r,f) := \displaystyle\int_0^r \frac{n(t,f)}{t} dt$, where $n(r,f)$ is the number of poles of f (counting multiplicities) in $|z| \le r$.
- The *Nevanlinna characteristic function* $T(r,f) = m(r,f) + N(r,f)$ measures "the affinity" of f for infinity.
- Similarly, $T\left(r, \dfrac{1}{f-a}\right) = m\left(r, \dfrac{1}{f-a}\right) + N\left(r, \dfrac{1}{f-a}\right)$

  measures "the affinity" of f for the value a.
- Jensen's formula becomes

$$T(r,f) = T(r, 1/f) + \log|f(0)|.$$

Elementary properties of $\log^+$

$$\log^+\left(\prod_{j=1}^{q} a_j\right) \le \sum_{j=1}^{q} \log^+ a_j,$$

$$\log^+\left(\sum_{j=1}^{q} a_j\right) \le \log^+\left(q \max_{1\le j\le q} a_j\right) \le \sum_{j=1}^{q} \log^+ a_j + \log q,$$

$$\log a = \log^+ a - \log^+(1/a),$$

$$|\log a| = \log^+ a + \log^+(1/a),$$

$$\log^+ a \le \log^+ b, \qquad \forall a \le b.$$

Elementary properties of the Nevanlinna functions

$$n\left(r, \sum_{j=1}^{q} f_j\right) \leq \sum_{j=1}^{q} n(r, f_j), \qquad n\left(r, \prod_{j=1}^{q} f_j\right) \leq \sum_{j=1}^{q} n(r, f_j),$$

$$N\left(r, \sum_{j=1}^{q} f_j\right) \leq \sum_{j=1}^{q} N(r, f_j), \qquad N\left(r, \prod_{j=1}^{q} f_j\right) \leq \sum_{j=1}^{q} N(r, f_j),$$

$$m\left(r, \sum_{j=1}^{q} f_j\right) \leq \sum_{j=1}^{q} m(r, f_j) + \log q, \quad m\left(r, \prod_{j=1}^{q} f_j\right) \leq \sum_{j=1}^{q} m(r, f_j),$$

$$T\left(r, \sum_{j=1}^{q} f_j\right) \leq \sum_{j=1}^{q} T(r, f_j) + \log q, \quad T\left(r, \prod_{j=1}^{q} f_j\right) \leq \sum_{j=1}^{q} T(r, f_j).$$

Nevanlinna's First Main Theorem

(Nevanlinna's First Main Theorem)

For any merormorphic function f and any $a \in \mathbb{C}$, we have

$$T\left(r, \frac{1}{f-a}\right) = T(r,f) + O(1), \qquad r \to \infty,$$

where $f \not\equiv a$.

Proof:

$$T(r, f-a) \leq T(r,f) + T(r,a) + \log 2.$$

Similarly

$$T(r,f) \leq T(r, f-a) + T(r,a) + \log 2.$$

Hence

$$|T(r,f-a) - T(r,f)| \leq T(r,a) + \log 2 = \log^+ |a| + \log 2.$$

So

$$\begin{aligned} \left| T(r,f) - T\left(r, \frac{1}{f-a}\right)\right| &\leq |T(r,f) - T(r,f-a)| + \left|T(r,f-a) - T\left(r, \frac{1}{f-a}\right)\right| \\ &\leq \log^+ |a| + \log 2 + \log^+ |f(a)|. \end{aligned}$$

□

Summary of the story so far

- The *proximity function* is $m(r,f) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta$, where $\log^+ x := \max(\log x, 0)$.
- The *enumerative function* is $N(r,f) := \displaystyle\int_0^r \frac{n(t,f)}{t} dt$, where $n(r,f)$ is the number of poles of f (counting multiplicities) in $|z| \leq r$.
- The *Nevanlinna characteristic function* $T(r,f) = m(r,f) + N(r,f)$ measures "the affinity" of f for infinity.
- Nevanlinna's First Main Theorem

  For $a \in \mathbf{C}$,

$$T\left(r, \frac{1}{f-a}\right) = T(r,f) + O(1), \qquad r \to \infty.$$

- The function $\exp(z)$ is never 0 or ∞ but it stays near these values on large parts of the circle $|z| = r$ for $r >> 1$.

The order of an meromorphic function

- For any meromorphic function f, $T(r,f)$ is continuous and nondecreasing.
- The *order* of a meromorphic function f is

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r},$$

- For an entire function, $T(r,f)$ behaves like $\log M(r,f)$ where $M(r,f) = \max_{|z|=r} |f(z)|$.
- **Theorem**

  Let f be a non-constant entire function. Let $r > 0$ be sufficiently large that $M(r,f) := \max_{|z|=r} |f(z)| \geq 1$. Then for all finite $R > r$ we have

$$T(r,f) \leq \log M(r,f) \leq \frac{R+r}{R-r} T(R,f).$$

Functions with a finite number of poles

- A meromorphic function has a finite number of poles if and only if $N(r, f) = O(\log r)$.
- A meromorphic function f is rational if and only if $T(r, f) = O(\log r)$.
- For any transcendental function f, we have $\log r = o(T(r, f))$.

The Lemma on the Logarithmic Derivative

We use $S(r, f)$ to denote any function of r that is $o(T(r, f))$ outside some set of finite linear measure.

Lemma on the Logarithmic Derivative

Let f be a nonconstant meromorphic function. Then

$$m(r, f'/f) = S(r, f).$$

Furthermore, if f has finite order then

$$m(r, f'/f) = O(\log r).$$

There are many methods available to deal with the *exceptional sets* that arise in Nevanlinna theory.

One simple corollary of the lemma is that $T(r, f') \leq 2T(r, f)$.

Application to the first Painlevé equation

Let y be a transcendental meromorphic solution of the first Painlevé equation,

$$P_{\mathrm{I}}: \qquad y'' = 6y^2 + z.$$

Then

$$y^2 = 6^{-1}\left(y\frac{y''}{y} - z\right).$$

An obvious property of the proximity function m is $m(r, y^2) = 2m(r, y)$. Hence

$$\begin{aligned}
2m(r, y) = m(r, y^2) = m\left(r, 6^{-1}\left(y\frac{y''}{y} - z\right)\right) &\leq m(r, 6^{-1}) + m\left(r, y\frac{y''}{y} - z\right) \\
&\leq m\left(r, y\frac{y''}{y}\right) + m(r, z) + \log 2 \\
&\leq m(r, y) + m\left(r, \frac{y''}{y}\right) + O(\log r) \\
&= m(r, y) + S(r, y) + O(\log r).
\end{aligned}$$

So

$$m(r, y) = S(r, y) + O(\log r).$$

Application to the first Painlevé equation

- Let y be a transcendental meromorphic solution of the first Painlevé equation,

$$P_{\mathrm{I}}: \qquad y'' = 6y^2 + z.$$

- Then $m(r,y) = S(r,y) + O(\log r)$.
- Suppose that y has only finitely many poles. Then $N(r,y) = O(\log r)$.
- Therefore

$$T(r,y) = m(r,y) + N(r,y) = S(r,y) + O(\log r).$$

- Recall that if y is transcendental then $\log r = o(T(r,y))$.
- Therefore our solution y satisfies $T(r,y) = S(r,y)$, which means that $T(r,y) = o(T(r,y))$ as $r \to \infty$ outside of some possible exceptional set E of finite linear measure, which is clearly a contradiction.

A useful identity

(Valiron-Mohon'ko))

Let

$$R(z, f(z)) := \frac{a_0(z) + a_1(z)f(z) + \cdots + a_p(z)f^p(z)}{b_0(z) + b_1(z)f(z) + \cdots + b_q(z)f^q(z)},$$

be a rational function of f of degree $d = \max(p, q)$ with coefficients a_i and b_j satisfying

$$T(r, a_i) = S(r, f) \text{ and } T(r, b_j) = S(r, f).$$

Then

$$T(r, R(z, f(z))) = d\,T(r, f) + S(r, f).$$

Malmquist's theorem

(Malmquist's theorem)

Let f be a meromorphic solution of the equation

$$f'(z) = R(z, f(z)) := \frac{a_0(z) + a_1(z)f(z) + \cdots + a_p(z)f^p(z)}{b_0(z) + b_1(z)f(z) + \cdots + b_q(z)f^q(z)}, \tag{1}$$

where the coefficients a_i and b_j satisfy

$$T(r, a_i) = S(r, f) \text{ and } T(r, b_j) = S(r, f),$$

and the degree of R as a function of f is $d = \max(p, q)$. Then equation (1) is the Riccati equation

$$f'(z) = a_0(z) + a_1(z)f(z) + a_2(z)f^2(z).$$

Proof: (Yosida)

Using the Valiron-Mohon'ko theorem and the fact that $(T(r, f') \leq 2T(r, f)$, we have

$$d\,T(r, f) + S(r, f) = T(r, R(z, f(z))) = T(r, f') \leq 2T(r, f) + S(r, f).$$

Hence $d \leq 2$.

Proof of Malmquist's theorem

- We have shown that if $f' = P(z,f)/Q(z,f)$, where P and Q are relatively prime in f, then $P(z,f) = a_0(z) + a_1(z)f + a_2(z)f^2$ and $Q(z,f) = b_0(z) + b_1(z)f + b_2(z)f^2$.
- It remains to show that Q is independent of f.
- Without loss of generality we assume that $a_0(z) \not\equiv 0$.
- Now the function $g := 1/f$ satisfies the equation

$$g' = -\frac{g^2(a_0g^2 + a_1g + a_2)}{b_0g^2 + b_1g + b_2} = \widetilde{R}(z,g) = \frac{\widetilde{P}(z,g)}{\widetilde{Q}(z,g)},$$

  where $\widetilde{P}(z,g) = -g^2(a_0g^2 + a_1g + a_2) = -g^4P(z,1/g)$ and $\widetilde{Q}(z,g) = g^2Q(z,1/g)$.
- From the First Main Theorem we have $T(r,g) = T(r,f) + O(1)$. Hence $T(r,a_i) = S(r,g)$ and $T(r,b_j) = S(r,g)$.
- So $\widetilde{R}$ has degree at most 2. Therefore two of the roots (counting multiplicites) of the quartic polynomial $\widetilde{P} = -g^4P(z,1/g)$ must be shared by $\widetilde{Q}(z,g) = z^2Q(z,1/g)$.
- Recall that P and Q are relatively prime and 0 is not a root of P (since $a_0 \not\equiv 0$). So g^2 must divide $\widetilde{Q}(z,1/g)$. Hence $b_1 = b_2 = 0$. $\square$

Nevanlinna's second main theorem

- **Nevanlinna's second main theorem**

  Let f be a nonconstant meromorphic function. For $q \geq 2$, let $a_1, \ldots, a_q \in \mathbb{C}$ be q distinct points. Then

$$(q-1)T(r,f) \leq N(r,f) + \sum_{j=1}^{q} N\left(r, \frac{1}{f-a_j}\right) - N_{\mathrm{ram}}(r,f) + S(r,f),$$

  where $N_{\mathrm{ram}}(r,f) = 2N(r,f) - N(r,f') + N(r,1/f')$.

- Using the ramification term, we have the following immediate corollary:

$$(q-1)T(r,f) \leq \bar{N}(r,f) + \sum_{j=1}^{q} \bar{N}\left(r, \frac{1}{f-a_j}\right) + S(r,f),$$

  where $\bar{N}(r,f)$ counts poles ignoring multiplicities.

- Corollary: **Picard's theorem**

  Let f be a meromorphic function missing three distinct values in $\mathbb{C} \cup \{\infty\}$. Then f is a constant.

  Let f be a meromorphic function which takes each of three distinct values in $\mathbb{C}\cup\{\infty\}$ at most finitely many times. Then f is rational.

Other corollaries/analogues

- Defect relations
- Totally ramified values
- Shared values
- New direction: Yamanoi
- Replacing f' by more general linear operators
- Vojta's dictionary

Summary of part 1(a): General Nevanlinna theory

- Nevanlinna characteristic:

$$T(r,f) = m(r,f) + N(r,f)$$

- First Main Theorem:

$$T\left(r, \frac{1}{f-a}\right) = T(r,f) + O(1), \qquad r \to \infty,$$

- Lemma on the lemma on the logarithmic derivative:

$$m(r, f'/f) = S(r,f).$$

- Simple applications to differential equations
- Second Main Theorem:

$$(q-1)T(r,f) \leq \bar{N}(r,f) + \sum_{j=1}^{q} \bar{N}\left(r, \frac{1}{f-a_j}\right) + S(r,f),$$

PART 1(b): Applications to differential equations

Clunie's Lemma

Let f be a transcendental solution of

$$f^N P(z,f) = Q(z,f),$$

where P and Q are differential polynomials in f with coefficients that are $S(r,f)$. If the total degree of Q is no greater than N, then

$$m\,(r, P(z,f)) = S(r,f).$$

- In particular, if the coefficient functions are rational and f is transcendental meromorphic, then $m\,(r, P(z,f)) = S(r,f)$.
- The result we derived earlier about the first Painlevé equation, $f'' = 6f^2 + z$, now follows immediately.

Mohonko's Theorem

- (A. Mohon'ko and V. Mohon'ko)
  Let f be a transcendental meromorphic solution of

$$P(z; f, f', \ldots, f^{(n)}) = 0, \tag{2}$$

  where P is a nonzero polynomial in all of its arguments. If the constant $a \in \mathbb{C}$ does not solve equation (2), then

$$m\left(r, \frac{1}{f-a}\right) = S(r, f).$$

- Application to P_I.

Second-order equations

- $y'' = P(z, y)$.
- $y'' = 6y^2 + f(z)$.
- $y'' = 2y^3 + f(z)y + g(z)$.

Extending Painlevé analysis to find particular solutions

- Suppose that a solution of
$$\frac{d^2y}{dz^2} = 6y^2 + f(z)$$
has a pole at a point z_0 where f is analytic.

- The series expansion of the solution is necessarily of the form
$$y(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^{n-2}, \qquad a_0 = 1.$$

- Substituting and equating coeffs gives $a_1 = a_2 = a_3 = 0$ and the recurrence relation
$$(n+1)(n-6)a_n = 6 \sum_{m=1}^{n-1} a_m a_{n-m} + \frac{1}{(n-4)!} f^{(n-4)}(z_0).$$

- There is a resonance at $n = 6$ which gives $f''(z_0) = 0$. If this is true for "enough" z_0 then
$$\frac{d^2y}{dz^2} = 6y^2 + Az + B,$$
where A and B are constants.

Using Nevanlinna theory to find particular solutions

- Nevanlinna theory has been used to find all solutions of Hayman's equation,

$$ww'' - w'^2 = \alpha(z)w + \beta(z)w' + \gamma(z)$$

  that are

  1. meromorphic, when α, β and γ are constants. (with Yik Man Chiang); and
  2. *admissible* meromorphic, when α, β and γ are meromorphic. (with Jun Wang)

- Further generalisations (with Khadija Al-Amoudi).

Algebroid solutions

- A function f is called *algebroid* if it is algebraic over the meromorphic functions, i.e., it satisfies

$$a_0(z) + a_1(z)f(z) + \cdots + a_{n-1}(z)f(z)^{n-1} + f(z)^n = 0,$$

for meromorphic functions $a_0, \ldots, a_{n-1}$.

- Malmquist actually showed that if $F(z, y, y') = 0$ has an algebroid solution, where F is rational, then the equation can be reduced to either a Riccati equation or the equation for the Weierstrass elliptic function.

- Thomas Kecker and I have shown that the only *admissible* degree 2 algebroid solutions of

$$y'' = c_0(z) + \cdots + c_4(z)y^4 + y^5$$

can be expressed in terms of either admissible solutions of Riccati equations or the fourth Painlevé equation (or its degenerations).

Differential equations

- **The Painlevé property**

  An ODE is said to possess the Painlevé property if all solutions are single-valued about all movable singularities.

- The only equation with this property of the form

$$\frac{dy}{dz} = R(z; y),$$

where R is rational in y, is the Riccati equation

$$\frac{dy}{dz} = p(z)y^2 + q(z)y + r(z).$$

The general solution is given by

$$y(z) = -\frac{1}{p(z)}\frac{w'(z)}{w(z)},$$

where

$$p(z)\frac{d^2w}{dz^2} - \left[\frac{d\,p(z)}{dz} + p(z)q(z)\right]\frac{dw}{dz} + r(z)p^2(z)w = 0.$$

The Painlevé Property and Integrability

An ODE is said to possess the Painlevé property if all solutions are single-valued about all movable singularities.

- Kowalevskaya (classical top)
- Painlevé, Gambier, Fuchs (classification)

$$y'' = F(y, y'; z)$$

- There are six Painlevé eqns. The first two are

$$P_I \quad y'' = 6y^2 + z$$
$$P_{II} \quad y'' = 2y^3 + zy + \alpha.$$

- Ablowitz, Ramani and Segur conjecture:

All ODE reductions of equations solvable by the inverse scattering transform (IST) possess the Painlevé property (possibly after a transformation of variables).

PART 2: Detecting integrability in discrete systems

- Singularity confinement
- Measures of complexity in discrete systems
  - Growth of meromorphic solutions (Nevanlinna theory)
  - Diophantine integrability
  - Algebraic entropy

Discrete equations: Singularity confinement

Grammaticos, Ramani and Papageorgiou (1991);

Ramani, Grammaticos and Hietarinta (1991)

$$y_{n+1} + y_{n-1} = \frac{a_n + b_n y_n}{1 - y_n^2}$$

$$\begin{aligned}
y_{n-1} &= k + o(1), \\
y_n &= \theta + \epsilon, \qquad\qquad \theta = \pm 1 \\
y_{n+1} &= -\frac{a_n + \theta b_n}{2\theta}\epsilon^{-1} + O(1), \\
y_{n+2} &= -\theta + \frac{2\theta b_{n+1} - \theta b_n - a_n}{a_n + \theta b_n}\epsilon + O(\epsilon^2), \\
y_{n+3} &= \frac{a_n + \theta b_n}{2\theta}\left\{\frac{(a_{n+2} - a_n) - \theta(b_{n+2} - 2b_{n+1} + b_n)}{\theta(2b_{n+1} - b_n) - a_n}\right\}\epsilon^{-1} + O(1).
\end{aligned}$$

Confinement:

$$y_{n+1} + y_{n-1} = \frac{\alpha + \beta(-1)^n + (\gamma n + \delta) y_n}{1 - y_n^2}$$

Example of Hietarinta and Viallet

$$y_{n+1} + y_{n-1} = y_n + \frac{a}{y_n^2}$$

$$\begin{aligned}
y_{n-1} &= k + o(1), \\
y_n &= \epsilon, \\
y_{n+1} &= \epsilon^{-2} - k + \epsilon + O(\epsilon^2), \\
y_{n+2} &= \epsilon^{-2} - k + \epsilon^4 + O(\epsilon^5), \\
y_{n+3} &= -\epsilon + 2\epsilon^4 + O(\epsilon^5), \\
y_{n+4} &= k + o(1).
\end{aligned}$$

First-Order Difference Equations

- Consider the difference equation

$$y(z+1) = R(y(z)). \tag{3}$$

- If R is rational then equation (3) admits a non-constant meromorphic solution.
- If R is polynomial then equation (3) admits a non-constant entire solution.
- An immediate consequence of this theorem is that the Logistic map,

$$y(z+1) = \alpha y(z)(1 - y(z)),$$

  has a non-constant entire solution, $y(z) = w(z)$.
- The logistic map has a family of entire solutions:

$$y(z) = w(z - p(z)), \quad \text{where } p \text{ is periodic.}$$

- Nevanlinna theory provides a concept of "nice" meromorphic functions: *functions of finite order.*

Nevanlinna Theory

- Nevanlinna characteristic $T(r, f)$.
- For an entire function f,

$$T(r, f) \sim \log M(r, f), \quad M(r, f) = \max_{|z|=r} |f(z)|.$$

- More generally, for a meromorphic function f,

$$T(r, f) = m(r, f) + N(r, f),$$

where $m(r, f)$ is a measure of how large f is on $|z| = r$ and $N(r, f)$ is a measure of how many poles f has in $D_r := \{z : |z| \leq r\}$.

- The order of f is $\limsup_{r \to \infty} \dfrac{\log(T(r, f))}{\log r}$.
- Examples of finite-order meromorphic functions:
  e^z, $\cos z$, $\tan z$, $\wp(z)$.
- Infinite-order:
  $\exp(\exp z)$.

Difference equations of Painlevé type

- (Ablowitz, H, Herbst) An analogue of the Painlevé property for difference equations is the existence of sufficiently many finite-order meromorphic solutions.
- **Theorem** (Yanagihara) If the difference equation

$$y(z+1) = R(z, y(z)),$$

where

$$R(z,y) = \frac{a_0(z) + a_1(z)y + \cdots + a_p(z)y^p}{b_0(z) + b_1(z)y + \cdots + b_q(z)y^q},$$

admits a finite-order non-rational meromorphic solution, then $\max(p,q) \leq 1$.

- This gives the difference Riccati equation

$$y(z+1) = \frac{\alpha(z)y(z) + \beta(z)}{\gamma(z)y(z) + \delta(z)},$$

which is linearized by

$$y(z) = \frac{\alpha(z-1)}{\gamma(z-1)} \left[\frac{w(z) - w(z-1)}{w(z)} \right].$$

- Necessary conditions for higher order equations studied by
  - Yanagihara;
  - Ablowitz, H, and Herbst;
  - Heittokangas, Korhonen, Laine, Rieppo, Tohge;
  - Grammaticos, Tamizhmani, Ramani, Tamizhmani.
- None of the above results give information about the z-dependence of the coefficient functions in the equations.

Theorem (H. and Korhonen, 2007)

If the equation $$\overline{w} + \underline{w} = R(z,w), \qquad (\dagger)$$

has an admissible meromorphic solution of finite order, then either w satisfies the discrete Riccati eqn $\overline{w} = (\overline{p}\,w + q)/(w + p)$, or $(\dagger)$ can be transformed by a linear change of variables to one of the following equations:

$$\overline{w} + w + \underline{w} = \frac{\pi_1 z + \pi_2}{w} + \kappa_1$$

$$\overline{w} - w + \underline{w} = \frac{\pi_1 z + \pi_2}{w} + (-1)^z \kappa_1$$

$$\overline{w} + \underline{w} = \frac{\pi_1 z + \pi_3}{w} + \pi_2$$

$$\overline{w} + \underline{w} = \frac{\pi_1 z + \kappa_1}{w} + \frac{\pi_2}{w^2}$$

$$\overline{w} + \underline{w} = \frac{(\pi_1 z + \kappa_1)w + \pi_2}{(-1)^{-z} - w^2}$$

$$\overline{w} + \underline{w} = \frac{(\pi_1 z + \kappa_1)w + \pi_2}{1 - w^2}$$

$$\overline{w}w + w\underline{w} = p$$

$$\overline{w} + \underline{w} = p\,w + q$$

where p, q, π_k, κ_k are "small" functions and π_k and κ_k are periodic with period k.

Theorem (H. and Korhonen, 2006)

Let f be a finite-order meromorphic function and $c \in \mathbb{C}$. Then

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o(T(r,f)),$$

for all $r \geq 1$ and $\delta < 1$, outside of a possible exceptional set of finite logarithmic measure. A similar result was obtained by Chiang and Feng, 2007.

- This plays an important role in theclassification of difference equations

- Corollaries include difference analogues for finite-order functions of the following

  1. Clunie's lemma and the Mohon'ko lemma

  2. Nevanlinna's second main theorem, Picard's theorem, defect relations and Nevanlinna's five values theorem

- There is a q-difference version of all of the above

- Holomorphic curves version

- Generalisation to other linear operators

Finite-order solutions and singularity (non-)confinement

Recall that $n(r,y)$ is the number of poles of y in $\{z : |z| \le r\}$.

For any admissible meromorphic solution y of

$$y(z+1)+y(z-1)=\frac{a(z)y(z)+b(z)}{y^2(z)},$$

it can be shown that —

if $n(r,1/y) \le \alpha n(r+1,y)$, where $\alpha < 1$, then y has infinite-order.

- If y has a zero of order k at $z=z_0$, then it has a pole of order at least $2k$ at either z_0+1 or z_0-1.
- Let $Z(z_0)=(z_0-m,\ldots,z_0-1,z_0,z_0+1,\ldots,z_0+n)$ be the longest sequence such that y has a zero of order k at each z_0+2j and a pole of order at least $2k$ at each z_0+2j+1. Let $R = \sharp$ poles$/\sharp$ zeros in $Z(z_0)$ (counting multiplicities).
- If $Z(z_0)$ has an even number of points then $R \ge 2$.
- If $Z(z_0)$ has an odd number l of points then there are at least $(l-1)k/2$ poles and at most $(l+1)k/2$ zeros. So $R \ge 2(l-1)/(l+1) \ge 4/3$ if $l \ge 5$.

Singularity confinement

$$y(z+1)+y(z-1)=\frac{a(z)y(z)+b(z)}{y^2(z)}$$

- Recall that if $n(r,1/y)\leq \alpha n(r+1,y)$, where $\alpha<1$, then y has infinite-order.
- We have just seen that either the chain of zeros and poles

$$Z(z_0)=(z_0-m,\ldots,z_0-1,z_0,z_0+1,\ldots,z_0+n)$$

  has exactly three points, or

$$R=\frac{\sharp\text{ poles in }Z(z_0)\text{ (counting multiplicities)}}{\sharp\text{ zeros in }Z(z_0)\text{ (counting multiplicities)}}\geq\frac{4}{3}.$$

- Nevanlinna theory shows that there are "a lot of" (infinitely many, in particular) poles of y.
- So if y is of finite order then there must be infinitely many points z_* such that y has zeros of some order k at z_*+1 and z_*-1, a pole of order $2k$ at z_* and the points z_*+2 and z_*-2 are either regular or poles of order less than $2k$.
- This gives the first (of two) levels of confinement.

The example of Hietarinta and Viallet

- Hietarinta and Viallet showed that the equation

$$y_{n+1} + y_{n-1} = y_n + \frac{a}{y_n^2}.$$

appears to possess the singularity confinement property and yet it exhibits chaos.

$$k, \quad \epsilon, \quad a\epsilon^{-2} - k + \epsilon, \quad a\epsilon^{-2} - k + O(\epsilon^4), \quad -\epsilon + O(\epsilon^4), \quad k + O(\epsilon)$$

- Now suppose that y is a meromorphic solution of

$$y(z+1) + y(z-1) = y(z) + \frac{a}{y^2(z)},$$

satisfying $y(0) = 0$ and $y(-1) = k \neq 0, \infty$. Then for z near 0,

$$\begin{aligned} y(z-1) &= k + O(z), \quad y(z) = O(z), \quad y(z+1) = ay^{-2}(z) - k + y(z) \\ y(z+2) &= ay^{-2}(z) - k + O(y^4(z)) \\ y(z+3) &= -y(z) + O(y^4(z)) \\ y(z+4) &= y(z-1) + O(y(z)) \end{aligned}$$

- So even if all of the singularities of are "confined", we have

$$n(r, 1/y) \leq \frac{1}{2} n(r+1, y) \quad \Rightarrow \quad T(r, y) = T(r, 1/y) \leq \frac{1}{2} T(r+1, y).$$

Differential-delay equations

- Several differential-delay equations have been obtained as similarity reductions of integrable equations.
- In 1992, Quispel, Capel and Sahadevan obtained the equation

$$w(z)[w(z+1) - w(z-1)] = aw(z) + bw'(z).$$

- Other reductions differential-delay equations have been obtained by Levi and Winternitz, and Joshi.
- How special is the value distribution of solutions of these equations?

Diophantine integrability

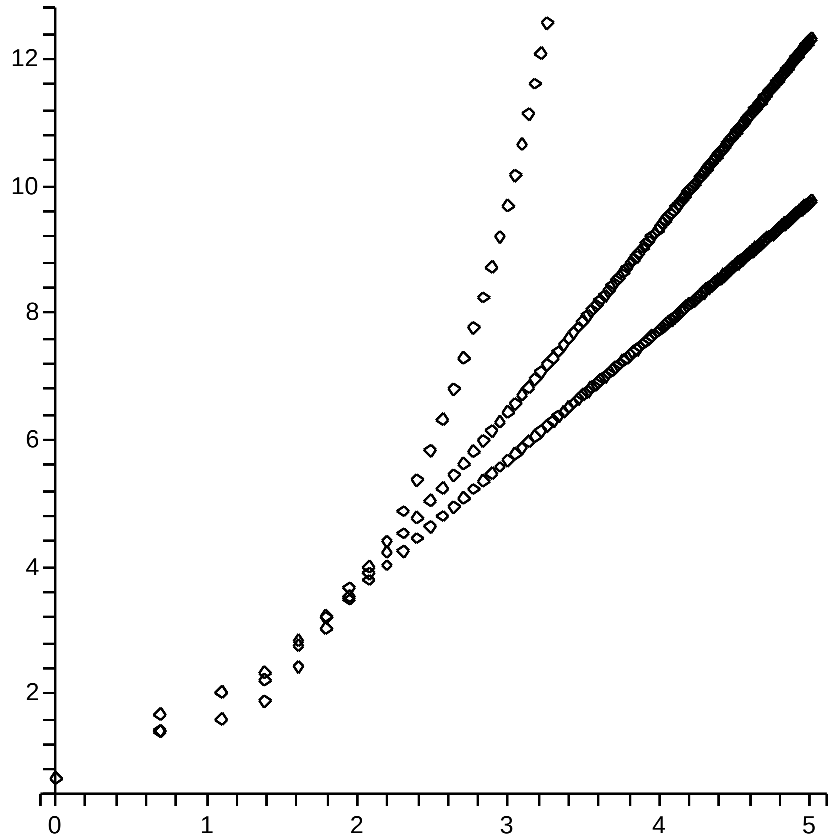
- Osgood noticed that there is a formal similarity between Nevanlinna theory and Diophantine approximation.
- Vojta devised a "dictionary" to translate defns and thms between these theories.
- Statements concerning the Nevanlinna characteristic of a meromorphic function correspond to statements about the heights of an infinite set of numbers.
- For $x = p/q \in \mathbb{Q}$, the height is $H(p/q) = \max(|p|, |q|)$.
- Next consider rational iterates of a discrete equation of the form

$$y_{n+1} + y_{n-1} = R(n, y_n). \qquad (A)$$

- Vojta's dictionary suggests the following definition.
  Equation (A) is *Diophantine integrable* if the logarithmic heights of the (rational) iterates y_n are bounded by a polynomial in n.
- This is very easy to check numerically.
- Heights have been used to numerically estimate the complexity of a map in Abarenkova, Anglès d'Auriac, Boukraa, Hassani and Maillard, 1999.

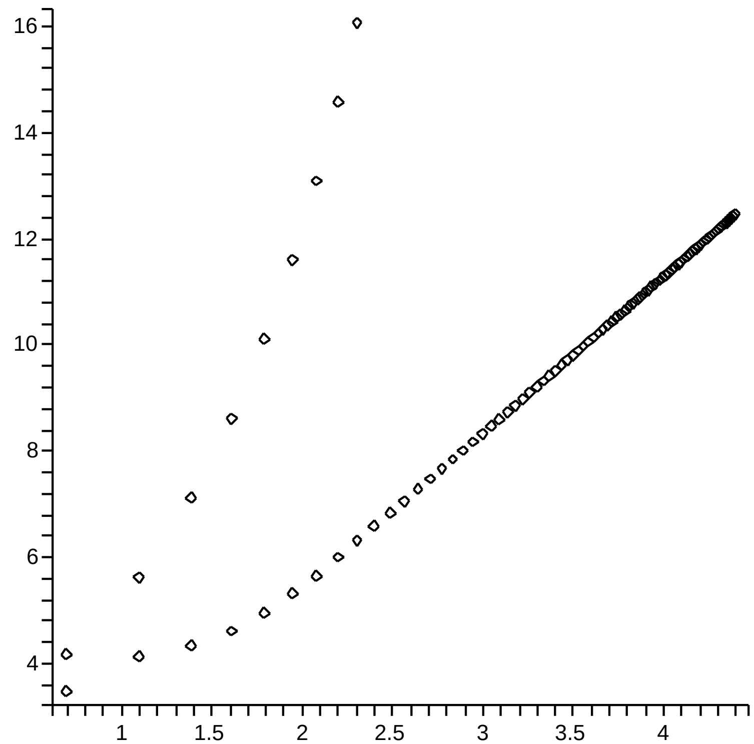
Log plots — $\log\log H(y_n)$ vs $\log n$

$$y_{n+1} + y_{n-1} = \frac{a_n}{y_n} + b_n$$



qP_{VI} — $\log \log H(y_n)$ vs $\log n$

$$\frac{f_n f_{n+1}}{cd} = \frac{g_{n+1} - \alpha q^{n+1}}{g_{n+1} - \gamma} \frac{g_{n+1} - \beta q^{n+1}}{g_{n+1} - \delta}, \qquad \frac{g_n g_{n+1}}{\gamma \delta} = \frac{f_n - a q^n}{f_n - c} \frac{f_n - b q^n}{f_n - d}.$$

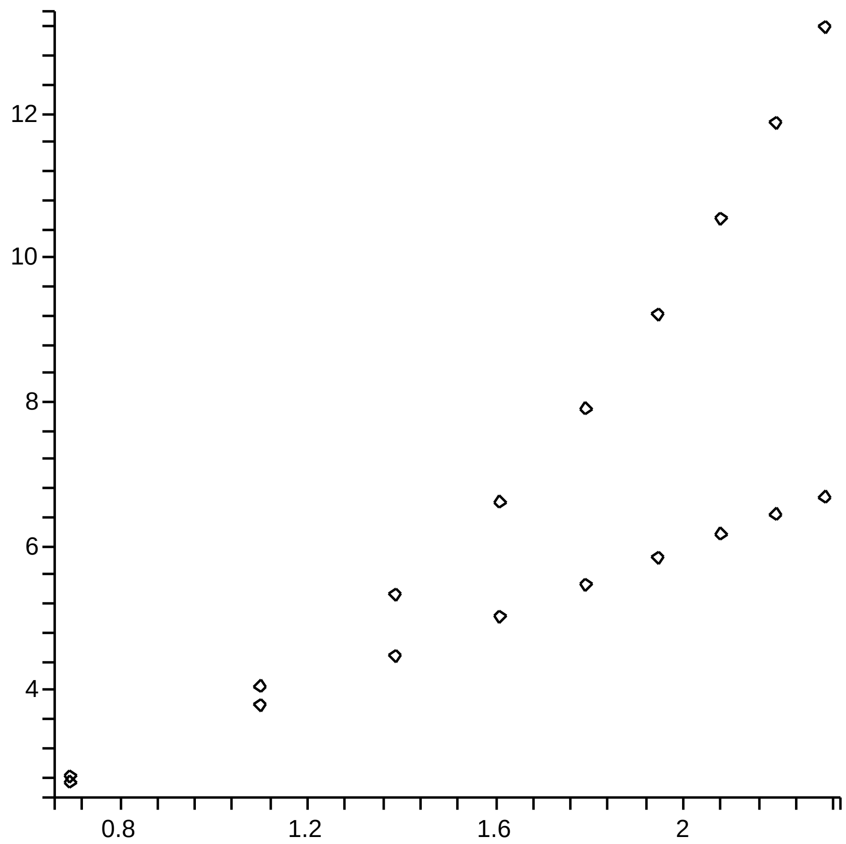


$$(\alpha, \beta, \gamma, \delta, a, b, c, d) = \left(\frac{15}{7}, \frac{4}{3}, \frac{1}{2}, 1, \frac{8}{7}, \frac{5}{7}, 2, \frac{1}{7}\right)$$

Integrable case: $q = \frac{ab\gamma\delta}{\alpha\beta cd} = \frac{1}{2}$. Other case: $q = 2$.

$\log \log H(y_n)$ vs $\log n$

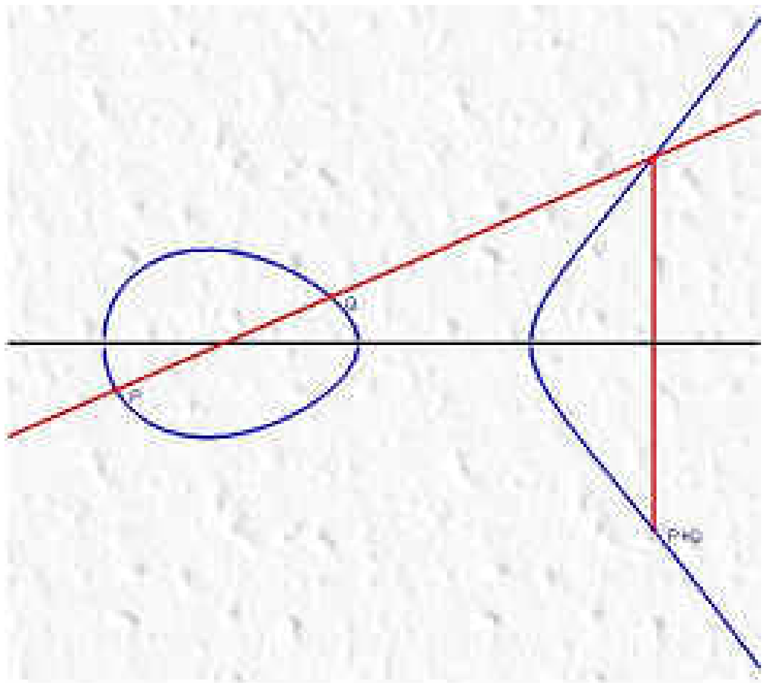
$$y_{m+1,n+1} = y_{m,n} + \frac{1}{y_{m,n+1}} - \frac{a}{y_{m+1,n}}$$



Integrable case: $a = 1$. Other case: $a = 2$.

Addition law on the cubic

$$y^2 = x^3 + ax + b$$



The elliptic curve is best considered in $\mathbb{P}^2$.

Let $x = X/Z$ and $y = Y/Z$, where $(X, Y, Z) \in \mathbb{P}^2$.

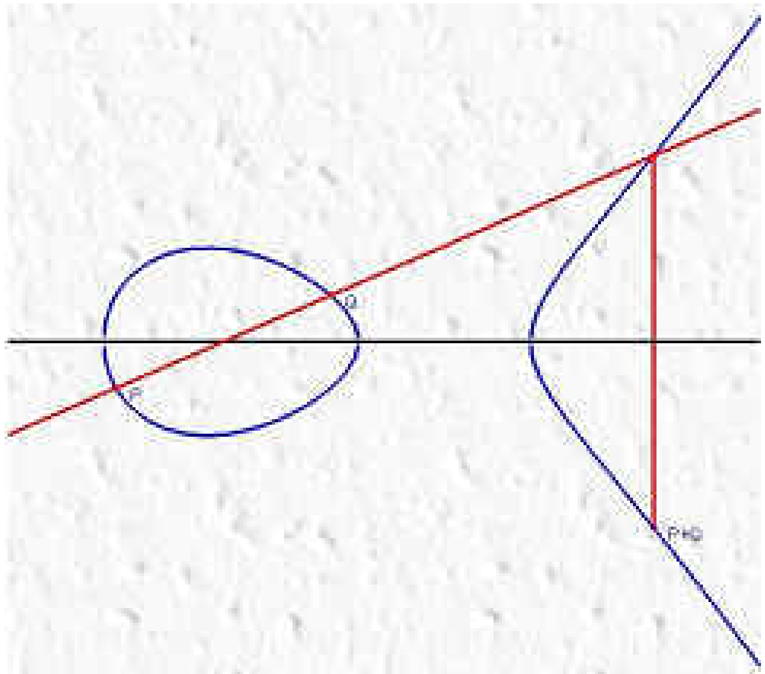
Then the elliptic curve $\mathcal{C}$ is $ZY^2 = X^3 + aXZ^2 + bZ^3$.

As well as points in $\mathbb{R}^2$, it includes the "point at infinity" $(0, 1, 0)$,

which is usually taken to be the zero element in the group on $\mathcal{C}$.

Addition law on the cubic

$$y^2 = x^3 + ax + b$$



Let $\widehat{P} = (\hat{x}, \hat{y})$ and $P_n = (x_n, y_n)$, $n \in \mathbb{Z}$ be points on $\mathcal{C}$ such that

$$P_n = P_0 + n\widehat{P}.$$

$$x_{n+1} + x_{n-1} = \frac{2(\hat{x}x_n + a)(x_n + \hat{x}) + 4b}{(x_n - \hat{x})^2}.$$

Mordell's theorem

- **Theorem** If a non-singular planar cubic has a rational point, then the group of rational points is finitely generated.
- One of the main ideas in the proof of Mordell's theorem is to consider the height of rational points on the curve obtained by repeatedly "adding" a given rational point on the curve.
- Recall that the height of a rational number $x = a/b$ is $H(x) = \max\{|a|, |b|\}$.
- The logarithmic height is $h(x) := \log H(x)$. The height of a rational point on a curve is defined to be the height of its x-coordinate.
- For an elliptic curve,

$$h(P_0 + n\widehat{P}) = O(n^2).$$

The symmetric QRT map

The symmetric Quispel-Roberts-Thompson map is

$$x_{n+1} = \frac{f_1(x_n) - x_{n-1} f_2(x_n)}{f_2(x_n) - x_{n-1} f_3(x_n)},$$

where

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = (\mathbf{A}_0 \mathbf{X}_n) \times (\mathbf{A}_1 \mathbf{X}_n), \quad \mathbf{X}_n = \begin{pmatrix} x_n^2 \\ x_n \\ 1 \end{pmatrix}, \quad \mathbf{A}_j = \begin{pmatrix} \alpha_j, & \beta_j, & \gamma_j \\ \beta_j & \epsilon_j & \zeta_j \\ \gamma_j & \zeta_j & \mu_j \end{pmatrix}, \quad j = 0, 1.$$

This system has the conserved quantity $K = \dfrac{\mathbf{X}^T \mathbf{A}_0 \mathbf{X}}{\mathbf{X}^T \mathbf{A}_1 \mathbf{X}}$.

This definition can be rewritten as

$$\alpha x_n^2 x_{n+1}^2 + \beta(x_n^2 x_{n+1} + x_n x_{n+1}^2) + \gamma(x_n^2 + x_{n+1}^2) + \epsilon x_n x_{n+1} + \zeta(x_n + x_{n+1}) + \mu = 0,$$

where $\alpha = \alpha_0 - K\alpha_1$, $\beta = \beta_0 - K\beta_1$, $\gamma = \gamma_0 - K\gamma_1$, etc.

It follows that

$$x_{n+1} + x_{n-1} = -\frac{\beta x_n^2 + \epsilon x_n + \zeta}{\alpha x_n^2 + \beta x_n + \gamma} \quad \text{and} \quad x_{n+1} x_{n-1} = \frac{\gamma x_n^2 + \zeta x_n + \mu}{\alpha x_n^2 + \beta x_n + \gamma}.$$

Height growth for the symmetric QRT

$$x_{n+1}+x_{n-1}=-\frac{\beta x_n^2+\epsilon x_n+\zeta}{\alpha x_n^2+\beta x_n+\gamma} \quad \text{and} \quad x_{n+1}x_{n-1}=\frac{\gamma x_n^2+\zeta x_n+\mu}{\alpha x_n^2+\beta x_n+\gamma}.$$

- In $\mathbb{QP}^2$, $H(u_0,u_1,u_2):=\max_{j=0,1,2}\{|u_j|\}$, where $u_j\in\mathbb{Z}$ and $\gcd\{u0,u_1,u_2\}=1$.

$$\begin{aligned} &H(1,x_{n+1}+x_{n-1},x_{n-1}x_{n+1}) \\ &= H(\alpha x_n^2+\beta x_n+\gamma, \beta x_n^2+\epsilon x_n+\zeta, \gamma x_n^2+\zeta x_n+\mu)\le cH(x_n)^2. \end{aligned}$$

- A standard identity for heights is $2H(1,u+v,uv)\ge H(u)H(v)$.
- We therefore have $H(x_{n+1})H(x_{n-1})\le \frac{c}{2}H(x_n)^2$.
- We see that the *logarithmic height*, $h(x_n):=\log H(x_n)$, satisfies

$$h(x_{n+1})-2h(x_n)+h(x_{n-1})\le \log(c/2).$$

- So $h(x)=O(n^2)$.
- This fact is used in the proof of Mordell's theorem.

Height growth and the discrete Painlevé equations

(Joint work with W. Morgan)

Define $h_r(y_n) := \sum_{n=r_0}^{r} h(y_n)$.

Let $(y_n) \subset \mathbb{Q} \setminus \{0\}$ be a solution of

$$y_{n+1} + y_{n-1} = \frac{\alpha_n + \beta_n y_n + \gamma_n y_n^2}{y_n^2}, \qquad (\dagger)$$

where $\alpha_n \not\equiv 0$, β_n and γ_n are in $\mathbb{Q}(n)$ and $\max\{h_r(\alpha_n), h_r(\beta_n), h_r(\gamma_n)\} = o(h_r(y_n))$. If

$$h(y_n) = O(n^\sigma),$$

for some σ, then

$$\alpha_n = a_0, \qquad \beta_n = b_0 + b_1 n, \qquad \text{and} \qquad \gamma_n = 0.$$

- If $b_1 = 0$ then equation $(\dagger)$ can be solved in terms of elliptic functions.
- If $b_1 \neq 0$ then equation $(\dagger)$ is the following discrete Painlevé equation,

$$y_{n+1} + y_{n-1} = \frac{A + n y_n}{y_n^2}.$$

Absolute values on $\mathbb{Q}$

An <u>absolute value</u> on a field k is a mapping $|\cdot| : k \to \mathbb{R}$ such that for all $x, y \in k$,

1. $|x| \geq 0$ with equality if and only if $x = 0$;
2. $|xy| = |x||y|$;
3. $|x + y| \leq |x| + |y|$.

The p-adic absolute value

Let p be a fixed prime. Any non-zero rational number x can be written as

$$x = p^r \frac{a}{b}, \qquad \text{where } p \not| \, ab.$$

The p-adic absolute value of x is $|x|_p := p^{-r}$.

Theorem (Ostrovski)

Any non-trivial absolute value on $\mathbb{Q}$ is equivalent to a p-adic absolute value, for some prime p, or to the usual absolute value (denoted by $|\cdot|_\infty$.)

The p-adic absolute value

If $x = p^r \frac{a}{b} \neq 0$, where $p \not| ab$, then $|x|_p := p^{-r}$.

Note that

$$\left| \sum_{n=0}^{N-1} 2^n + 1 \right|_2 = |2^N|_2 = 2^{-N} \to 0.$$

So

$$1 + 2 + 2^2 + \cdots + 2^n + \cdots = -1,$$

with respect to the 2-adic absolute value.

Another important property of the p-adic absolute value is that it is non-Archimedean, i.e.,

$$|x + y|_p \leq \max\{|x|_p, |y|_p\}, \qquad \forall x, y \in \mathbb{Q}.$$

The usual absolute value, $| \cdot |_\infty$, is Archimedean.

The logarithmic height and absolute values on $\mathbb{Q}$

Again suppose that

$$x = \pm \frac{p_1^{r_1} \cdots p_m^{r_m}}{q_1^{s_1} \cdots q_n^{s_n}} = \frac{a}{b} \neq 0,$$

where $p_1, \ldots, p_m; q_1, \ldots, q_n$ are prime.

The logarithmic height of x is given by

$$\begin{aligned} h(x) &= \log H(x) = \max\{\log |a|_\infty, \log |b|_\infty\} \\ &= \log |b|_\infty + \max\{\log |a|_\infty - \log |b|_\infty, 0\} = \log |b|_\infty + \log^+ |a/b|_\infty, \end{aligned}$$

where $\log^+ \eta := \max\{\log \eta, 0\}$.

So

$$h(x) = \log q_1^{s_1} + \log q_2^{s_2} + \cdots + \log q_n^{s_n} + \log^+ |x|_\infty = \sum_{p \leq \infty} \log^+ |x|_p = h(1/x).$$

Singularity confinement using absolute values

(with Will Morgan)

For each absolute value $|\cdot|$ and for sufficiently small $\delta > 0$, we can define a "scale" given by ϵ_n, which depends on nearby values of α_j and β_j such that if (y_n) satisfies

$$y_{n+1} + y_{n-1} = \frac{\alpha_n + \beta_n y_n}{y_n^2}$$

and if for some particular k, $|y_k| < \epsilon_k$ and $|y_{k-1}| \leq |y_k|^{-1/2}$, then

1. $y_{k+1} = \frac{\alpha_k}{y_k^2} + \frac{\beta_k}{y_k} + A_k$, where $|A_k| \leq |y_k|^{-1/2}$.

2. $y_{k+2} = -y_k + \frac{\beta_{k+1}}{\alpha_k} y_k^2 + B_k$, where $|B_k| \leq |y_k|^{3-4\delta}$

3. $y_{k+3} = \frac{\alpha_{k+2}-\alpha_k}{y_{k+2}^2} + \frac{\beta_{k+2} - 2\frac{\alpha_{k+2}}{\alpha_k}\beta_{k+1} + \beta_k}{y_{k+2}} + C_k$, where
   $|C_k| \leq \max\left\{ \left|\frac{\alpha_{k+2}-\alpha_k}{\alpha_k}\right| |y_{k+2}|^{1-\delta}, |y_{k+2}|^{-1/2} \right\}$ for non-Archimedean absolute values and
   $|C_k| \leq 2\left|\frac{\alpha_{k+2}-\alpha_k}{\alpha_k}\right| |y_{k+2}|^{1-\delta} + 3|y_k|^{-1/2}$ for Archimedean absolute values.

Lemma

Let $(x_n) \subset \mathbb{Q} \setminus \{0\}$ be a solution of

$$x_{n+1} + x_{n-1} = \frac{\alpha_n + \beta_n x_n + \gamma_n x_n^2}{x_n^2},$$

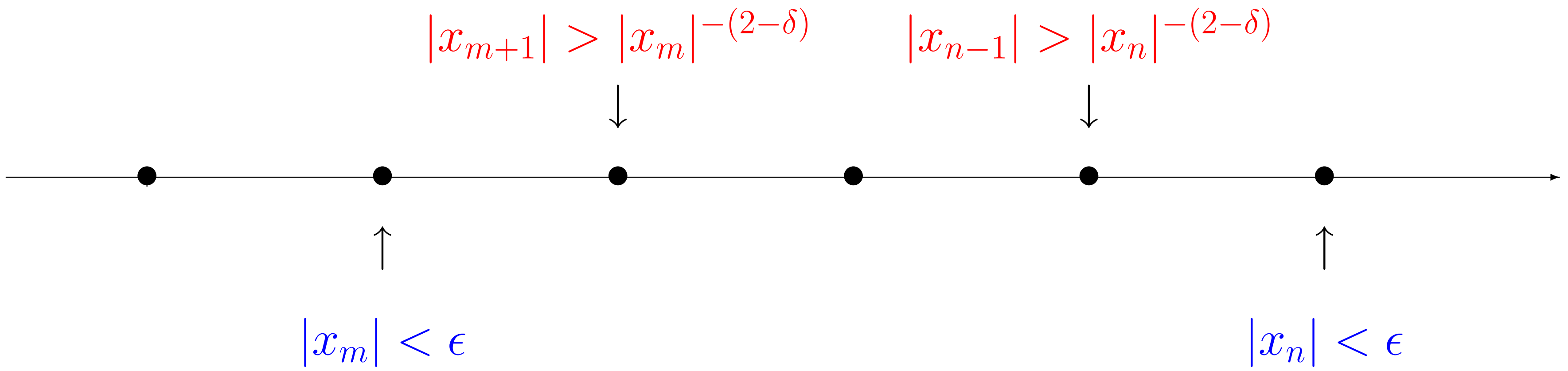
where $\alpha_n \gamma_n \neq 0$ for all $n \geq r_0$. Choose a prime $p \leq \infty$, $\delta \in (0, 1/2)$ and define $\epsilon_p > 0$ by

$$\epsilon_p^{-\delta} := C_p \max\{1, |\alpha_n|_p^{-1}, |\alpha_{n-1}|_p, |\alpha_{n+1}|_p, |\beta_n|_p, |\beta_{n-1}|_p, |\beta_{n+1}|_p,$$
$$|\gamma_n|_p, |\gamma_{n-1}|_p^{-1}, |\gamma_{n+1}|_p^{-1}\},$$

where $C_p = 1$ if $p < \infty$ and $C_\infty = 3$. Then for any $n \in \mathbb{Z}$ such that $|x_n|_p < \epsilon_p$, either

$$|x_{n+1}|_p \geq |x_n|_p^{-(2-\delta)} \text{ and } |x_{n+2}|_p \geq \epsilon_p \quad \text{or} \quad |x_{n-1}|_p \geq |x_n|_p^{-(2-\delta)} \text{ and } |x_{n-2}|_p \geq \epsilon_p.$$

Associating large and small iterates



Exponential growth

Let $S_1(r) := \{n \in \mathbb{Z} : r_0 \le n \le r \text{ and } |x_n| < \epsilon\}$ and $S_2(r) := [r_0, r] \setminus S_1(r)$.

$$\sum_{n=r_0}^{r} \log^+ |x_n|_p^{-1} = \sum_{n\in S_1(r)} \log^+ |x_n|_p^{-1} + \sum_{n\in S_2(r)} \log^+ |x_n|_p^{-1}$$

Now

$$\sum_{n\in S_1(r)} \log^+ |x_n|_p^{-1} \le \sum_{n=r_0-1}^{r+1} \log^+ |x_n|_p^{\frac{1}{2-\delta}},$$

$$\sum_{n\in S_2(r)} \log^+ |x_n|_p^{-1} \le \sum_{n=r_0}^{r} \log \epsilon_p^{-1} \le \delta^{-1} \sum_{n=r_0}^{r} \log \left[C_p \max\{1, |\alpha_n|_p, |\alpha_{n-1}|_p^{-1}, |\alpha_{n+1}|_p^{-1}, \ldots\}\right]$$

$$\le \delta^{-1} \sum_{n=r_0}^{r} \left[\log C_p + \log^+ |\alpha_n|_p + \log^+ |\alpha_{n-1}|_p^{-1} + \log^+ |\alpha_{n+1}|_p^{-1} + \cdots\right]$$

Define $h_r(x_n) = \sum_{n=r_0}^{r} h(x_n) = \sum_{p\le\infty} \sum_{n=r_0}^{r} \log^+ |x_n|_p$. Then

$$h_r(x_n) = h_r(1/x_n) \le \frac{1}{2-\delta} h_{r+1}(x_n) + \text{log heights of coefficients.}$$

Height growth and a discrete Painlevé equation

(Joint work with Asma Al-Ghassani)

Let $(y_n) \subset \mathbb{Q} \setminus \{-1, 1\}$ be an admissible solution of

$$y_{n+1} + y_{n-1} = \frac{a_n + b_n y_n + c_n y_n^2}{1 - y_n^2}, \tag{4}$$

where a_n, b_n and c_n are rational functions of n with coefficients in $\mathbb{Q}$ and the right hand side of (4) is irreducible. Then either

1. $a_n = \alpha$, $b_n = \beta n + \gamma$, $c_n = 0$ for constants α, β, γ; or
2. y_n is also an admissible solution of the difference Riccati equation
$$y_{n+1} = \frac{1/2(a_n + \theta b_n - 2\theta) + y_n}{1 - \theta y_n}, \text{ where } \theta = -1 \text{ or } 1; \text{ or}$$
3. $$\limsup_{r\to\infty} \frac{\log\log \sum_{n=r_0}^{r} h(y_n)}{\log r} \geq 1.$$

Heights on number fields

- A number field k is a finite extension of the rational numbers, e.g. $\mathbb{Q}(\sqrt{2+5^{1/3}})$.
- A place on k is an equivalence class of absolute values.
- Let M_k be the set of places on k.
- There are $[k:\mathbb{Q}] < \infty$ Archimedian places on k.
- For any $x \in k \setminus \{0\}$, the Artin-Whaples product formula is

$$\prod_{v \in M_k} |x|_v = 1.$$

- The (absolute) logarithmic height of x is

$$h(x) = \frac{1}{[k:\mathbb{Q}]} \sum_{v \in M_k} \log^+ |x|_v$$

$$
\begin{aligned}
H(x_1 + x_2) &\leq 2H(x_1)H(x_2); \\
H(x_1)H(x_2) &\leq H(x_1)H(x_2); \\
H(x_1x_2 + x_2x_3 + x_3x_1) &\leq 3H(x_1)H(x_2)H(x_3).
\end{aligned}
$$

Taking log of the above expressions gives

$$
\begin{aligned}
h(x_1 + x_2) &\leq h(x_1) + h(x_2) + \log 2, \\
h(x_1x_2) &\leq h(x_1) + h(x_2), \\
h(x_1x_2 + x_2x_3 + x_3x_1) &\leq h(x_1) + h(x_2) + h(x_3) + \log 3.
\end{aligned}
$$

Compare with expressions from Nevanlinna theory:

$$
\begin{aligned}
T(r, f + g) &\leq T(r, f) + T(r, g) + \log 2; \\
T(r, fg) &\leq T(r, f) + T(r, g); \\
T(r, fg + gh + hf) &\leq T(r, f) + T(r, g) + T(r, h) + \log 3.
\end{aligned}
$$

Estimates involving rational functions

Let

$$R := \frac{a_0 + a_1 x + \cdots + a_p x^p}{b_0 + b_1 x + \cdots + b_q x^q},$$

be an irreducible rational function of x of degree

$$d = \max\{p, q\}.$$

Then

$$C_1 H(x)^d \leq H(R) \leq C_2 H(x)^d,$$

where C_1 and C_2 are polynomials in the heights of the coefficients a_i, b_j.

So the logarithmic height $h(\,\cdot\,) = \log H(\,\cdot\,)$ satisfies

$$|h(R) - d\,h(x)| \leq \log C,$$

where C is a polynomial in $H(a_i)$ and $H(b_j)$.

Heights and discrete equations

- Consider the equation

$$y_{n+1} = \frac{a_0(n) + a_1(n)y_n + \cdots + a_p(n)y_n^p}{b_0(n) + b_1(n)y_n + \cdots + b_q(n)y_n^q},$$

  where the a_i's and b_j's are polynomials in n.

- Taking the logarithmic height gives

$$h(y_{n+1}) = d\,h(y_n) + O(\log n).$$

- So if $H(y_n)$ grows faster than any polynomial in n then $h(y_n)$ grows exponentially unless $d \leq 1$.

- Similarly, if

$$y_{n+1} + y_{n-1} = R(n, y_n) \text{ or } y_{n+1}y_{n-1} = R(n, y_n)$$

  then $d := \deg_{y_n}(R(n, y_n)) \leq 2$.

- Are there deeper connections between the (discrete) Painlevé equations and number theory (esp. arithmetic geometry)?

Algebraic entropy

- Integrability as low complexity: Arnold (1990), Veselov (1992)
- Algebraic entropy: Falqui and Viallet (1993), Bellon and Viallet (1999)

$$\lim_{n\to\infty} \frac{\log d_n}{n}$$

- Example with confinement and positive algebraic entropy: Hietarinta and Viallet (1998)
- Regularised map using blow-ups: Takenawa (2001)
- Upper bound by looking for cancellations: van der Kamp (2012)

Standard methods of calculating algebraic entropy: heuristic vs rigorous.

Degree of a rational function

There are two equivalent definitions of the degree of a rational function.

Let $R(z) = \dfrac{P(z)}{Q(z)}$, where P and Q are polynomials with no common factors. Then

1. $\deg(R) = \max\{\deg(P(z)), \deg(Q(z))\}$.
2. Let a be any number in the extended complex plane $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$. Then the $\deg(R)$ is the number of pre-images of a in $\mathbb{CP}^1$ counting multiplicities.

For example, the degree of the rational function

$$\frac{2x^5 - 4x^4 + 2x^3 + x + 1}{x(x-1)^2} = \frac{x+1}{x(x-1)^2} + 2x^2$$

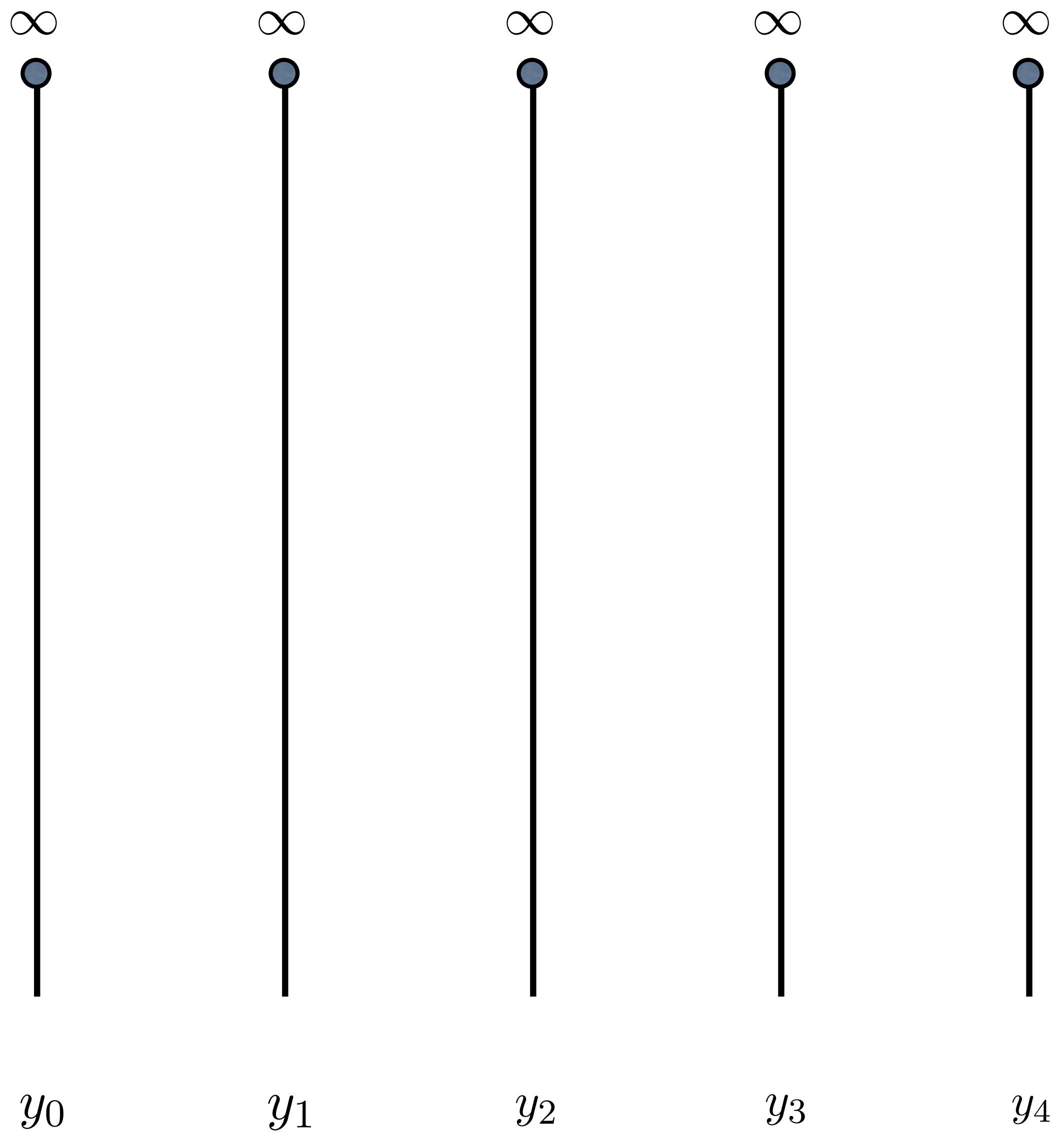
is 5.

Singularity confinement revisited

$$y_{n+1} + y_{n-1} = \frac{a_n + b_n y_n}{1 - y_n^2}$$

$$\begin{aligned}
y_{n-1} &= k + o(1), \\
y_n &= \theta + \epsilon, \qquad \theta = \pm 1, \quad \epsilon = (z - z_0)^p f(z), \quad f \text{ analytic at } z_0,\ f(z_0) \neq 0 \\
y_{n+1} &= -\frac{a_n + \theta b_n}{2\theta} \epsilon^{-1} + O(1), \\
y_{n+2} &= -\theta + \frac{2\theta b_{n+1} - \theta b_n - a_n}{a_n + \theta b_n} \epsilon + O(\epsilon^2), \\
y_{n+3} &= \frac{a_n + \theta b_n}{2\theta} \left\{ \frac{(a_{n+2} - a_n) - \theta(b_{n+2} - 2b_{n+1} + b_n)}{\theta(2b_{n+1} - b_n) - a_n} \right\} \epsilon^{-1} + O(1).
\end{aligned}$$

Also, if $y_{n-1} \sim \alpha z$ and $y_n \sim \beta z$ as $z \to \infty$, then $y_{n+1} \sim -\alpha z$.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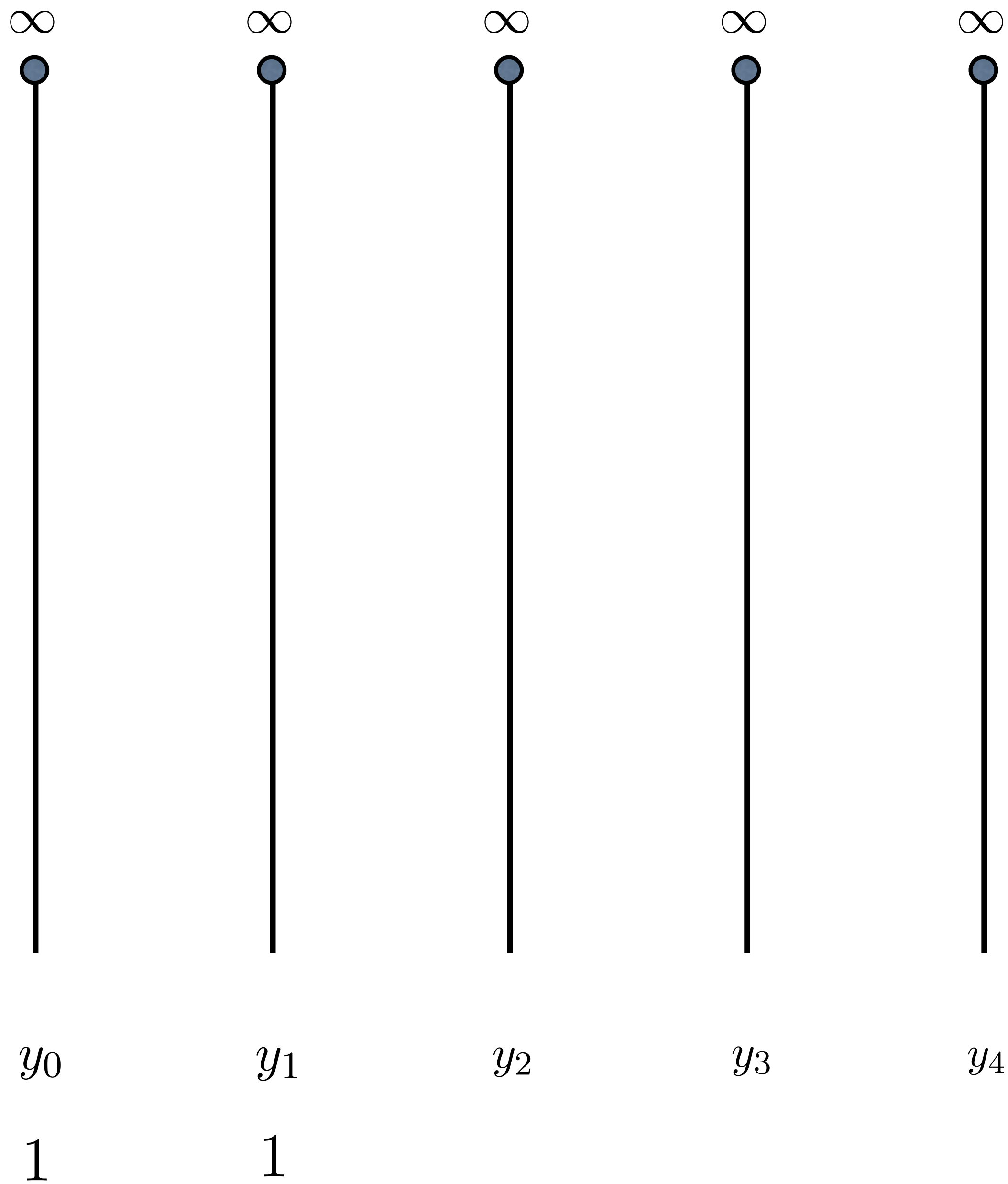

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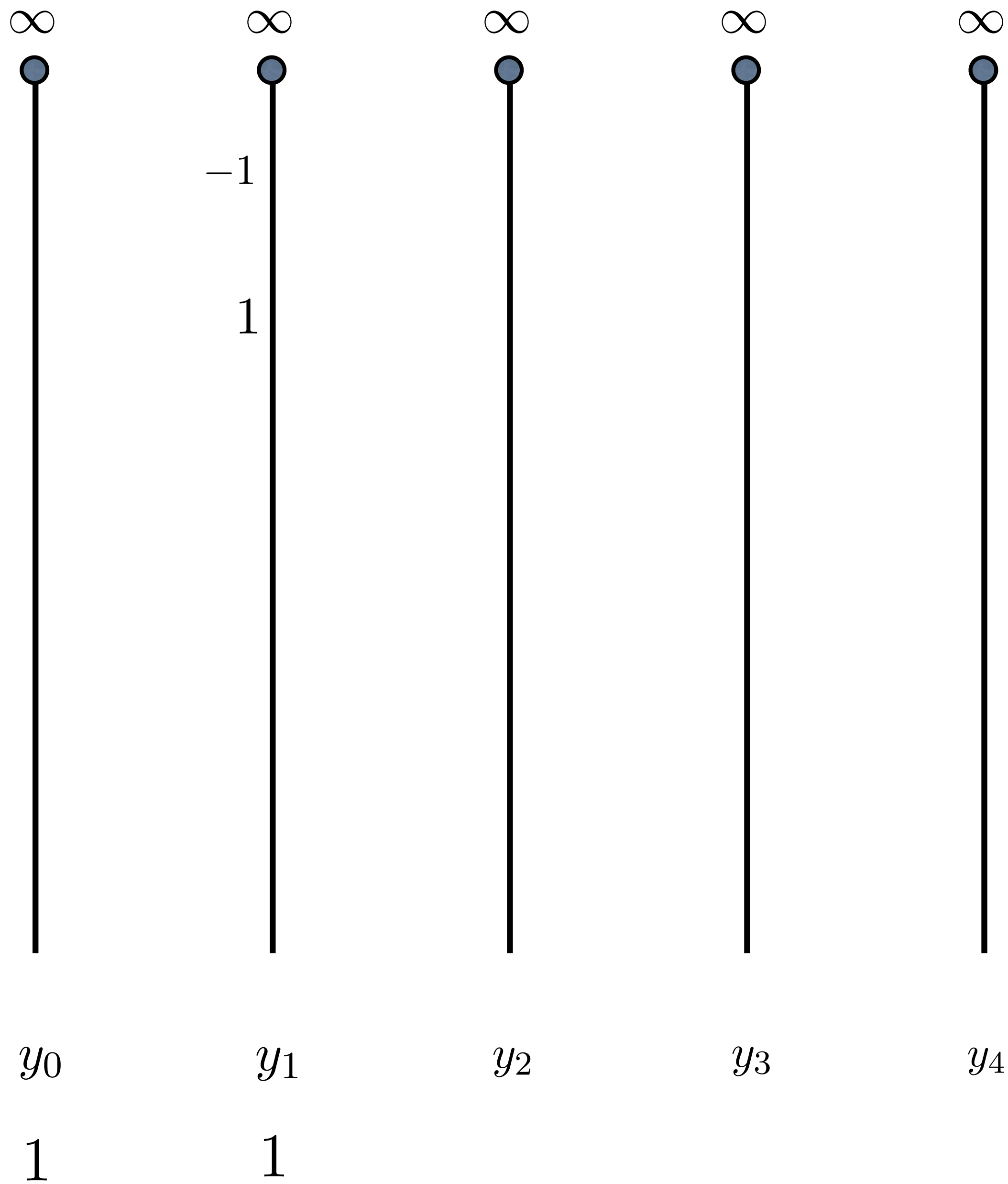

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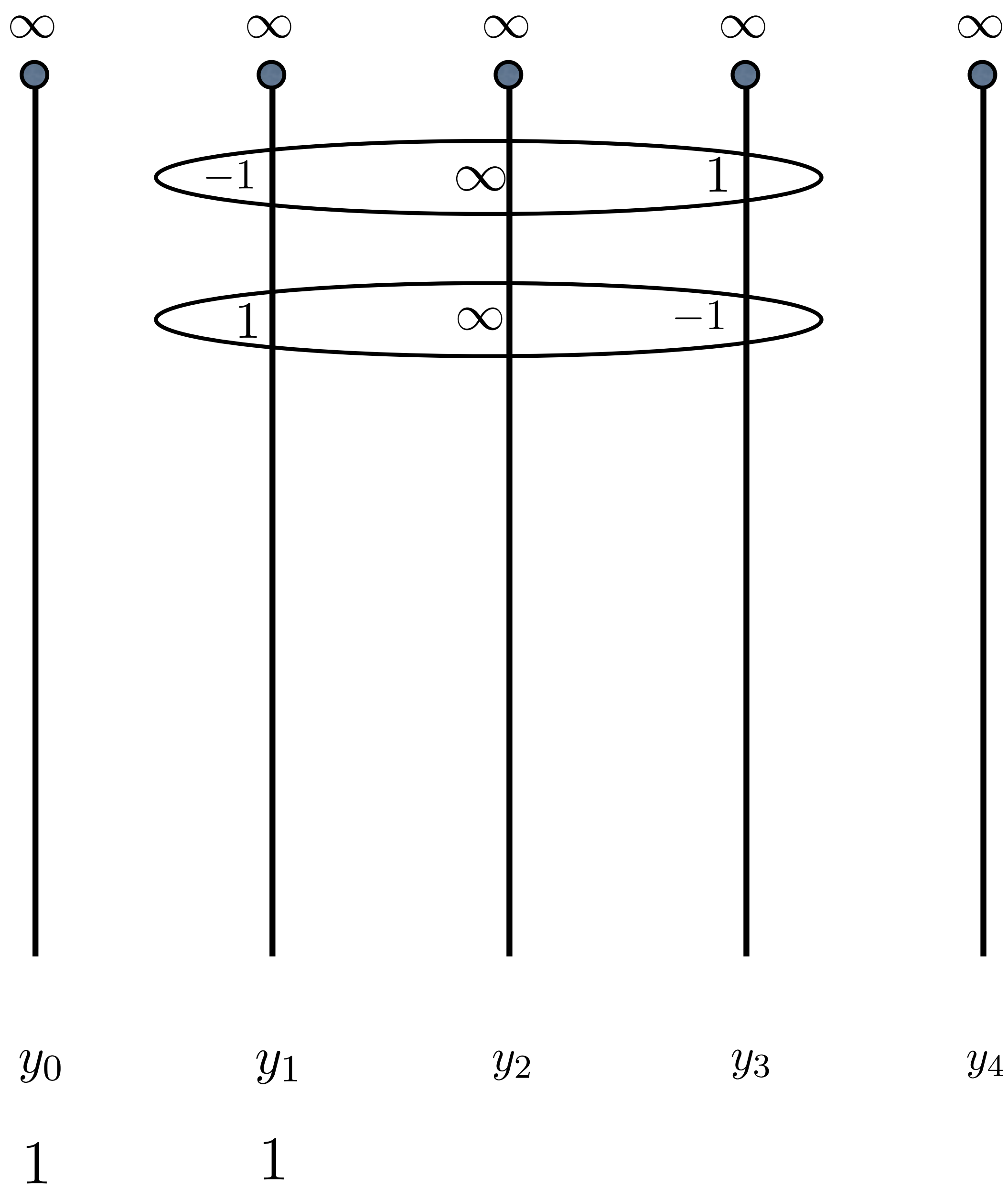

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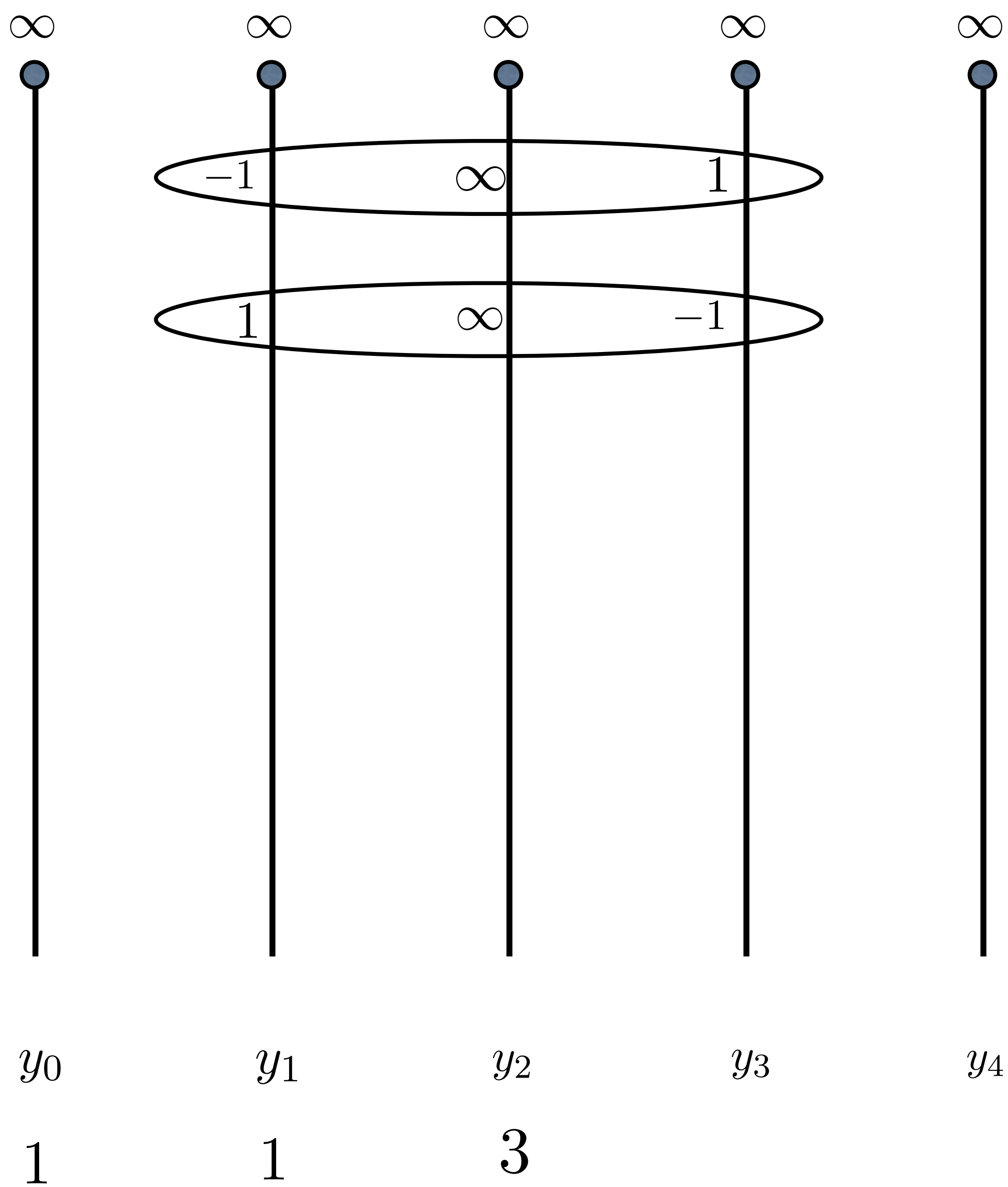
1 ∞ −1
y_0 y_1 y_2 y_3 y_4
1 1



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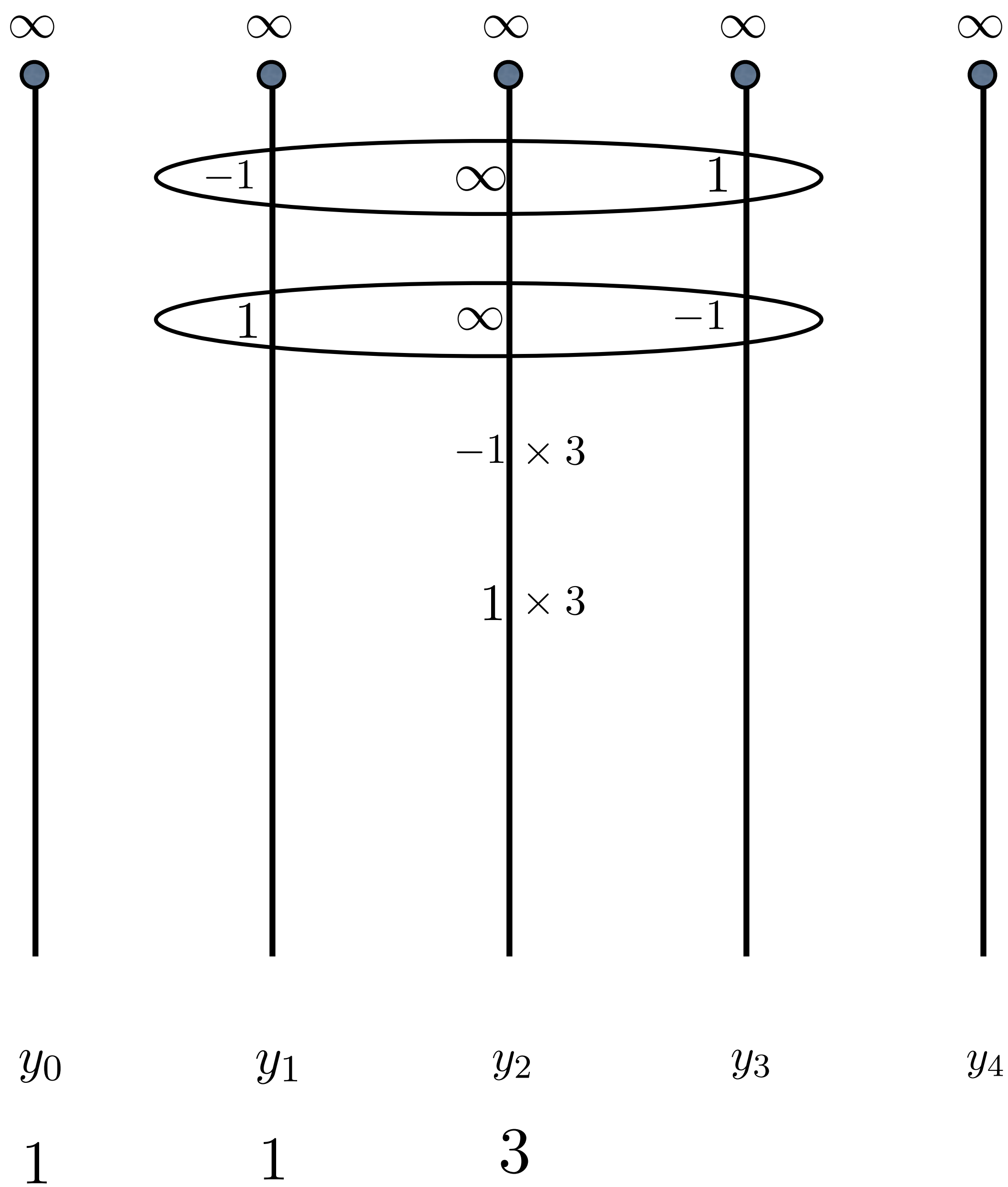
3



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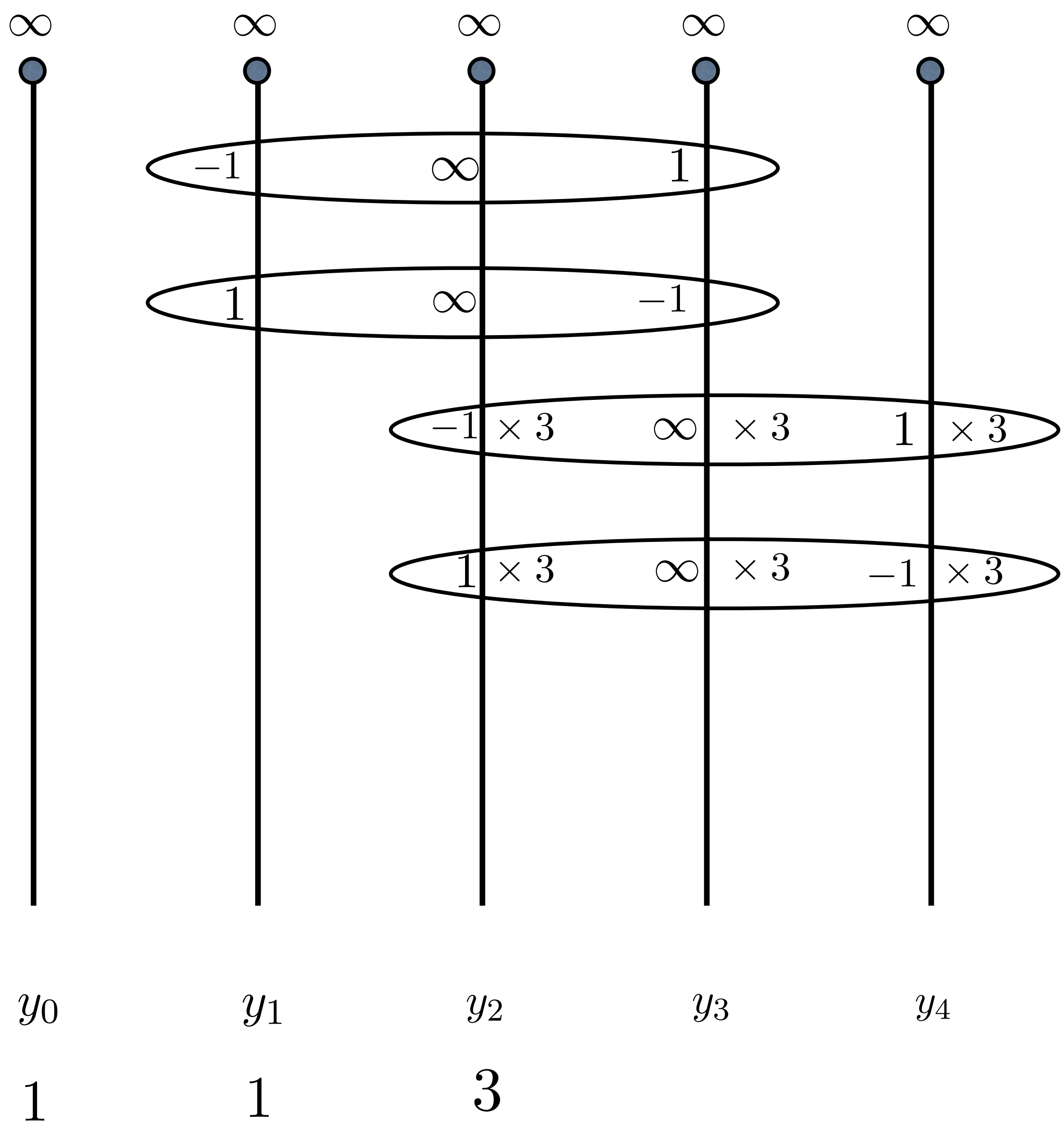

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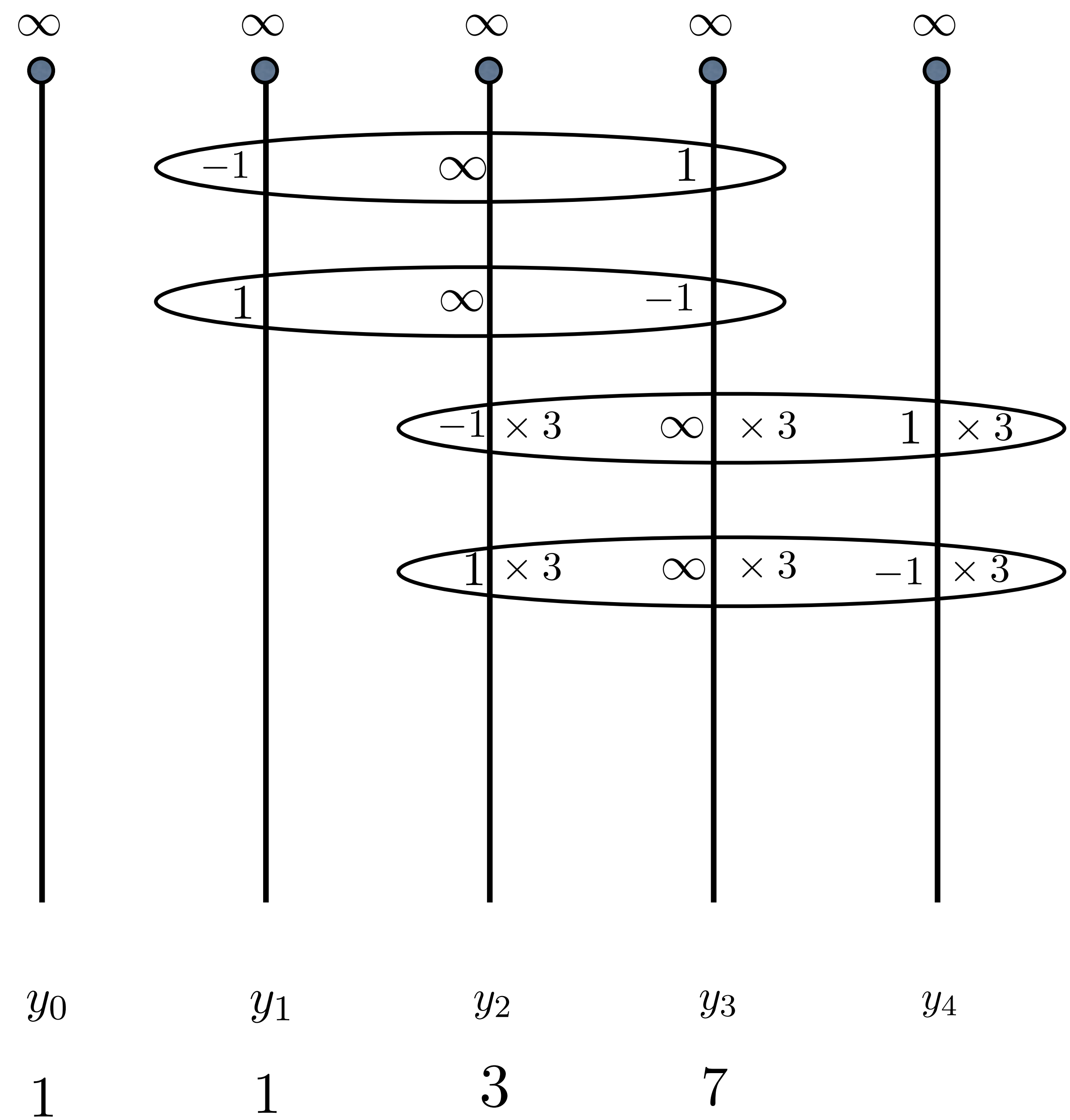

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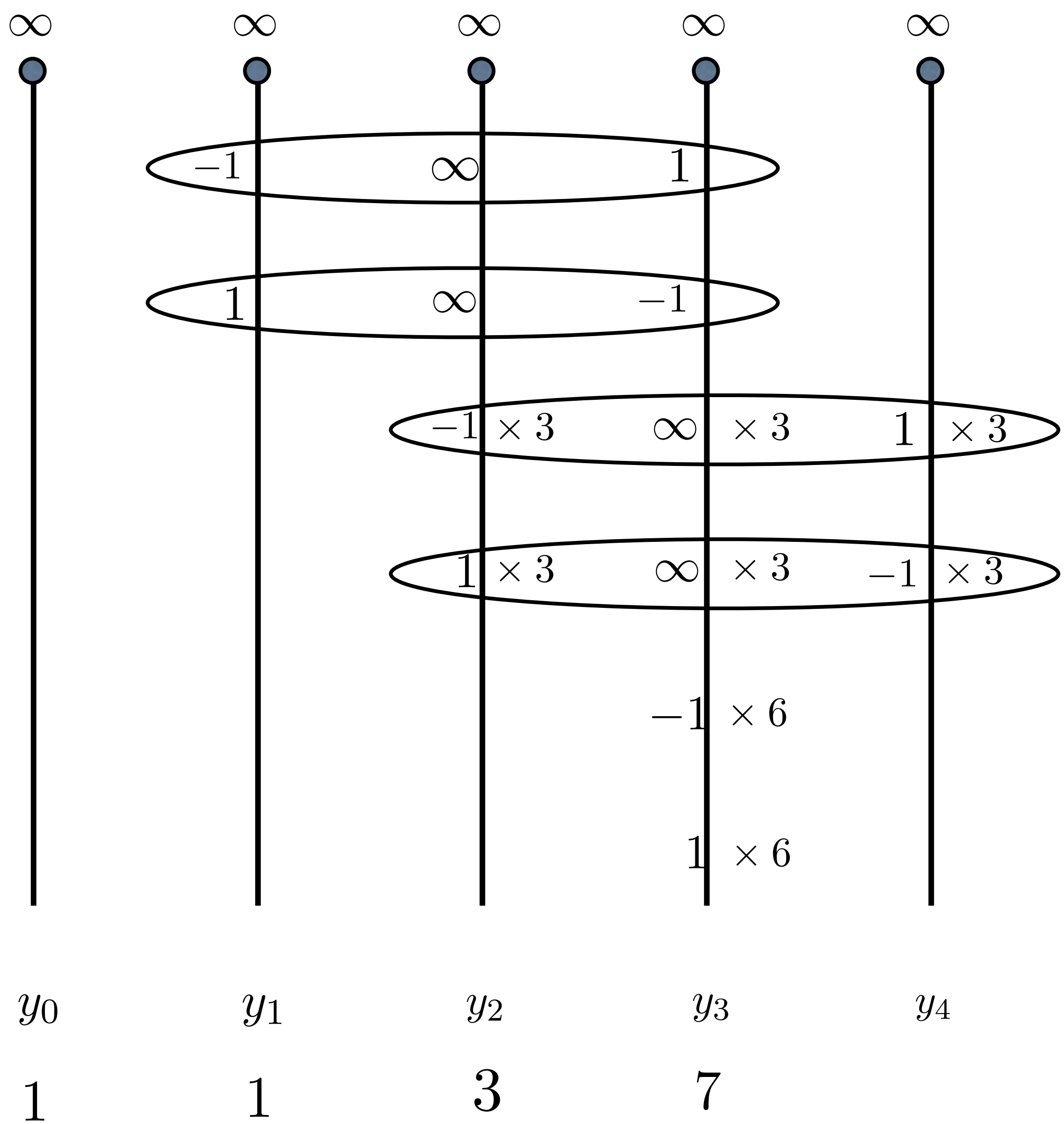
1 × 3 ∞ × 3 −1 × 3
y_0 y_1 y_2 y_3 y_4
1 1 3 7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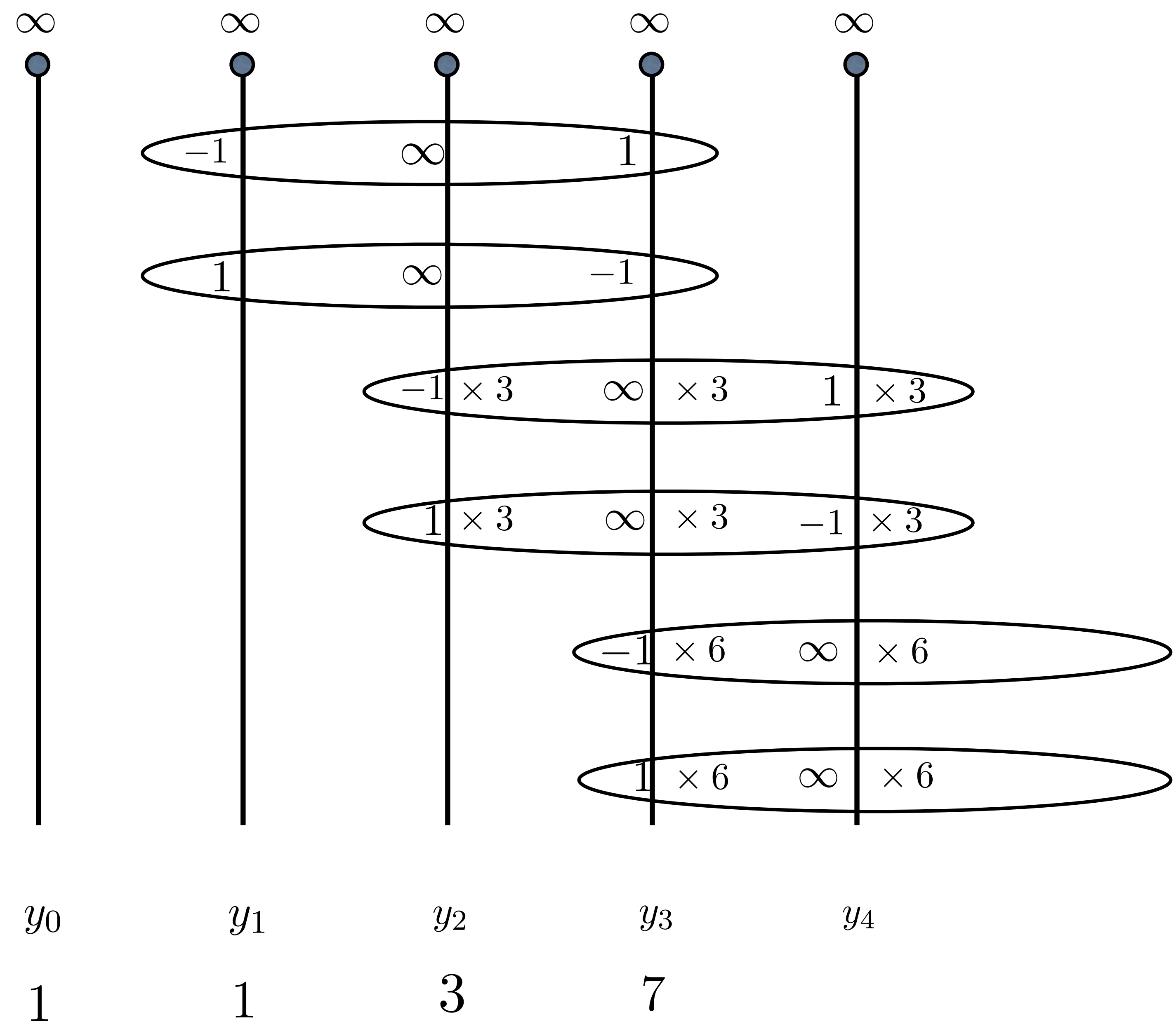
y_0 y_1 y_2 y_3 y_4
1 1 3 7



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−1 × 3
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∞ × 6
1 × 6
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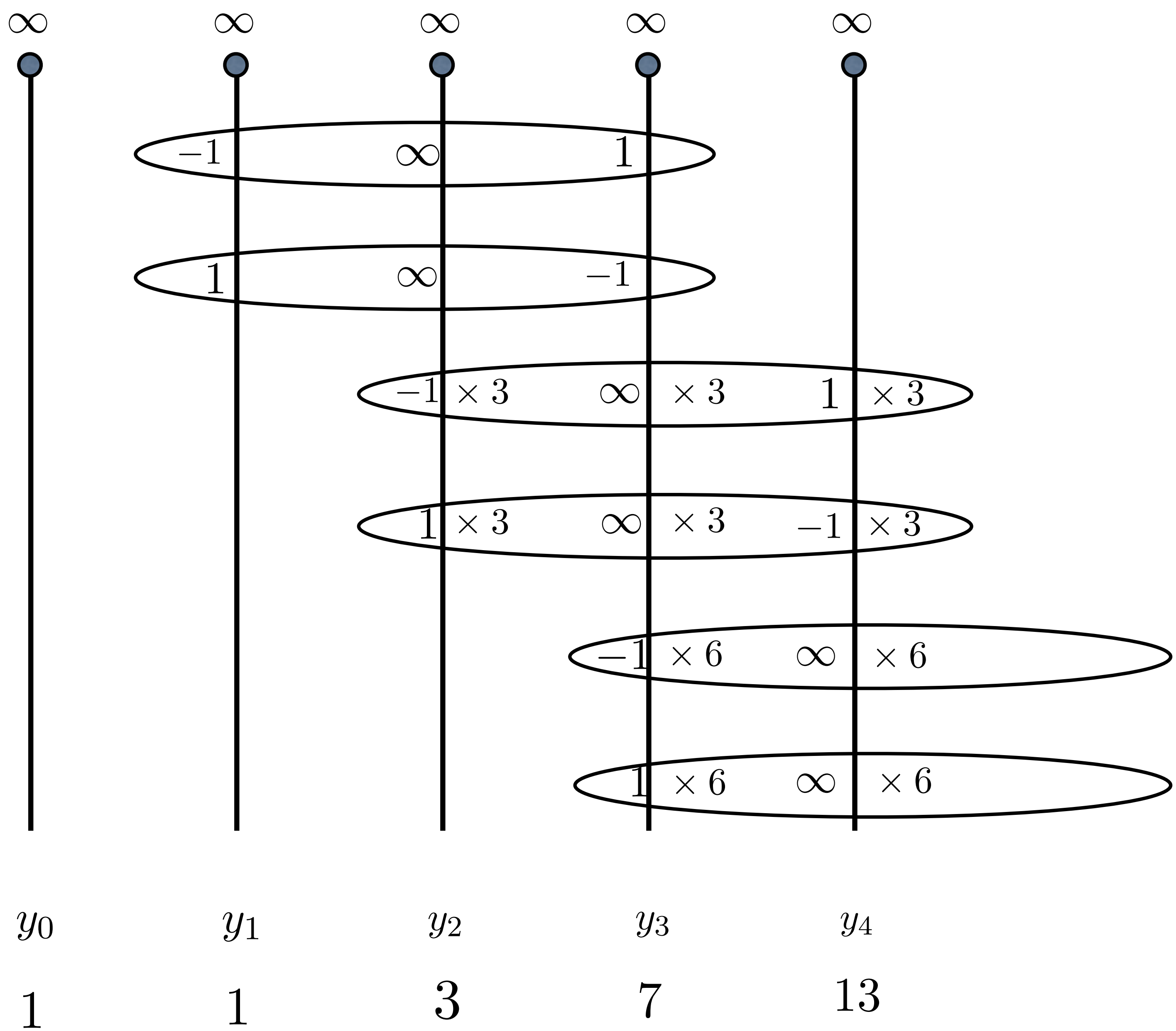
7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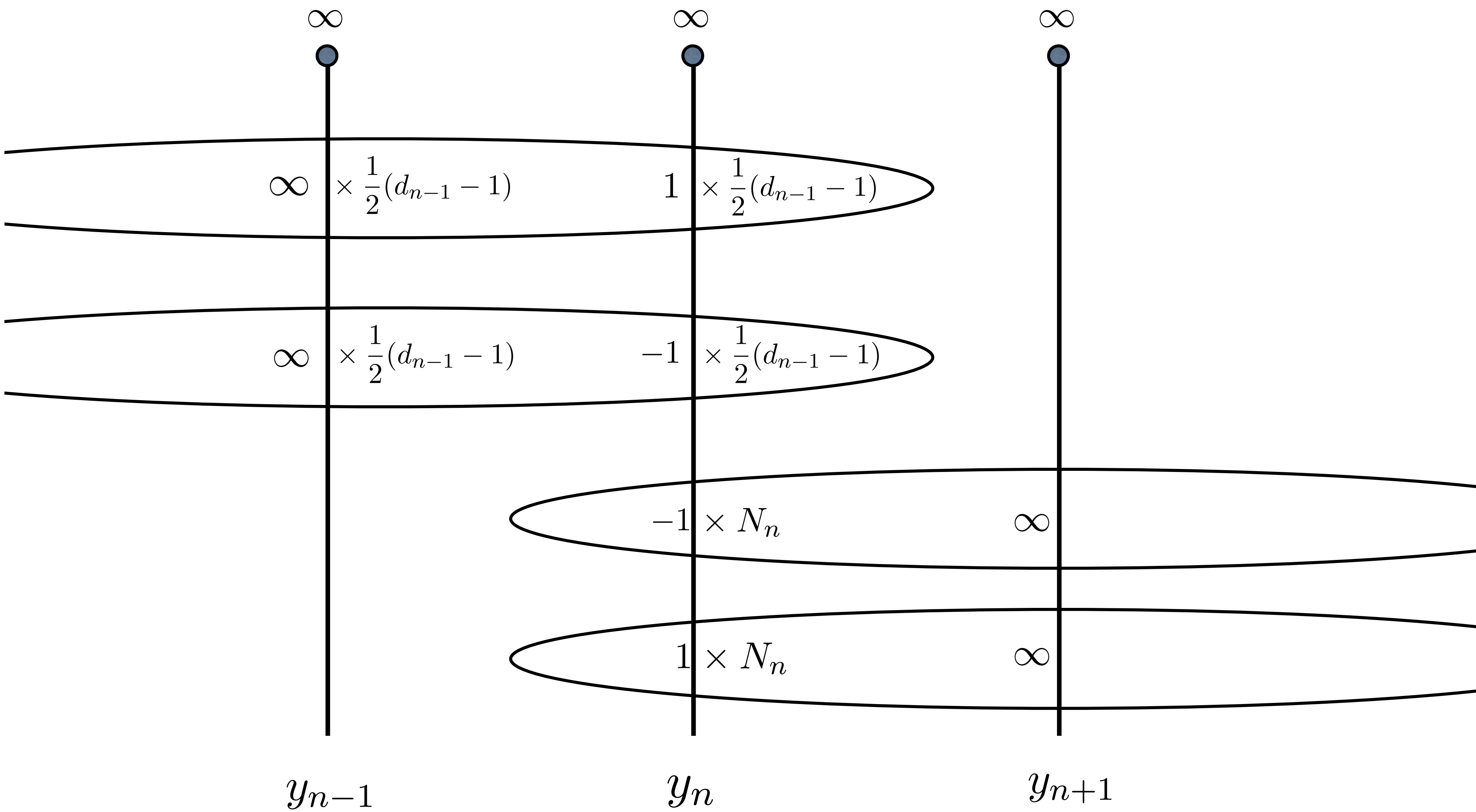
∞ × 6
y0
y1
y2
y3
y4
1
1
3
7
13



$$d_n = N_n + \frac{1}{2}\left(d_{n-1} - 1\right) \qquad d_{n+1} = 2N_n + 1$$

Exact formula for degrees

We have

$$d_{n+1} = 2N_n + 1 \quad \text{and} \quad d_n = N_n + \frac{1}{2}\left(d_{n-1} - 1\right).$$

Eliminating N_n gives

$$d_{n+1} - 2d_n + d_{n-1} = 2.$$

We also have the initial conditions $d_0 = d_1 = 1$. Hence

$$d_n = \frac{n(n-1)}{2} + 1.$$

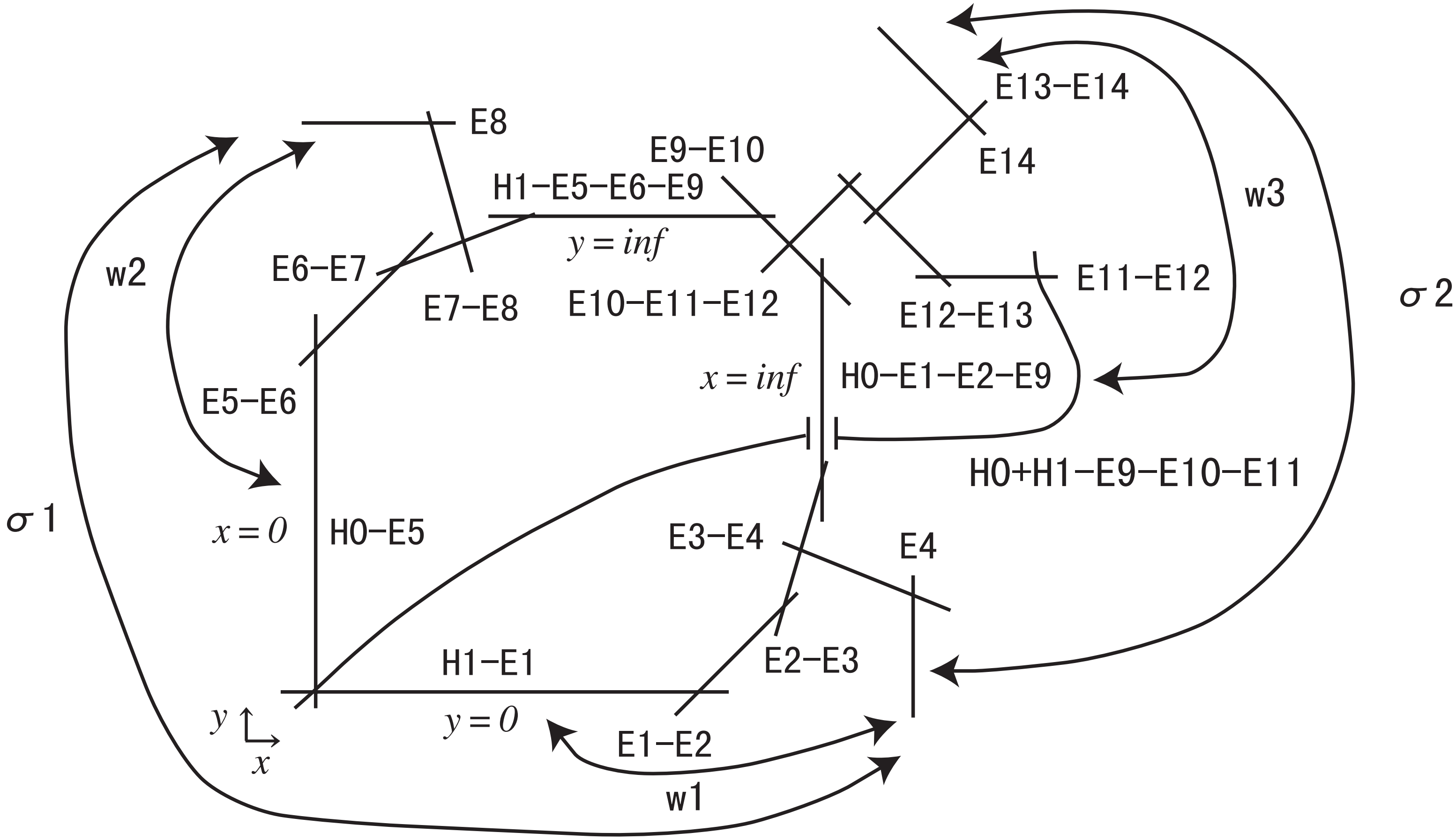
Example of Hietarinta and Viallet revisited

$$y_{n+1} + y_{n-1} = y_n + \frac{a}{y_n^2}$$

$$\begin{aligned} y_{n-1} &= k + o(1), \\ y_n &= \epsilon, \\ y_{n+1} &= \epsilon^{-2} - k + \epsilon + O(\epsilon^2), \\ y_{n+2} &= \epsilon^{-2} - k + \epsilon^4 + O(\epsilon^5), \\ y_{n+3} &= -\epsilon + 2\epsilon^4 + O(\epsilon^5), \\ y_{n+4} &= k + o(1). \end{aligned}$$

We will choose $y_0 \sim \alpha z + \beta$ and $y_1 \sim \gamma z + \delta$ as $z \to \infty$, where $\alpha\gamma(\alpha - \gamma) \neq 0$. Then y_n has a simple pole at $z = \infty$ for all n.

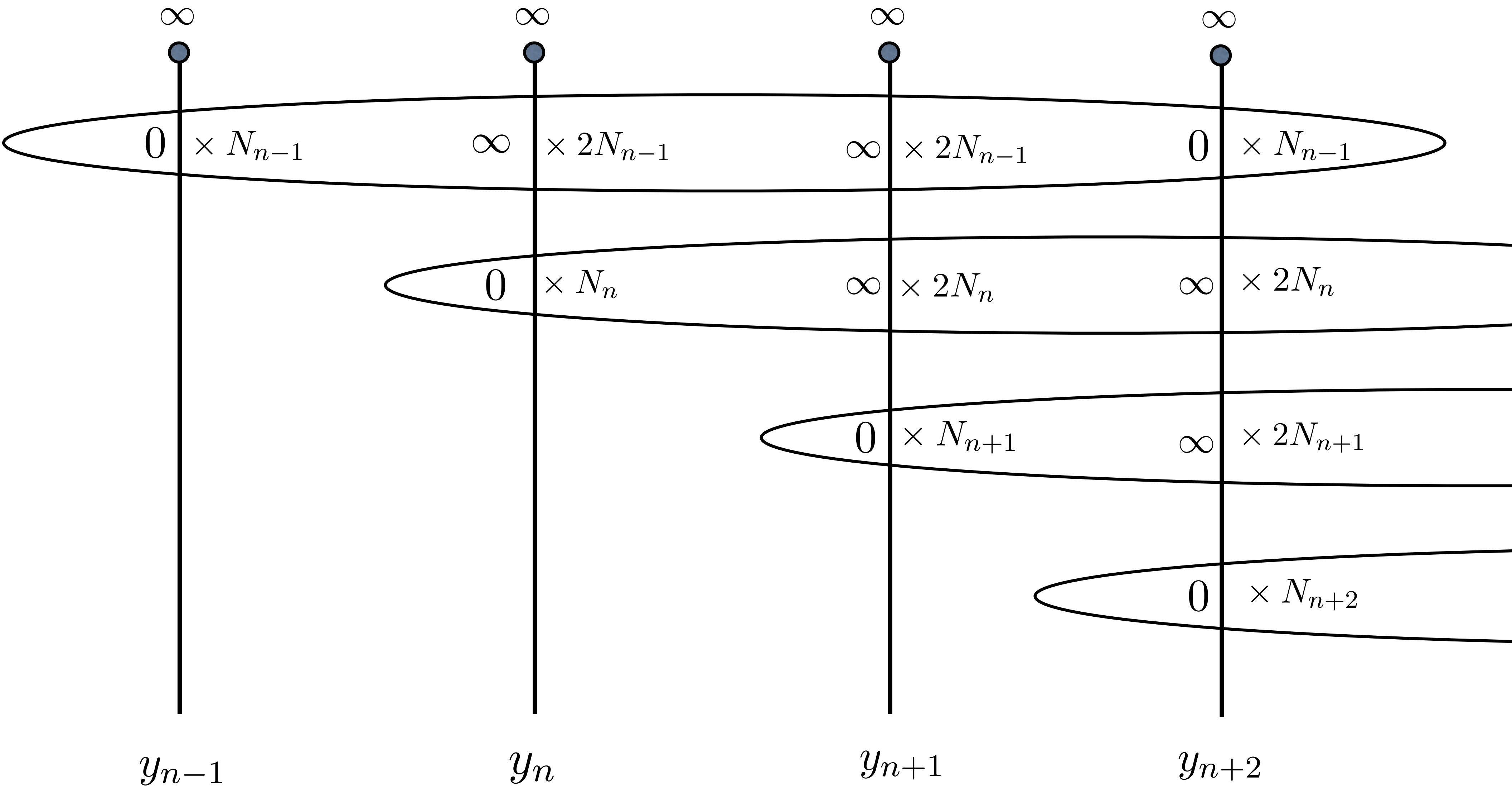
Takenawa's sequence of blow-ups for the Hietarinta-Viallet equation



He provided a rigorous proof that the algebraic entropy is

$$\frac{3+\sqrt{5}}{2}.$$

This value had been calculated using more heuristic methods by Hietarinta and Viallet.



$$d_{n+2} = N_{n+2} + N_{n-1}$$

$$d_{n+1} = 2(N_n + N_{n-1}) + 1$$

Substituting

$$N_n + N_{n-1} = (d_{n+1} - 1)/2 \quad \text{and} \quad N_{n+2} + N_{n-1} = d_{n+2}$$

in

$$(N_n + N_{n-1}) - (N_n + N_{n-3}) + (N_{n-2} + N_{n-3}) - (N_{n-1} + N_{n-2}) = 0$$

gives

$$d_{n+1} - 3d_n + d_{n-1} = 1.$$

Together with the initial conditions $d_0 = d_1 = 1$, this gives

$$d_n = \frac{\sqrt{5} - 1}{\sqrt{5}} \left(\frac{3 + \sqrt{5}}{2} \right)^n + \frac{\sqrt{5} + 1}{\sqrt{5}} \left(\frac{3 - \sqrt{5}}{2} \right)^n - 1.$$

It follows that the algebraic entropy is

$$\frac{3 + \sqrt{5}}{2}.$$

Summary