The Hirota-Miwa equation

– its integrability and reductions –

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Overview of the course

- Main objective:

  To present the Hirota-Miwa equation from 3 different points of view.

  - the Laurent property
  - explicit solutions (Hirota)
  - symmetries (Miwa)

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  To present the Hirota-Miwa equation from 3 different points of view.

  - the Laurent property
  - explicit solutions (Hirota)
  - symmetries (Miwa)

- In the second part:

  To discuss reductions of the HM equation to lower dimensional lattice equations and (if time permits) to explain the construction of Yang-Baxter maps (and, if time still permits, to present some new results on symmetry constraints for the HM equation).

The Laurent phenomenon

Definition: An initial value problem for a discrete (rational) equation has the *Laurent property* if its general solution is a Laurent polynomial of the initial data. [**S. Fomin & A. Zelevinsky, Adv. Appl. Math. 2002**]

For example: $$\begin{cases} f_m = \dfrac{f_{m-1}^2 + 1}{f_{m-2}}, \\ f_0 = X,\ f_1 = Y \end{cases}$$

The first several iterates are $f_2 = \dfrac{Y^2+1}{X}$, $\quad f_3 = \dfrac{\left(\frac{Y^2+1}{X}\right)^2 + 1}{Y} = \dfrac{(Y^2+1)^2 + X^2}{X^2 Y}$,

$$f_4 = \frac{(Y^2+1)^3 + 2X^2(Y^2+1) + X^4}{X^3Y^2}, \qquad f_5 = \frac{(Y^2+1)^4 + 3X^2(Y^2+1)^2 + 2X^4(Y^2+1) + X^6 + X^4}{X^4Y^3}$$

and we see that f_2, f_3, f_4, f_5 are Laurent polynomials of X and Y.

In fact: all f_n Laurent polynomials of X and Y $\quad\Rightarrow\quad$ the Laurent phenomenon !

The Laurent phenomenon

This phenomenon requires highly non-trivial factorizations to occur in the numerator and denominator of the general solution.

For example:
$$\begin{cases} f_m = \dfrac{f_{m-1} + f_{m-2} + 1}{f_{m-1}}, \\ f_0 = X,\ f_1 = Y. \end{cases}$$

for which the property fails immediately: $f_2 = \dfrac{X+Y+1}{Y}$, $\quad f_3 = \dfrac{Y^2 + X + 2Y + 1}{X+Y+1}$

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The Laurent phenomenon also occurs for lattice equations

($\leftrightarrow$ the Dodgson scheme...)

The Laurent phenomenon

For example ($\alpha \in \mathbb{C}^\times$):

$$
\begin{cases} \dfrac{f_{\ell,m-1}f_{\ell-1,m}+\alpha}{f_{\ell-1,m-1}} & (\ell, m > 0), \\ X_{\ell m} & (\ell = 0 \text{ or } m = 0) \end{cases}
$$

$$f_{11} = \frac{f_{10}f_{01}+\alpha}{f_{00}} = \frac{X_{10}X_{01}+\alpha}{X_{00}},$$

$$f_{21} = \frac{f_{20}f_{11}+\alpha}{f_{10}} = \frac{X_{10}X_{01}X_{20}+\alpha X_{20}+\alpha X_{00}}{X_{00}X_{10}},$$

$$f_{12} = \frac{f_{11}f_{02}+\alpha}{f_{01}} = \frac{X_{10}X_{01}X_{02}+\alpha X_{02}+\alpha X_{00}}{X_{00}X_{01}},$$

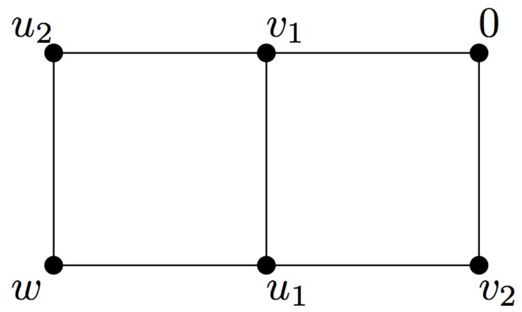
$$f_{22} = \frac{f_{21}f_{12}+\alpha}{f_{11}} = \frac{X_{10}X_{01}X_{20}X_{02}+\alpha X_{20}X_{02}+\alpha X_{00}X_{20}+\alpha X_{00}X_{02}+\alpha X_{00}^2}{X_{00}X_{10}X_{01}}$$

$$\vdots$$

The Laurent phenomenon

The discrete KdV equation: $$f_h = \frac{\alpha f_{h+v_1} f_{h+u_1} + \beta f_{h+v_2} f_{h+u_2}}{f_{h+w}},$$

$$v_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix},\ u_1 = \begin{pmatrix} -1 \\ -1 \end{pmatrix},\ v_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix},\ u_2 = \begin{pmatrix} -2 \\ 0 \end{pmatrix},\ w = \begin{pmatrix} -2 \\ -1 \end{pmatrix} \quad \in \mathbb{Z}^2$$



has the Laurent property when defined on a so-called *good domain*.

[S. Fomin & A. Zelevinsky, Adv. Appl. Math. 2002, T. Mase 2013]

The Laurent phenomenon

A good domain :

- the initial value problem is well-posed, i.e.: the fundamental recursion is defined everywhere in the domain and is uniquely defined in terms of the initial data which lie on the border of the domain
- the evolution exhibits a kind of hyperbolicity, i.e.: an evolved point only depends on finitely many initial data

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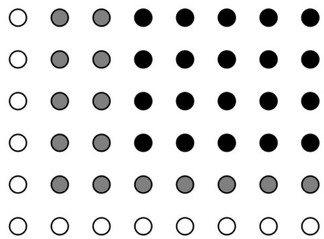

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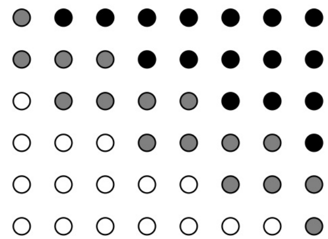

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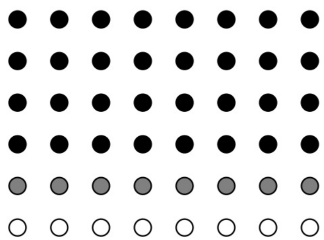

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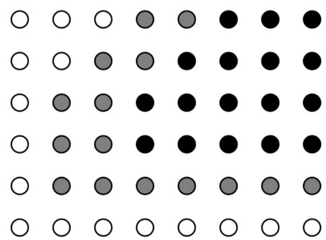

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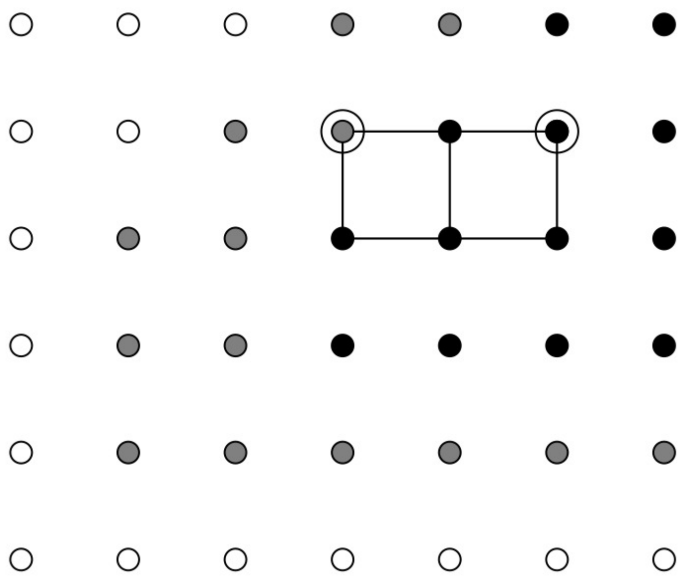
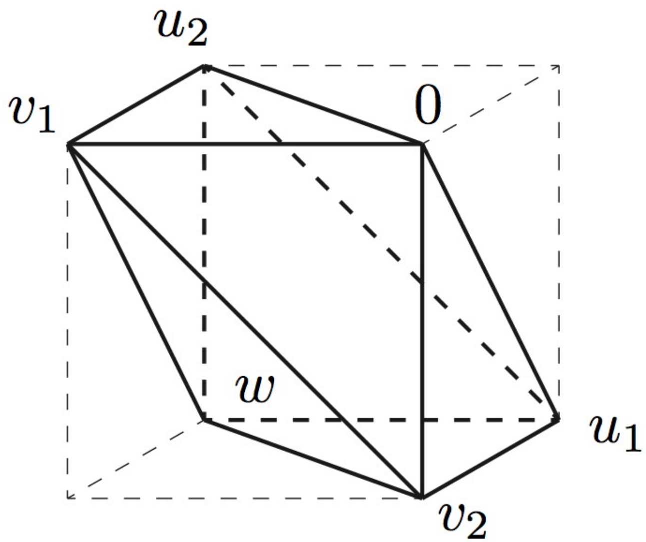


The Laurent phenomenon

The Hirota-Miwa equation: $$f_h = \frac{\alpha f_{h+v_1} f_{h+u_1} + \beta f_{h+v_2} f_{h+u_2}}{f_{h+w}},$$

$$v_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, u_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, u_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, w = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \quad \in \mathbb{Z}^3$$



has the Laurent property when defined on a good domain.

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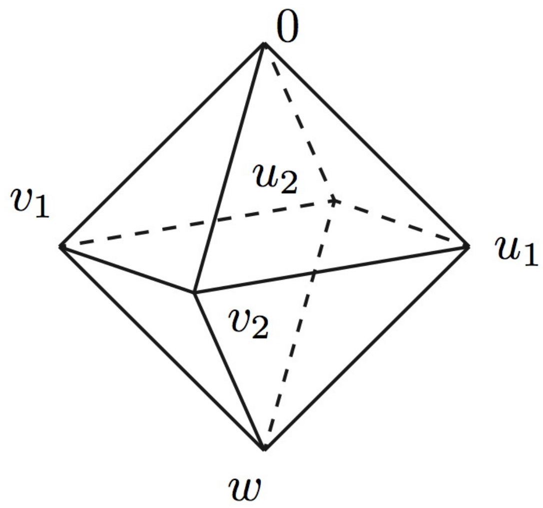
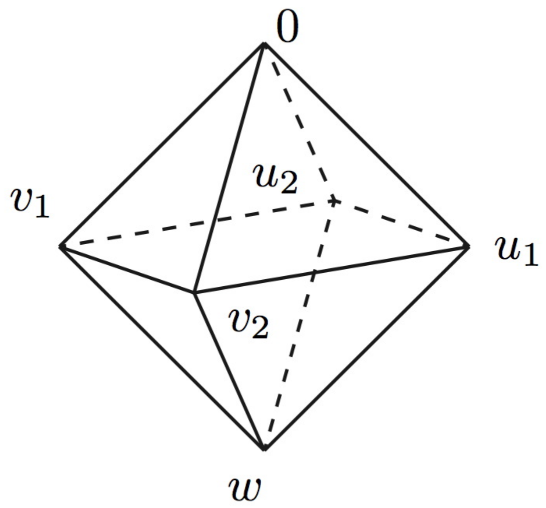


Theorem **[T. Mase 2013]**

The Hirota-Miwa equation and all its reductions (down to the 1-dimensional level) have the Laurent property, when defined on a good domain.

Furthermore, this property makes it possible to explicitly calculate the degree growth for the general solution.

The Laurent phenomenon



Theorem **[T. Mase 2013]**

The *non-autonomous* Hirota-Miwa equation $f_h = \dfrac{\alpha_h f_{h+v_1} f_{h+u_1} + \beta_h f_{h+v_2} f_{h+u_2}}{f_{h+w}}$

$(\alpha_h, \beta_h \in \mathbb{C}^\times)$ has the Laurent property, if and only if its coefficients satisfy

$$\alpha_h \alpha_{h+w} \beta_{h+v_1} \beta_{h+u_1} = \beta_h \beta_{h+w} \alpha_{h+v_2} \alpha_{h+u_2}$$

which is a necessary and sufficient condition for the existence of a gauge transformation to the autonomous HM equation. **[B. Grammaticos & A. Ramani, 2000]**

The Hirota-Miwa equation

$$(b-c)\,\tau(l+1,m,n)\,\tau(l,m+1,n+1) + (c-a)\,\tau(l,m+1,n)\,\tau(l+1,m,n+1) + (a-b)\,\tau(l,m,n+1)\,\tau(l+1,m+1,n) \;=\; 0$$

is a discrete version of KP in bilinear form, i.e. the continuum limit:

$$\boxed{x_1 = x_1^0 + \frac{l}{a} + \frac{m}{b} + \frac{n}{c}\,, \qquad x_2 = x_2^0 + \frac{l}{2a^2} + \frac{m}{2b^2} + \frac{n}{2c^2}\,, \qquad x_3 = x_3^0 + \frac{l}{3a^3} + \frac{m}{3b^3} + \frac{n}{3c^3}}$$

as $|a|, |b|, |c| \to \infty$ yields: $\left(4D_{x_3}D_{x_1} - D_{x_1}^4 - 3D_{x_2}^2\right)\tau(x_1,x_2,x_3)\cdot\tau(x_1,x_2,x_3) = 0$

The KP equation: $u_{x_2} = v_{x_1}\,, \quad u_{x_3} = \frac{1}{4}(u_{3x_1} + 12uu_{x_1}) + \frac{3}{4}v_{x_2}$

(where $u = \big(\log\tau\big)_{x_1x_1}$ and $v = \big(\log\tau\big)_{x_1x_2}$)

The Hirota-Miwa equation

Hirota discovered that the HM equation has discrete solitons, similar to those of the KP equation, when considering the equation on the cubic lattice $\mathbb{Z}^3$.

$$\tau_{1\text{-sol}} = 1 + d\left(\frac{a-q}{a-p}\right)^{\ell}\left(\frac{b-q}{b-p}\right)^{m}\left(\frac{c-q}{c-p}\right)^{n}$$

$$\sim (a-p)^{\ell}(b-p)^{m}(c-p)^{n} + d\,(a-q)^{\ell}(b-q)^{m}(c-q)^{n}$$

$$\tau_{2\text{-sol}} = 1 + d_1\, e^{\xi(p_1,q_1)} + d_2\, e^{\xi(p_2,q_2)} + d_1 d_2 \frac{(p_1-p_2)(q_1-q_2)}{(p_1-q_2)(q_1-p_2)}\, e^{\xi(p_1,q_1)+\xi(p_2,q_2)}$$

$$\text{with} \quad e^{\xi(p_i,q_i)} := \left(\frac{a-q_i}{a-p_i}\right)^{\ell}\left(\frac{b-q_i}{b-p_i}\right)^{m}\left(\frac{c-q_i}{c-p_i}\right)^{n}$$

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Casorati determinants
$$\tau = \begin{vmatrix} f^{(1)} & \Delta_\ell f^{(1)} & \dots & \Delta_\ell^{N-1} f^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ f^{(N)} & \Delta_\ell f^{(N)} & \dots & \Delta_\ell^{N-1} f^{(N)} \end{vmatrix}$$

$$\text{Eg.:} \quad f^{(i)} = \left(1-\frac{p_i}{a}\right)^\ell \left(1-\frac{p_i}{b}\right)^m \left(1-\frac{p_i}{c}\right)^n + \gamma_i \left(1-\frac{p_i'}{a}\right)^\ell \left(1-\frac{p_i'}{b}\right)^m \left(1-\frac{p_i'}{c}\right)^n$$

Gram determinants
$$\tau = \det\Big(\Omega(\varphi(i), \varphi^{*(j)})\Big)_{i,j=1..N}$$

$$\Delta_\ell\, \Omega^{(i,j)} = \varphi^{*(j)}\, S_\ell\left(\varphi^{(i)}\right), \ \Delta_m \Omega^{(i,j)} = \varphi^{*(j)}\, S_m\left(\varphi^{(i)}\right), \ \Delta_n \Omega^{(i,j)} = \varphi^{*(j)}\, S_n\left(\varphi^{(i)}\right)$$

$$\text{Eg.:} \quad \Omega^{(i,j)} = C + \frac{1}{q_j - p_j} \left(\frac{a-p_i}{a-q_j}\right)^\ell \left(\frac{b-p_i}{b-q_j}\right)^m \left(\frac{c-p_i}{c-q_j}\right)^n$$

The Hirota-Miwa equation

When considered on a cubic lattice, contrary to the case of octahedral symmetry for the Dodgson scheme, the entries of these determinants are no longer arbitrary but have to satisfy certain dispersion relations:

$$\boxed{\Delta_\ell\varphi^* = \Delta_m\varphi^* = \Delta_n\varphi^*}$$

$$\big(\Delta_\ell := a(S_\ell - 1),\ \Delta_m := b(S_m - 1),\ \Delta_n := c(S_n - 1)\big)$$

$$\rightarrow \qquad \boxed{\varphi^* \sim \left(1-\frac{p}{a}\right)^\ell \left(1-\frac{p}{b}\right)^m \left(1-\frac{p}{c}\right)^n}$$

or, similarly, for back-shifts: $\boxed{a\big(1-S_\ell^{-1}\big)\varphi = b\big(1-S_m^{-1}\big)\varphi = c\big(1-S_n^{-1}\big)\varphi}$

$$\boxed{\varphi \sim \left(1-\frac{q}{a}\right)^{-\ell} \left(1-\frac{q}{b}\right)^{-m} \left(1-\frac{q}{c}\right)^{-n}}$$

The discrete KP hierarchy

- These determinants also satisfy the bilinear equations proposed by Ohta et al. as the discrete KP hierarchy:

$$
{}^{\forall}N = 3, \ldots, M+1 \ : \qquad \begin{vmatrix} \tau_{\ell_1}\tau_{\widehat{\ell}_1} & \tau_{\ell_2}\tau_{\widehat{\ell}_2} & \cdots & \tau_{\ell_N}\tau_{\widehat{\ell}_N} \\ 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_N \\ \vdots & \vdots & & \vdots \\ a_1^{N-2} & a_2^{N-2} & \cdots & a_N^{N-2} \end{vmatrix} = 0
$$

$$
\tau_{\widehat{\ell}_k} = \tau(\ell_1 + 1, \ldots, \ell_{k-1} + 1, \ell_k, \ell_{k+1} + 1, \ldots, \ell_N + 1)
$$

for N-tuples $(\ell_1, \ldots, \ell_N)$, and parameters a_j $(j = 1, \ldots, N)$

[Y. Ohta et al., J. Phys. Soc. Jpn. 1993]

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- However, it can be shown that the HM-equations in the set

$$
\mathbf{E} = \{\ \mathrm{E}(\ell, m_j, m_k) \big|\ m_j, m_k \in \{m_1, \dots, m_M\},\ m_j \neq m_k\}
$$

generate all the bilinear equations in the dKP hierarchy.

The discrete KP hierarchy

$$\text{Lattice equation of order } M+1: \qquad \sum_{n=1}^{M+1} \frac{\tau_{\ell_n}\,\tau_{\widehat{\ell_n}}}{\prod_{k\neq n}(a_n-a_k)} \;=\; 0$$

$$\Leftrightarrow \quad \sum_{n=1}^{M+1} \operatorname*{Res}_{\lambda=a_n}\left[\tau(\mathbf{x}-\boldsymbol{\varepsilon}[\lambda^{-1}])\,\tau(\mathbf{x}'+\boldsymbol{\varepsilon}[\lambda^{-1}])\,\prod_{k=1}^{M+1}(a_k-\lambda)^{-1}\right] \;=\; 0$$

Miwa-transf.: $\mathbf{x}=(x_1,x_2,\ldots,x_{M+1}):=\sum_{n=1}^{M+1}\ell_n\,\boldsymbol{\varepsilon}[a_n^{-1}]\,,\quad \boldsymbol{\varepsilon}[\eta]=(\zeta,\frac{\zeta^2}{2},\ldots,\frac{\zeta^{M+1}}{M+1})$

$$\Leftrightarrow \quad \oint_{\lambda=\infty}\frac{\mathrm{d}\lambda}{2\pi i}\,\tau(\mathbf{x}-\boldsymbol{\varepsilon}[\lambda^{-1}])\tau(\mathbf{x}'+\boldsymbol{\varepsilon}[\lambda^{-1}])\,e^{\sum_{n=1}^{+\infty}(x_n-x_n')\lambda^n} \;=\; 0 \qquad (\mathbf{x}'=\mathbf{x}-\sum_{n=1}^{M+1}\boldsymbol{\varepsilon}[a_n^{-1}])$$

yields the KP bilinear identity at $M\to+\infty$, but this requires a certain 'smoothness' of τ

The discrete KP hierarchy

$$\tau : \mathbb{Z}^r \to \mathbb{C}\,, \qquad \boxed{(\mu-\nu)\,\tau_l\tau_{mn} + (\nu-\lambda)\,\tau_m\tau_{ln} + (\lambda-\mu)\,\tau_n\tau_{lm} = 0 \qquad \text{(HM)}}$$

for $l, m, n \in \{\ell_1, \ell_2, \ldots, \ell_r\}$ (all distinct) and $\lambda, \mu, \nu \in \{a_1, a_2, \ldots, a_r\}$ ($a_j \in \mathbb{C}^\times$, distinct)

where: $\tau_l := \tau\big|_{l\to l+1}\,,\quad \tau_{lm} := \tau\big|_{l\to l+1, m\to m+1}\,,$ etc.

Definition: The discrete KP hierarchy is the set of all HM-equations (HM) on the infinite dimensional lattice that is obtained as $r \to \infty$.

(※ the equations in this hierarchy generate the one proposed in [**Y. Ohta et al. 1993**].)

Definition: A tau function for the discrete KP hierarchy is a solution τ of all HM-equations in the hierarchy, defined up to the equivalence

$$^\forall c_j \in \mathbb{C}^\times \quad : \quad c_0 c_1^l c_2^m c_3^n\, \tau(\ell) \sim \tau(\ell)\,,$$

for any triple (l, m, n) of distinct directions on the lattice.

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is generated by the linear system

$$\psi_{lm} = \frac{1}{\lambda-\mu}\frac{\tau_l\tau_m}{\tau\tau_{lm}}\left[\lambda\,\psi_m - \mu\,\psi_l\right],$$

$$\psi_{mn} = \frac{1}{\mu-\nu}\frac{\tau_m\tau_n}{\tau\tau_{mn}}\left[\mu\,\psi_n - \nu\,\psi_m\right],$$

$$\psi_{ln} = \frac{1}{\lambda-\nu}\frac{\tau_l\tau_n}{\tau\tau_{ln}}\left[\lambda\,\psi_n - \nu\,\psi_l\right]$$

or by its adjoint:

$$\psi^* = \frac{1}{\lambda-\mu}\frac{\tau_l\tau_m}{\tau\tau_{lm}}\left[\lambda\,\psi^*_l - \mu\,\psi^*_m\right] = \frac{1}{\mu-\nu}\frac{\tau_m\tau_n}{\tau\tau_{mn}}\left[\mu\,\psi^*_m - \nu\,\psi^*_n\right] = \frac{1}{\lambda-\nu}\frac{\tau_l\tau_n}{\tau\tau_{ln}}\left[\lambda\,\psi^*_l - \nu\,\psi^*_n\right].$$

Symmetries of the discrete KP hierarchy

Theorem **[J.J.C. Nimmo 1997, RW et al. 1997]**

Given a tau function τ for the discrete KP hierarchy and an associated (adjoint) eigenfunction (ψ^*) ψ , the result of the map $\tau \mapsto \tilde{\tau} = \tau \times \psi$ $(\tau \mapsto \tilde{\tau} = \tau \times \psi^*)$ is a tau function, i.e.: $\tilde{\tau}$ satisfies the dKP hierarchy.

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Darboux transformations in fact give rise to a covariance of the linear problem:

$$\phi \;\mapsto\; \quad a\,\frac{\psi\, S_\ell\, \phi - \phi\, S_\ell\, \psi}{\psi}$$

which, for generic eigenfunctions ϕ related to τ, yields a solution to the linear problem associated to $\tilde{\tau} = \tau\,\psi$.

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These *Darboux* transformations of tau functions are the essential building blocks of the dKP hierarchy: they generate Casorati-type solutions for the HM equations and, most importantly, they are in fact equivalent to lattice-shifts:

E.g., defining $\psi^* := \dfrac{\tau_1}{\tau} \displaystyle\prod_{s=2}^{n} \left(\dfrac{a_s - a_1}{a_s} \right)^{\ell_s}$, one obtains a solution to all adjoint linear equations that do not involve the 1-direction: $\psi^* = \dfrac{1}{a_k - a_j} \dfrac{\tau_j \tau_k}{\tau \tau_{jk}} \left[a_k \psi_k^* - a_j \psi_j^* \right]$ $(j, k \neq 1)$

Hence, for this Darboux transformation: $\tau \mapsto \tau\, \psi^* \sim \tau_1$.

Symmetries of the discrete KP hierarchy

Theorem **[J.J.C. Nimmo 1997, RW et al. 1997]**

Given a tau function τ for the discrete KP hierarchy and an associated *squared eigenfunction potential* $\Omega(\psi, \psi^*)$, the map

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Theorem **[J.J.C. Nimmo 1997, RW et al. 1997]**

For any tau function τ for the dKP hierarchy, given an eigenfunction ψ and an adjoint eigenfunction ψ^* associated to τ, there exists a squared eigenfunction potential $\Omega(\psi, \psi^*)$ defined by the relations

$$\Delta_{\ell_j}\Omega(\psi, \psi^*) = \psi^* \, \psi_{\ell_j} \qquad (j = 1, 2, \ldots)\,,$$

(up to an additive constant). Δ_{ℓ_j} : $\Delta_{\ell_j} f(\ell_j) = a_j\big[f(\ell_j + 1) - f(\ell_j)\big]\,.$

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Such *binary Darboux* transformations $\tau \mapsto \tau\,\Omega$:

- generate Grammian determinant solutions to the HM equations
- realize the general action of $GL(\infty)$ on the Sato Grassmannian
- act in exactly the same way at the continuum limit, at which the squared eigenfunction potential is defined by the exact differential

$$d\Omega(\psi, \psi^*) \;=\; \psi\psi^* dx + (\psi_{x_1}\psi^* - \psi\psi^*_{x_1})dx_2 + \cdots$$

- explain why $(\psi\psi^*)_{x_1}$ generates the symmetry algebra for the KP hierarchy:

$$\hat{u} \;=\; \partial^2_{x_1} \log(\tau\Omega + \tau/\varepsilon) \;=\; u + \partial^2_{x_1} \log(1 + \varepsilon\,\Omega) = u + \varepsilon\,(\psi\psi^*)_{x_1} + \mathcal{O}(\varepsilon^2)$$

binary Darboux transformations

Consider:

$$\begin{array}{ccc} \tau & \xrightarrow{\ \phi\ } & \tilde{\tau} \\ {\scriptstyle \phi^*}\Big\downarrow & & \Big\uparrow{\scriptstyle \hat{\phi}} \\ \tau^* & \xleftarrow[\ \hat{\phi}^*\ ]{} & \hat{\tau} \end{array}$$

consistency condition:

$$\begin{array}{c} \tau\phi = \tilde{\tau} = \hat{\tau}\hat{\phi} \\ \tau\phi^* = \tau^* = \hat{\tau}\hat{\phi}^* \end{array} \quad \Rightarrow \quad \hat{\phi} \equiv \frac{\phi\hat{\phi}^*}{\phi^*}$$

One can show that if ϕ^* is an adjoint eigenfunction for τ, then $\dfrac{1}{\phi^*}$ is an eigenfunction associated to the tau function $\tau^* = \tau\phi^*$.

binary Darboux transformations

Hence the Bianchi diagram:

$$\begin{array}{ccc} \tau & \xleftarrow{\;1/\phi\;} & \tilde{\tau} \\ {\scriptstyle \phi^*}\big\downarrow & & \big\downarrow {\scriptstyle 1/\hat{\phi} \,\equiv\, \dfrac{\phi^*}{\phi\,\hat{\phi}^*}} \\ \tau^* & \xleftarrow[\;\hat{\phi}^*\;]{} & \hat{\tau} \end{array}$$

Require that ϕ^* is the Darboux transform of $\dfrac{1}{\hat{\phi}}$ by $\dfrac{1}{\phi}$:

$$\phi^* = a\left[S_\ell\left(\frac{\phi^*}{\phi\,\hat{\phi}^*}\right) - \frac{\phi}{S_\ell\phi}\,\frac{\phi^*}{\phi\,\hat{\phi}^*}\right] \quad \Leftrightarrow \quad a(S_\ell - 1)\left(\frac{\phi^*}{\hat{\phi}^*}\right) = \phi^* S_\ell \phi$$

and define $\Omega(\phi, \phi^*) := \dfrac{\phi^*}{\hat{\phi}^*} \equiv \dfrac{\phi}{\hat{\phi}}$ such that $\boxed{\Delta_\ell \Omega \equiv a(S_\ell - 1)\Omega = \phi^* S_\ell \phi}$

binary Darboux transformations

In fact, $\Omega(\phi,\phi^*)$ is a well-defined *eigenfunction* potential:

$$\boxed{a(S_\ell - 1)\Omega = \phi^*\, S_\ell\phi \ , \quad a_j(S_j - 1)\Omega = \phi^*\, S_j\phi \quad ({}^\forall j = 2,\cdots)}$$

as all cross-differences are equal.

Hence, all functions in the Bianchi-diagram are defined in terms of τ,ϕ and ϕ^* and in particular one has that:

$$\boxed{\hat{\tau} = \tau\,\Omega(\phi,\phi^*)}$$

Together with the transformation for the eigenfunctions and adjoint eigenfunctions this defines the binary Darboux transformation:

$$\psi \ \mapsto \ \psi - \phi\,\frac{\Omega(\psi,\phi^*)}{\Omega(\phi,\phi^*)}\ , \qquad \psi^* \ \mapsto \ \psi^* - \phi^*\,\frac{\Omega(\phi,\psi^*)}{\Omega(\phi,\phi^*)}$$

Special function solutions: $a = z^{-1}$, $b = 1$, $c = 0$

$$\boxed{\tau(m+1)\tau(\ell+1,n+1) - z\,\tau(\ell+1)\tau(m+1,n+1) + (z-1)\tau(n+1)\tau(\ell+1,m+1) = 0}$$

A linear system is obtained after a gauge transformation: $\phi \mapsto (-c)^{-n}\phi$

$$\begin{aligned}
\phi &= \frac{1}{z-1}\,\frac{\tau(\ell+1)\tau(m+1)}{\tau\,\tau(\ell+1,m+1)}\left[z\,\phi(m+1) - \phi(\ell+1)\right] \\
&= \frac{\tau(\ell+1)\tau(n+1)}{\tau\,\tau(\ell+1,n+1)}\left[\phi(\ell+1) + z\,\phi(n+1)\right] \\
&= \frac{\tau(m+1)\tau(n+1)}{\tau\,\tau(m+1,n+1)}\left[\phi(m+1) + \phi(n+1)\right]
\end{aligned}$$

If $\tau \equiv 1$, one obtains: $\boxed{z^{-1}\left[\phi - \phi(\ell+1)\right] = \phi - \phi(m+1) = \phi(n+1)}$

i.e., from $\varphi_\zeta \sim \zeta^n (1-\zeta)^m (1 - z\,\zeta)^\ell$ one can build Casorati determinant solutions.

Special function solutions: $a = z^{-1},\ b = 1,\ c = 0$

$$\boxed{\tau(m+1)\tau(\ell+1,n+1) - z\,\tau(\ell+1)\tau(m+1,n+1) + (z-1)\tau(n+1)\tau(\ell+1,m+1) = 0}$$

Trick: take arbitrary linear combinations of $\varphi_\zeta \sim \zeta^n(1-\zeta)^m(1-z\,\zeta)^\ell$.

In particular, φ_ζ can be used to define the hypergeometric function ${}_2\mathbf{F}_1$:

$${}_2\mathbf{F}_1\left({a\,,b \atop c} \,;z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \mathrm{d}t\ \ t^{b-1}\,(1-t)^{c-b-1}\,(1-zt)^{-a} \,.$$

Hence, using $\Phi_{\alpha,\beta,\gamma} := \displaystyle\int_0^1 \varphi_\zeta\ \zeta^{\beta-1}(1-\zeta)^{\gamma-\beta-1}\,(1-z\,\zeta)^{-\alpha}\mathrm{d}\zeta$ one obtains

$$\boxed{\tau(\ell,m,n) = \frac{\Gamma(n+\beta)\Gamma(m+\gamma-\beta)}{\Gamma(m+n+\gamma)}\ {}_2\mathbf{F}_1\left({\alpha-\ell\,,\,\beta+n \atop \gamma+m+n} \,;z\right)}$$

after one Darboux transformation and, using the freedom in α,β,γ , one can generate Casorati determinant solutions in terms of the $\Phi_{\alpha,\beta,\gamma}$.

a special continuum limit:

Take $z=\delta^2$ and rotate the axes as: $\boxed{\lambda=\ell\,,\ \mu=-m\,,\ \nu=n+\ell}$

$$\Rightarrow\quad \tau(\lambda+1,\nu+1)\tau(\mu-1,\nu-1)-\delta^2\tau(\lambda+1)\tau(\mu-1)+(\delta^2-1)\tau\,\tau(\lambda+1,\mu-1)=0$$

Introducing $\boxed{x=\frac{\mu}{\delta}\,,\ y=\frac{\lambda}{\delta}}$ and taking the limit $\delta\to\infty$ one obtains the 2D Toda lattice:

$$\tau(\nu+1)\tau(\nu-1)+\frac{\partial\tau}{\partial x}\frac{\partial\tau}{\partial y}-\tau\frac{\partial^2\tau}{\partial x\partial y}-\tau^2=0$$

$$\Leftrightarrow\qquad \frac{1}{2}D_xD_y\tau\cdot\tau\;=\;\tau(\nu+1)\tau(\nu-1)-\tau^2$$

Its linear system is obtained from $\phi\mapsto\delta^{-\nu}(-\delta^2)^\lambda\,\phi$

$$\Rightarrow\qquad \frac{\partial}{\partial x}\phi=\left(\frac{\partial}{\partial x}\log\frac{\tau(\nu+1)}{\tau}\right)\phi+\phi(\nu+1)\,,\quad \frac{\partial}{\partial y}\phi=-\frac{\tau(\nu+1)\tau(\nu-1)}{\tau^2}\,\phi(\nu-1)$$

Hence, for $\tau\equiv1$ one obtains, $\underline{\varphi\sim(\zeta\delta)^\nu\,(1-\zeta)^{-x\delta}(1-\frac{1}{\delta^2\zeta})^{y\delta}}\quad\xrightarrow[\zeta=\eta/\delta]{\delta\to\infty}\quad\underline{\eta^\nu e^{\eta x-y/\eta}}$

2DToda and Bessel:

$$\boxed{\frac{1}{2}D_xD_y\tau\cdot\tau \;=\; \tau(\nu+1)\tau(\nu-1)-\tau^2}$$

Taking linear combinations of $\varphi_\eta = \eta^\nu e^{\eta x - y/\eta}$ one can construct

$$\Phi := \frac{1}{2\pi i}\int_{-\infty}^{(0+)} \eta^{-1-s}\varphi_\eta\,\mathrm{d}\eta \qquad (xy>0)$$

and if one takes $\eta = t\sqrt{\frac{y}{x}}$ then, comparing to the Bessel function $J_s(z)$,

$$J_s(z) = \frac{1}{2\pi i}\int_{-\infty}^{(0+)} t^{-1-s} e^{\frac{z}{2}(t-\frac{1}{t})}\,\mathrm{d}t \qquad (z>0)$$

one sees that the function $\Phi^{(s)}$ can be used in Wronskian determinant solutions for the 2D Toda lattice:

$$\Phi^{(s)} = \left(\frac{y}{x}\right)^{\frac{\nu-s}{2}}\left[\frac{1}{2\pi i}\int_{-\infty}^{(0+)} t^{\nu-s-1} e^{\sqrt{xy}(t-\frac{1}{t})}\,\mathrm{d}t\,\right]$$

$$\Leftrightarrow \qquad \boxed{\Phi^{(s)}(\nu;x,y) \;\equiv\; \left(\frac{y}{x}\right)^{\frac{\nu-s}{2}} J_{s-\nu}(2\sqrt{xy})}$$

Reductions of the HM equation

Let $\tau(l,m,n)$ satisfy the HM equation with parameters λ, μ and ν.

$$\text{Impose}: \quad \tau_{mn} \equiv \tau(l,m+1,n+1) = \tau(l,m,n)$$

$$\therefore \quad \tau + (\tfrac{1}{\mu} + \tfrac{1}{\nu})\tau_{x_1} + \tfrac{1}{2}(\tfrac{1}{\mu} + \tfrac{1}{\nu})^2\,\tau_{2x_1} + \tfrac{1}{2}(\tfrac{1}{\mu^2} + \tfrac{1}{\nu^2})\tau_{x_2} + \cdots = \tau$$

Reductions of the HM equation

Let $\tau(l,m,n)$ satisfy the HM equation with parameters λ,μ and ν.

Impose : $\quad \tau_{mn} \equiv \tau(l,m+1,n+1) = \tau(l,m,n) \quad$ (with $\mu+\nu=0$)

$$\therefore \quad \tau + \frac{1}{\mu^2}\tau_{x_2} + \mathcal{O}(\tfrac{1}{\mu^3}) = \tau \qquad \text{or} \quad |\mu| \to \infty : \quad \tau_{x_2} = 0 \qquad \text{(KdV)}$$

Reductions of the HM equation: dKdV

$$\boxed{S_mS_n\tau = \tau\,,\quad S_mS_n\Psi = \zeta\Psi\;;\quad \Phi := S_n\Psi} \qquad (\alpha_1 = \lambda/\mu\,,\ \alpha_2 = -\lambda/\mu)$$

2x2 Lax pair:
$$\begin{cases} S_\ell\begin{pmatrix}\Psi\\ \Phi\end{pmatrix} = \begin{pmatrix}(1-\alpha_2^{-1})u & \alpha_2^{-1}\\ \zeta\alpha_1^{-1} & (1-\alpha_1^{-1})u^{-1}\end{pmatrix}\cdot\begin{pmatrix}\Psi\\ \Phi\end{pmatrix} \\ S_m\begin{pmatrix}\Psi\\ \Phi\end{pmatrix} = \begin{pmatrix}(1-\alpha_1)\,v & \alpha_1\alpha_2^{-1}\\ \zeta & 0\end{pmatrix}\cdot\begin{pmatrix}\Psi\\ \Phi\end{pmatrix}\end{cases}$$

with compatibility condition:

$$\boxed{S_mu = \frac{1}{v+\delta u}\,,\qquad S_\ell\, v = \frac{v\,S_mu}{u}} \qquad \left(\delta = \tfrac{\alpha_1(\alpha_2-1)}{\alpha_2(\alpha_1-1)}\right)$$

which is the dKdV equation : $\dfrac{1}{S_\ell S_m u} - \dfrac{1}{u} = \delta\big(S_\ell\, u - S_m u\big)$ in disguise...

Reductions of the HM equation: dKdV

1. The map $(u, v) \mapsto (S_m u, S_\ell\, v)$ **is birational**

and in fact even "quadrirational" [**Adler et al. Comm. Anal. Geom. 12 (2004) 967**]

i.e.: the map $(u, S_\ell\, v) \;\;\mapsto\;\; (S_m u, v)$ is also birational !

2. The map $(u, S_\ell\, v) \mapsto (S_m u, v)$ **is in fact related to a Yang-Baxter map**

$$R(\lambda, \mu): \quad (u, S_\ell\, v) \;\;\mapsto\;\; \left((S_\ell\, v)\, \frac{u\, S_\ell\, v - 1}{\mu - \lambda u\, S_\ell\, v},\; u\, \frac{\mu - \lambda u\, S_\ell\, v}{u\, S_\ell\, v - 1} \right)$$

Yang-Baxter maps

i.e., it is related to certain "set-theoretical solutions" to the Yang-Baxter equation
[A. Veselov, MSJ Mem. 17 (2007) 145]

I.e., to a map $X \times X \to X \times X$ (where X is a set)

$$\begin{array}{rcccc} R & : & X \times X & \to & X \times X \\ & & \Cup & & \Cup \\ & & (x, y) & \mapsto & (f(x,y), g(x,y)) \end{array}$$

that satisfies the Yang-Baxter relation (with spectral parameters)

$$\boxed{\begin{aligned} & R_{12}(\lambda_1, \lambda_2) R_{13}(\lambda_1, \lambda_3) R_{23}(\lambda_2, \lambda_3) \\ & \qquad = R_{23}(\lambda_2, \lambda_3) R_{13}(\lambda_1, \lambda_3) R_{12}(\lambda_1, \lambda_2) \end{aligned}}$$

$$\begin{array}{rcccc} R_{ij} & : & X \times X \times \cdots \times X & \to & X \times X \times \cdots \times X \\ & & \Cup & & \Cup \\ & & (\ldots, x_i, \ldots, x_j, \ldots) & \mapsto & (\ldots, f(x_i, x_j), \ldots, g(x_i, x_j), \ldots) \end{array}$$

Reductions of the HM equation: dKdV

1. This Yang-Baxter map as the "companion" map

$$\widetilde{R}(\lambda,\mu):\ (u,v)\mapsto\left(\frac{1}{v}\frac{v+\mu u}{v+\lambda u},\ \frac{1}{u}\frac{v+\mu u}{v+\lambda u}\right)$$

$$\text{for which:}\quad \widetilde{R}(\delta,0):\ (u,v)\mapsto\left(\frac{1}{v+\delta u},\ \frac{1}{u}\frac{v}{v+\delta u}\right)\ \equiv\ (S_m u, S_\ell v)$$

2. The map $\widetilde{R}$ is obtained from the Type II (1,1) reduction $(S_n S_k \tau = \tau(\ell,m,n,k))$

$$\begin{cases} T_\ell\begin{pmatrix}\Psi\\ \Phi\end{pmatrix} = \begin{pmatrix}(1-\alpha_2^{-1})u & \alpha_2^{-1}\\ \zeta\alpha_3^{-1} & (1-\alpha_3^{-1})u^{-1}\end{pmatrix}\cdot\begin{pmatrix}\Psi\\ \Phi\end{pmatrix}\\ T_1\begin{pmatrix}\Psi\\ \Phi\end{pmatrix} = \begin{pmatrix}(1-\alpha_1\alpha_2^{-1})v & \alpha_1\alpha_2^{-1}\\ \zeta\alpha_1\alpha_3^{-1} & (1-\alpha_1\alpha_3^{-1})v^{-1}\end{pmatrix}\cdot\begin{pmatrix}\Psi\\ \Phi\end{pmatrix}\end{cases}$$

c.c.: $$\boxed{(S_m u, S_\ell v) = R\left(\frac{\alpha_3(\alpha_2-1)}{\alpha_2(\alpha_3-1)},\frac{\alpha_3(\alpha_1-\alpha_3)(\alpha_1-\alpha_2)}{\alpha_2\alpha_1^2(\alpha_3-1)^2}\right)(u,v)}$$

(for $v\to\frac{\alpha_1\alpha_2(\alpha_3-1)}{\alpha_3(\alpha_1-\alpha_2)}v$)

General reduction $(n \geq 2)$ $S_2 \cdots S_{n+1}\Psi = \zeta\Psi$, $S_2 \cdots S_{n+1}\tau = \tau$

$$S_\ell \boldsymbol{\Psi} = L(\mathbf{u},\zeta)\cdot\boldsymbol{\Psi}, \qquad S_1\boldsymbol{\Psi} = M(\mathbf{v},\zeta)\cdot\boldsymbol{\Psi}$$

$$L(\mathbf{u},\zeta) = \begin{pmatrix} (1-\alpha_2^{-1})u_2 & \alpha_2^{-1} & 0 & \dots & 0 \\ 0 & (1-\alpha_3^{-1})u_3 & \alpha_3^{-1} & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & & \alpha_n^{-1} \\ \zeta\alpha_{n+1}^{-1} & 0 & \dots & & (1-\alpha_{n+1}^{-1})u_{n+1} \end{pmatrix}$$

$$M(\mathbf{u},\zeta) = \begin{pmatrix} (1-\alpha_1\alpha_2^{-1})v_2 & \alpha_1\alpha_2^{-1} & 0 & \dots & 0 \\ 0 & (1-\alpha_1\alpha_3^{-1})v_3 & \alpha_1\alpha_3^{-1} & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & & \alpha_1\alpha_n^{-1} \\ \zeta\alpha_1\alpha_{n+1}^{-1} & 0 & \dots & & (1-\alpha_1\alpha_{n+1}^{-1})v_{n+1} \end{pmatrix}$$

$$\text{with} \qquad \prod_{k=1}^{n} u_k = \prod_{k=1}^{n} v_k = 1$$

General reduction $(n \geq 2)$ $S_2 \cdots S_{n+1}\Psi = \zeta\Psi$, $S_2 \cdots S_{n+1}\tau = \tau$

- The compatibility condition of this Lax pair yields a quadrirational map

$$\widetilde{R}(\alpha) \; : \; (\mathbf{u}, \mathbf{v}) \; \mapsto \; (S_1\mathbf{u}, S_\ell\mathbf{v})$$

which is the companion map of a **Yang-Baxter** map

$$R(\alpha) \; : \; (\mathbf{u}, S_\ell\mathbf{v}) \; \mapsto \; (S_1\mathbf{u}, \mathbf{v})$$

related to the Yang-Baxter maps constructed by Etingof in relation to geometric crystals of type A_{n-1}. [**P. Etingof, Comm. Algebra 31 (2003) 1961**]

- The dynamical systems given by the maps R and $\widetilde{R}$ can always be ultradiscretized for appropriate boundary conditions.

Example: $n = 3$

$$\boxed{S_1 u_j = \frac{u_j\left(\mu_{j+1}u_{j+1} - \nu_{j+1}v_{j+1}\right)}{\mu_j u_j - \nu_j v_j}\,, \qquad S_\ell v_j = \frac{v_j\, T_1 u_j}{u_1}} \qquad (j = 1, 2, 3)$$

for $\mu_j = 1 - \alpha_{j+1}^{-1}$, $\nu_j = \alpha_1^{-1} - \alpha_{j+1}^{-1}$, defined cyclically $(u_4, v_4, \mu_4, \nu_4) = (u_1, v_1, \mu_1, \nu_1)$

$$\Rightarrow \qquad \widetilde{R}(\mu, \nu) : \ (\mathbf{u}, \mathbf{v}) \ \mapsto \ (\widetilde{\mathbf{u}}, \hat{\mathbf{v}})$$

$$\boxed{\tilde{u}_j = \frac{u_j\left(\mu_{j+1}u_{j+1} - \nu_{j+1}v_{j+1}\right)}{\mu_j u_j - \nu_j v_j}\,, \quad \hat{v}_j = \frac{v_j\left(\mu_{j+1}u_{j+1} - \nu_{j+1}v_{j+1}\right)}{\mu_j u_j - \nu_j v_j}}$$

$$\text{and} \qquad R(\mu, \nu) : \ (\mathbf{u}, \hat{\mathbf{v}}) \ \mapsto \ (\widetilde{\mathbf{u}}, \mathbf{v})$$

$$\tilde{u}_j = \frac{u_j\left(\mu_{j-1}\mu_{j+1}u_{j-1}u_{j+1} + \mu_{j-1}\nu_j u_{j-1}\hat{v}_j + \nu_j\nu_{j+1}\hat{v}_j\hat{v}_{j+1}\right)}{\mu_{j-1}\mu_j u_{j-1}u_j + \mu_j\nu_{j+1}u_j\hat{v}_{j+1} + \nu_{j-1}\nu_{j+1}\hat{v}_{j-1}\hat{v}_{j+1}}$$

$$v_j = \frac{\hat{v}_j\left(\mu_{j-1}\mu_j u_{j-1}u_j + \mu_j\nu_{j+1}u_j\hat{v}_{j+1} + \nu_{j-1}\nu_{j+1}\hat{v}_{j-1}\hat{v}_{j+1}\right)}{\mu_{j-1}\mu_{j+1}u_{j-1}u_{j+1} + \mu_{j-1}\nu_j u_{j-1}\hat{v}_j + \nu_j\nu_{j+1}\hat{v}_j\hat{v}_{j+1}}$$

General idea:

The Lax equation $\big(S_1 L(\mathbf{u})\big)\cdot M(\mathbf{v}) = \big(S_\ell\, M(\mathbf{v})\big)\cdot L(\mathbf{u})$ can be interpreted as the factorization problem $\boxed{A(\tilde{\mathbf{x}})\cdot A(\tilde{\mathbf{y}}) = A(\mathbf{y})\cdot A(\mathbf{x})}$ for a matrix $A(\mathbf{x}) = \sum_{i=1..n} \big(x_i E_{i,i} + \alpha_{i+1} E_{i,i+1}\big)$ with $(E_{i,j})_{kl} = \delta_{ik}\delta_{jl}$ and

$$x_j = (1-\alpha_{j+1}^{-1})u_{j+1}\ ,\quad y_j = (\alpha_1^{-1}-\alpha_{j+1}^{-1})S_\ell\, v_{j+1}$$
$$x_j = (1-\alpha_{j+1}^{-1})S_1\, u_{j+1}\ ,\quad y_j = (\alpha_1^{-1}-\alpha_{j+1}^{-1})v_{j+1}$$

Such factorizations, if uniquely solvable, are known to give rise to Yang-Baxter maps $R:\ (\mathbf{x},\mathbf{y}) \mapsto (\tilde{\mathbf{x}},\tilde{\mathbf{y}})$, i.e.:

$$R:\quad (\mathbf{u}, S_\ell\,\mathbf{v}) \ \mapsto\ (S_1\mathbf{u}, \mathbf{v})$$

The dynamical system obtained from the Lax equation corresponds to its companion map

$$\widetilde{R}:\quad (\mathbf{u}, \mathbf{v}) \ \mapsto\ (S_1\mathbf{u}, S_\ell\,\mathbf{v})$$

[A. Veselov, Phys. Lett. A 314 (2003) 214]

General idea:

The Lax equation yields a unique evolutionary system, defined in terms of $\widetilde{R}$, with conserved quantities

$$X := \prod_{j=1}^{n} x_j = \prod_{j=1}^{n}(1-\alpha_{j+1}^{-1}) \ , \quad Y := \prod_{j=1}^{n} y_j = \prod_{j=1}^{n}(1-\alpha_1\alpha_{j+1}^{-1})$$

The factorization problem has a trivial solution $\tilde{x}_j = y_j,\ \tilde{y}_j = x_j$, which violates the conservation law if $\alpha_1 \neq 1$. Hence there is a unique factorization compatible with the solitonic evolution, given by the Yang-Baxter map:

$$R \ : \qquad (\mathbf{x},\mathbf{y}) \ \mapsto \ (\tilde{\mathbf{x}},\tilde{\mathbf{y}})$$

$$\tilde{x}_j = \frac{P_{j-1}}{P_j}\, x_j \ , \quad \tilde{y}_j = \frac{P_j}{P_{j-1}}\, y_j$$

$$\text{with} \quad P_j = \sum_{\ell=1}^{n} \prod_{k=1}^{\ell-1} y_{j+k} \prod_{k=\ell+1}^{n} x_{j+k}$$

[K. Kajiwara et al., Lett. Math. Phys. 60 (2002) 211], [P. Etingof, Comm. Algebra 31 (2003) 1961], [Y. Suris & A. Veselov, J. Nonl. Math. Phys. 10 (2003) 223]

General reduction $(n \geq 2)$ $\quad S_2 \cdots S_{n+1}\Psi = \zeta\Psi\ ,\quad S_2 \cdots S_{n+1}\tau = \tau$

- There exists a systematic construction of (ultradiscretizable) companion maps to Yang-Baxter maps.

- These Yang -Baxter maps are identical to those that arise for A_n-type geometric crystals.

- They are obtained from general reductions of the dKP hierarchy;

  Other reductions like e.g.

$$S_1 \dots S_n \tau = \tau$$

  are obtained through degeneracy of the general case: $\alpha_{n+1} = \alpha_1$.

- However, not all reductions lead to simple maps. E.g.:

$$\tau(\ell + 1, m_1 + 1, m_2 + 1) = \tau(\ell, m_1, m_2)$$

Symmetry constraints

[B. Konopelchenko & W. Strampp 1991]

“ NLS is obtained from the KP hierarchy by symmetry-constraint... ”

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$u(x_1, x_2, x_3)$: solution to the KP equation

$\psi(x_1, x_2, x_3)$: solution to $\psi_{x_2} = \psi_{x_1 x_1} + 2u\psi\,, \ \cdots$

$\psi^*(x_1, x_2, x_3)$: solution to $\psi^*_{x_2} = -(\psi^*_{x_1 x_1} + 2u\psi^*)\,, \ \cdots$

symmetry constraint : $u_{x_1} = (\psi\psi^*)_{x_1} \quad \Rightarrow \quad \begin{cases} \psi_{x_2} = \psi_{x_1 x_1} + 2(\psi\psi^*)\psi \\ -\psi^*_{x_2} = \psi^*_{x_1 x_1} + 2(\psi\psi^*)\psi^* \end{cases}$

(as $(\psi\psi^*)_{x_1}$ generates the (generalized) symmetries for the KP hierarchy)

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(as $(\psi\psi^*)_{x_1}$ generates the (generalized) symmetries for the KP hierarchy)

Various problems concerning such reductions :

- How to obtain Lax pairs ? **[Y. Cheng 1991]**
- How to obtain solutions, especially for $u = c + \psi\psi^*$? **[RW & I. Loris 1999]**

Reductions of the HM equation: discrete NLS

Let $\tau(l,m,n)$ satisfy the HM equation with parameters λ,μ and ν ;
let ϕ and ϕ^* be an eigenfunction and adjoint eigenfunction associated to τ.

Impose : $\boxed{\nu\,\tau_n = \tau \times \Omega(\phi,\phi^*)}$

$$\text{or :}\quad \nu\left(\tau_n - \tau\right) = \tau\times\Omega \;\Rightarrow\quad \tau_{x_1} + \mathcal{O}(\tfrac{1}{\nu}) = \tau\,\Omega$$

$$|\nu|\to\infty :\quad \tau_{x_1} = \tau\,\Omega \;\Rightarrow\quad \partial^2_{x_1}\log\tau = \Omega_{x_1} \equiv \phi\phi^*$$

$$\therefore\quad \text{“}\; u_{x_1} = \left(\phi\phi^*\right)_{x_1}\text{”}\quad \text{(NLS)}$$

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Impose : $\boxed{\nu\,\tau_n = \tau \times \Omega(\phi, \phi^*)}$

$$\Rightarrow \quad \Delta_m\Big(\frac{\tau_n}{\tau}\Big) \equiv \frac{1}{\nu}\,\phi^*\phi_m \quad \text{or :} \quad \boxed{\frac{\tau_m\tau_n}{\tau\tau_{mn}} \equiv 1 - \frac{1}{\mu\nu}\phi^*\varphi_{mn}} \quad \text{with} \quad \varphi_n := \frac{\tau\phi}{\tau_n}$$

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$$\phi^* = \frac{1}{\mu-\nu}\,\frac{\tau_m\tau_n}{\tau\tau_{mn}}\big(\mu\phi^*_m - \nu\phi^*_n\big) \quad \Leftrightarrow \quad \boxed{\mu\phi^*_m - \nu\phi^*_n = \frac{(\mu-\nu)\phi^*}{1-\frac{1}{\mu\nu}\phi^*\varphi_{mn}}}$$

Reductions of the HM equation: discrete NLS

Let $\tau(l,m,n)$ satisfy the HM equation with parameters λ, μ and ν ;
let ϕ and ϕ^* be an eigenfunction and adjoint eigenfunction associated to τ.

Impose : $\boxed{\nu\,\tau_n = \tau \times \Omega(\phi, \phi^*)}$

$$\Rightarrow \quad \Delta_m\Big(\frac{\tau_n}{\tau}\Big) \equiv \frac{1}{\nu}\,\phi^*\phi_m \quad \text{or :} \quad \boxed{\frac{\tau_m\tau_n}{\tau\tau_{mn}} \equiv 1 - \frac{1}{\mu\nu}\phi^*\varphi_{mn}} \quad \text{with} \quad \varphi_n := \frac{\tau\phi}{\tau_n}$$

$$\phi^* = \frac{1}{\mu-\nu}\,\frac{\tau_m\tau_n}{\tau\tau_{mn}}\big(\mu\phi^*_m - \nu\phi^*_n\big) \quad \Leftrightarrow \quad \boxed{\mu\phi^*_m - \nu\phi^*_n = \frac{(\mu-\nu)\phi^*}{1-\frac{1}{\mu\nu}\phi^*\varphi_{mn}}}$$

$$\phi_{mn} = \frac{1}{\mu-\nu}\,\frac{\tau_m\tau_n}{\tau\tau_{mn}}\big(\mu\phi_n - \nu\phi_m\big) \qquad \text{but what does } \varphi \text{ satsify ?}$$

Reductions of the HM equation: discrete NLS

Let $\tau(l,m,n)$ satisfy the HM equation with parameters λ, μ and ν ;
let ϕ and ϕ^* be an eigenfunction and adjoint eigenfunction associated to τ.

Casorati solutions to HM : $$\tau^{(N)} = \left|\Delta^{j-1} f^{(i)}(\ell)\right|_{i,j=1..N}$$

(where the $f^{(i)}(\ell)$ satisfy $\Delta_{\ell_j} f^{(i)}(\ell) = \Delta_{\ell_k} f^{(i)}(\ell)$ for all possible directions on the lattice)

then, in general: $$\phi^* = \frac{\tau^{(N+1)}}{\tau^{(N)}} \quad \text{and} \quad \phi = \frac{\tau^{(N-1)}}{\tau^{(N)}}$$

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However, think of ϕ as $\phi := \dfrac{\tau'_n}{\tau}$ $\Rightarrow$ $\varphi_n = \dfrac{\tau'_n}{\tau_n}$, or: $\varphi = \dfrac{\tau'}{\tau}$, which is also

an eigenfunction for τ : $$\boxed{\varphi_{mn} = \frac{1}{\mu-\nu}\frac{\tau_m\tau_n}{\tau\tau_{mn}}\big(\mu\varphi_n - \nu\varphi_m\big)}$$

discrete NLS: solutions

For example, for $N = 2$:

$$\varphi = \frac{f^{(1)}}{\begin{vmatrix} f^{(1)} & \Delta f^{(1)} \\ f^{(2)} & \Delta f^{(2)} \end{vmatrix}} \quad \text{and} \qquad \phi = \frac{f_n^{(1)}}{\begin{vmatrix} f^{(1)} & \Delta f^{(1)} \\ f^{(2)} & \Delta f^{(2)} \end{vmatrix}}$$

must both be eigenfunctions... Hence: $f^{(2)} \equiv f_n^{(1)}$!!

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Solutions are therefore given by bi-directional Casorati determinants:

$$\tau_{bid}^{(N)} := \left| \Delta_n^{i-1} \Delta^{j-1} f(\ell) \right|_{i,j=1..N}$$

(where f satisfies $\Delta_{\ell_j} f = \Delta_{\ell_k} f$ for all possible directions on the lattice)

and:

$$\phi^* = \frac{\tau_{bid}^{(N+1)}}{\tau_{bid}^{(N)}}, \quad \varphi = \frac{\tau_{bid}^{(N-1)}}{\tau_{bid}^{(N)}} \qquad \left(\phi = \frac{(\tau_{bid}^{(N-1)})_n}{\tau_{bid}^{(N)}} \right)$$

Reductions of the HM equation: discrete NLS

Let $\tau(l,m,n)$ satisfy the HM equation with parameters λ, μ and ν ;
let ϕ and ϕ^* be an eigenfunction and adjoint eigenfunction associated to τ.

Impose : $\boxed{\nu\,\tau_n = \tau \times \Omega(\phi,\phi^*)}$

$$\Rightarrow \quad \Delta_m\Big(\frac{\tau_n}{\tau}\Big) = \frac{1}{\nu}\,\Delta_m\Omega \equiv \frac{1}{\nu}\,\phi^*\phi_m \quad \text{or :} \quad \boxed{\frac{\tau_m\tau_n}{\tau\tau_{mn}} \equiv 1 - \frac{1}{\mu\nu}\phi^*\varphi_{mn}}$$

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$$\varphi_{mn} = \frac{1}{\mu-\nu}\,\frac{\tau_m\tau_n}{\tau\tau_{mn}}\big(\mu\varphi_n - \nu\varphi_m\big) \quad \Leftrightarrow \quad \boxed{\mu\varphi_n - \nu\varphi_m = \frac{(\mu-\nu)\varphi_{mn}}{1-\frac{1}{\mu\nu}\phi^*\varphi_{mn}}}$$

discrete NLS

$$\boxed{\mu\phi^*_m - \nu\phi^*_n = \frac{(\mu-\nu)\phi^*}{1-\frac{1}{\mu\nu}\phi^*\varphi_{mn}}\ , \qquad \mu\varphi_n - \nu\varphi_m = \frac{(\mu-\nu)\varphi_{mn}}{1-\frac{1}{\mu\nu}\phi^*\varphi_{mn}}} \qquad \text{(dNLS)}$$

has solutions $\phi^* = \dfrac{\tau_{bid}^{(N+1)}}{\tau_{bid}^{(N)}}\ ,\quad \varphi = \dfrac{\tau_{bid}^{(N-1)}}{\tau_{bid}^{(N)}}$ (with $\tau_{bid}^{(N)}$ a bi-directional Casoratian)

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$$\mu\phi^*_m - \nu\phi^*_n = \frac{(\mu-\nu)\phi^*}{1-\frac{1}{\mu\nu}\phi^*\varphi_{mn}} \ , \qquad \mu\varphi_n - \nu\varphi_m = \frac{(\mu-\nu)\varphi_{mn}}{1-\frac{1}{\mu\nu}\phi^*\varphi_{mn}} \qquad \text{(dNLS)}$$

has solutions $\phi^* = \dfrac{\tau^{(N+1)}_{bid}}{\tau^{(N)}_{bid}}$, $\varphi = \dfrac{\tau^{(N-1)}_{bid}}{\tau^{(N)}_{bid}}$ (with $\tau^{(N)}_{bid}$ a bi-directional Casoratian)

and a Lax pair: $(\Psi : \ \mathbb{Z}^2 \longrightarrow \mathbb{C}^2)$

$$\Psi_m = \frac{1}{\mu-\kappa}\begin{pmatrix} \mu & -\varphi_m \\ \phi^* & \mu-\kappa-\frac{1}{\mu}\phi^*\varphi_m \end{pmatrix} \cdot \Psi$$

$$\Psi_n = \frac{1}{\nu-\kappa}\begin{pmatrix} \nu & -\varphi_n \\ \phi^* & \nu-\kappa-\frac{1}{\nu}\phi^*\varphi_n \end{pmatrix} \cdot \Psi$$

[E. Date etal. 1983]

semi-discrete NLS

$$\mu\phi^*_m - \nu\phi^*_n = \frac{(\mu-\nu)\phi^*}{1-\frac{1}{\mu\nu}\phi^*\varphi_{mn}}\ , \qquad \mu\varphi_n - \nu\varphi_m = \frac{(\mu-\nu)\varphi_{mn}}{1-\frac{1}{\mu\nu}\phi^*\varphi_{mn}} \qquad \text{(dNLS)}$$

At the limit $|\nu| \to \infty$ one obtains a time-discretisation of the NLS equation:

$$\begin{cases} \mu(\phi^*_m - \phi^*) = \phi^*_{x_1} - \dfrac{1}{\mu}(\phi^*)^2\varphi_m \\[2ex] \mu(\varphi_m - \varphi) = (\varphi_m)_{x_1} + \dfrac{1}{\mu}\phi^*(\varphi_m)^2 \end{cases}$$

[E. Date etal. 1983]

with Lax pair

$$\Psi_{x_1} = \begin{pmatrix} \kappa & -\varphi \\ \phi^* & 0 \end{pmatrix} \cdot \Psi$$

$$\Psi_m = \frac{1}{\mu-\kappa} \begin{pmatrix} \mu & -\varphi_n \\ \phi^* & \mu-\kappa-\frac{1}{\mu}\phi^*\varphi_n \end{pmatrix} \cdot \Psi$$

NLS

$$\boxed{\mu\phi^*_m - \nu\phi^*_n = \frac{(\mu-\nu)\phi^*}{1-\frac{1}{\mu\nu}\phi^*\varphi_{mn}}\ , \qquad \mu\varphi_n - \nu\varphi_m = \frac{(\mu-\nu)\varphi_{mn}}{1-\frac{1}{\mu\nu}\phi^*\varphi_{mn}}} \qquad \text{(dNLS)}$$

$$|\mu|,|\nu|\to\infty: \qquad \begin{cases} \phi^*_{x_2} = -\big(\phi^*_{x_1x_1} + 2(\phi^*)^2\varphi\big) \\ \varphi_{x_2} = \varphi_{x_1x_1} + 2\phi^*\varphi^2 \end{cases} \qquad \text{(NLS)}$$

with Lax pair

$$\Psi_{x_1} = \begin{pmatrix} \kappa & -\varphi \\ \phi^* & 0 \end{pmatrix}\cdot\Psi$$

$$\Psi_{x_2} = \begin{pmatrix} \kappa^2+\varphi\phi^* & -\varphi_{x_1}-\kappa\varphi \\ \kappa\phi^*-\phi^*_{x_1} & -\phi^*\varphi \end{pmatrix}\cdot\Psi$$

and solutions $\tau^{(N)} := \left|\left(\frac{\partial}{\partial_{x_1}}\right)^{i+j-2} f\right|_{i,j=1..N}, \quad \varphi = \frac{\tau^{(N-1)}}{\tau^{(N)}}, \quad \phi^* = \frac{\tau^{(N+1)}}{\tau^{(N)}}.$

discrete Broer-Kaup

[R. Willox & M. Hattori 2014]

This discrete NLS equation is intimately related to:

$$\begin{cases} H_{mn} = \dfrac{H_n U_n(\mu - \nu H_m)}{U_m(\mu - \nu H_n)} \\[2ex] U = \dfrac{\mu\nu(H_n - H_m)}{(\mu - \nu H_n)(\mu - \nu H_m)} + \dfrac{\mu U_m}{(\mu - \nu H_m)} - \dfrac{\nu U_n H_n}{(\mu - \nu H_n)} \end{cases}$$

the continuum limit ($|\mu|, |\nu| \to \infty$) of which is the Broer-Kaup system

$$\begin{cases} h_{x_2} = \left(h_{x_1} + 2u + h^2\right)_{x_1} \\ u_{x_2} = \left(2uh - u_{x_1}\right)_{x_1} \end{cases}$$

in the dependent variables $u = (\log \tau)_{x_1 x_1}$ and $h = (\log \varphi)_{x_1}$, obtained from the ansatz $H := \dfrac{\varphi_m}{\varphi_n} = 1 + \dfrac{\mu - \nu}{\mu\nu} h$ and $U := \dfrac{\tau \tau_{mn}}{\tau_m \tau_n} = 1 + \dfrac{1}{\mu\nu} u$.

Symmetry constraints of the discrete KP hierarchy

Theorem **[R. Willox & M. Hattori 2014]**

Let S represent an arbitrary shift on the discrete KP lattice, and $\gamma \in \mathbb{C}^\times$.

The constraint $\boxed{\gamma\, S\big(\tau\big) \;=\; \tau\,\Omega(\phi,\phi^*)}$ on a tau function τ (and ϕ and ϕ^*),

is compatible with the discrete KP hierarchy.

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is compatible with the discrete KP hierarchy.

The constrained hierarchy has solutions τ, ϕ^* and ϕ:

$$\tau_{bid}^{(N)} := \left|\Delta_S^{i-1}\Delta^{j-1} f(\ell)\right|_{i,j=1..N}, \quad \phi^* = \frac{\tau^{(N+1)}}{\tau^{(N)}}, \quad \phi = \frac{S\big(\tau^{(N-1)}\big)}{\tau^{(N)}} \quad \left(\tau^{(0)} := 1\right)$$

where $\Delta_S := \gamma(S-1)$ and $f(\ell)$ satisfies $\Delta_{\ell_j} f(\ell) = \Delta_{\ell_i} f(\ell)$ for all ℓ_i, ℓ_j on the lattice.

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- For any constrained τ, the function $\varphi := \dfrac{S^{-1}\big(\tau\,\phi\big)}{\tau}$ is a dKP eigenfunction
- $^\exists$ a systematic procedure to construct Lax pairs for the constrained systems

discrete Yajima-Oikawa

[R. Willox & M. Hattori 2014]

Choosing $\gamma = -\mu\nu$ and $S:\ S\big(f(m,n)\big) \equiv f(m+1,n+1)$, one obtains:

$$\begin{cases} 2\mu^3(U_{m'n'} - U) = \varphi_m \phi^*_{n'} - \varphi_n \phi^*_{m'} \\ \phi^*_m + \phi^*_n = 2U\phi^* \\ \varphi_m + \varphi_n = 2U\varphi_{mn} \end{cases}$$

for $\nu = -\mu$ and $U = \dfrac{\tau\,\tau_{mn}}{\tau_m \tau_n}$, which is a discretisation of the Yajima-Oikawa system

$$\begin{cases} u_{x_2} = (\phi^*\varphi)_{x_1} \\ \phi^*_{x_2} = -(\phi^*_{x_1x_1} + 2u\phi^*) \\ \varphi_{x_2} = \varphi_{x_1x_1} + 2u\varphi \end{cases}$$

※ 'primed' subscripts denote downshifts.

Interesting questions / open problems

- Does the Laurent phenomenon for a bilinear equation, give any (useful) information on the behaviour of the singularities or of the solutions of the (nonlinear) systems that can be obtained from it ?

- Is there any information that can be obtained regarding the (Lie) algebraic structure of reductions of the HM equation (or of the discrete KP hierarchy) ?

- What about B-type discrete systems, e.g., the Miwa-equation and its reductions ?

  Or what about discrete versions of integrable systems with C or D type symmetries, such as Kaup-Kuperschmidt or Toda lattices of C and D-type ?