A Pollaczek-Khinchine formula for multidimensional ruin problem

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Cramer-Lundberg network without treaty

For $i = 1, \cdots, d$, surplus of Company i, in the absence of risk diversifying treaty:

$$H^{(i)}(t) \quad = \quad a_i + c_i t - \sum_{\ell=1}^{N^{(i)}(t)} J_\ell^{(i)}, \quad t \geq 0, \tag{1}$$

$a_i \geq 0$: initial capital, $c_i > 0$: constant premium rate,
$N^{(i)}(\cdot)$: Poisson process (λ_i), $J_\ell^{(i)}, \ell \geq 1$: i.i.d. claim sizes,
$\{N^{(i)}(t)\}, \{J_\ell^{(j)}\}$, $1 \leq i, j \leq d$ are independent families

$H^{(1)}(\cdot), \ldots, H^{(d)}(\cdot)$ are d independent *Cramer-Lundberg risk processes*

C-L network: contd.

$J^{(1)}, \cdots, J^{(d)}$ independent generic r.v.'s s.t. $J^{(k)} =^d J_1^{(k)}, \forall k$;
$N(t) = \sum_{i=1}^d N^{(i)}(t), t \geq 0$: Poisson process (λ), $\lambda = \sum_{i=1}^d \lambda_i$;
$d-$dimensional r.v. $J = (0, \cdots, 0, J^{(i)}, 0, \cdots, 0)$ with prob. $\frac{1}{\lambda}\lambda_i$.

Vector claim sizes $X_\ell, \ell \geq 1$ i.i.d. $\mathbb{R}^d-$valued r.v.s $=^d J$

$a = (a_1, \cdots, a_d)$ $c = (c_1, \cdots, c_d)$

$$\begin{aligned} H^{(a)}(t) &\triangleq a + tc - \sum_{\ell=1}^{N(t)} X_\ell \\ &=^d (H^{(1)}(t), \cdots, H^{(d)}(t)), \ t \geq 0, \end{aligned} \qquad (2)$$

as processes

C-L network: contd.

- J takes value in $\partial G =$ boundary of $d-$dimensional positive orthant G
- Though $J^{(i)}, 1 \leq i \leq d$ are independent, marginals $(J)_i, 1 \leq i \leq d$ of J are not independent
- Even if $J^{(i)}, 1 \leq i \leq d$ are absolutely continuous, J is not
- Marginals $(J)_i$ have atom at 0

Assumptions on reflection matrix R

- **(H1)** $R = ((R_{ij})) = I - P^t$ constant $d \times d$ matrix s.t. $P_{ii} = 0,\ P_{ij} \geq 0,\ i \neq j, \forall 1 \leq i, j \leq d$; and spectral radius of P is strictly less than 1. So

$$R^{-1} = I + P^t + (P^t)^2 + (P^t)^3 + \cdots \tag{3}$$

  is a matrix with nonnegative entries, with diagonal entries ≥ 1.
- **(H2)** $\exists k \in \{1, 2, \cdots, d\}$ s.t. $(R^{-1})_{ik} > 0, \forall 1 \leq i \leq d$; so at least one column vector of R^{-1} has strictly positive entries.

For insurance models, besides (H1), natural to assume also that $\sum_{j \neq i} P_{ij} \leq 1, \forall i$, that is P is a substochastic matrix
(H2) is satisfied if P is irreducible

Stochastic assumptions

- **(H3)** $A_i, i = 1, 2, \ldots$ one dimensional i.i.d. random variables such that $A_i > 0$; (scalar interarrival times)
- **(H4)** $X_\ell, \ell = 1, 2, \ldots$ i.i.d. $\mathbb{R}_+^d$-valued random variables; (vector claim sizes)
- **(H5)** $\{A_i : i \geq 1\}, \{X_\ell : \ell \geq 1\}$ are independent families of random variables.
- **(H6)** For each $\ell = 1, 2, \cdots$ and $i = 1, 2, \cdots, d$, $\mathbb{P}((X_\ell)_i > x) > 0, \quad \forall x \geq 0$; i.e., marginal claim sizes have unbounded support.
- **(H7)** $c = ((c)_1, \cdots, (c)_d) \gg 0$ with $(c)_i$ denoting constant premium rates. $A_1, (X_1)_i, 1 \leq i \leq d$ have finite expectations, and $\mathbb{E}[(c)_i A_1 - (X_1)_i] > 0, \quad 1 \leq i \leq d$; this is coordinatewise net profit condition.

Skorokhod Problem (SP) in an orthant

To describe joint dynamics of the d companies under risk diversifying treaty:

$G = \{x \in \mathbb{R}^d : x_i > 0, 1 \leq i \leq d\}$ denotes the $d-$dimensional positive orthant, and $\bar{G}$ its closure

We seek processes $\{Y^{(a)}(t) = (Y_1^{(a)}(t), \ldots, Y_d^{(a)}(t)) : t \geq 0\}, \{Z^{(a)}(t) = (Z_1^{(a)}(t), \ldots, Z_d^{(a)}(t)) : t \geq 0\}$ satisfying the following

(S0) $Y^{(a)}(0) = 0, Z^{(a)}(0) = a = (a_1, \cdots, a_d)$.

(S1) *[Constraint]* $Z_i^{(a)}(t) \geq 0, \quad t \geq 0, 1 \leq i \leq d;$
so $Z^{(a)}(\cdot)$ is a $\bar{G}-$valued process

SP:contd.

(S2) For $1 \leq i \leq d$ the *Skorokhod equation* holds, that is,

$$Z_i^{(a)}(t) \;=\; a_i + c_i t - \sum_{\ell=1}^{N(t)} (X_\ell)_i + Y_i^{(a)}(t) + \sum_{j \neq i} R_{ij} Y_j^{(a)}(t); \quad (4)$$

or equivalently in vector notation

$$Z^{(a)}(t) \;=\; H^{(a)}(t) + R \cdot (Y(t) - Y(0), \quad t \geq 0. \qquad (5)$$

(S3) *[Minimality]* For $1 \leq i \leq d$, $Y_i^{(a)}(\cdot)$ is a nondecreasing process and $Y_i^{(a)}(\cdot)$ can increase only when $Z_i^{(a)}(\cdot) = 0$; that is,

$$Y_i^{(a)}(t) - Y_i^{(a)}(s) = \int_{(s,t]} \mathbf{1}_{\{0\}}(Z_i^{(a)}(r)) dY_i^{(a)}(r), \quad t \geq s \geq 0. \quad (6)$$

An interpretation

Let Company i need at some instant of time an amount dy_i *to avert ruin.* For $j \neq i$, Company j is required to give a preassigned fraction $|R_{ji}|dy_i = P_{ij}dy_i$.

If $\sum_{j \neq i} |R_{ji}| < 1$, shortfall $\left(1 - \sum_{j \neq i} |R_{ji}|\right) dy_i$ to be procured by shareholders of Company i as "capital injection"; ('open' system)
Objective of treaty: To keep surplus of each company ≥ 0 in an optimal fashion.
The set up leads naturally to a d-person dynamic game with state space constraints and Skorokhod problem provides the optimal solution

C-L network with treaty

(H1) ensures that SP in the orthant can be solved uniquely 'path-by-path'

Under optimality, a company can borrow, invoking the treaty, only when its reserve is zero/ it is in the red, and the amount borrowed should be just enough to keep it afloat

$Y_i^{(a)}(t)$ = optimal cumulative amount obtained by Co.i as capital injection by its shareholders, and from other companies during $[0, t]$ specifically for the purpose of averting ruin

$Z_i^{(a)}(t)$ = optimal current surplus of Co.i at time t

In (S0)-(S3) note that the only interaction among d companies is through risk reducing treaty

A random walk in $\mathbb{R}^d$

Interarrival times for $\{N(t)\}$: $A_\ell, \ell \geq 1$ Exp(λ) i.i.d. r.v.'s.
$T_0 = 0,\ T_k = \sum_{\ell=1}^{k} A_\ell,\ k \geq 1$ claim arrival times for network
On $[T_k, T_{k+1})$ each component of $H^{(a)}(\cdot)$, and hence of $Z^{(a)}(\cdot)$, strictly increasing; so need for capital injection only at $T_k, k \geq 1$; also ruin can occur only at an arrival time.

$$
\begin{aligned}
H^{(a)}(T_n) &= a + \sum_{\ell=1}^{n} A_\ell c - \sum_{\ell=1}^{n} X_\ell \\
&= a + \sum_{\ell=1}^{n} U_\ell, n = 1, 2, \cdots \qquad (7)
\end{aligned}
$$

a random walk in $\mathbb{R}^d$ starting at a, as $\{U_\ell\}$ are i.i.d. $\mathbb{R}^d-$valued r.v.'s.

$SP(\{a + \sum_\ell U_\ell\}, R)$

$\{Y_n^{(a)}\}, \{Z_n^{(a)}\}$ satisfy $Y_0^{(a)} = 0, Z_n^{(a)} = a$,

$$(Z_n^{(a)})_i = (a)_i + \sum_{\ell=1}^{n} (U_\ell)_i + (Y_n^{(a)})_i + \sum_{j \neq i} R_{ij} (Y_n^{(a)})_j, \quad (8)$$

for $n \geq 1, 1 \leq i \leq d$; equivalently

$$Z_n^{(a)} = Z_{n-1}^{(a)} + U_n + R(\Delta Y_n^{(a)}), \;\; n \geq 1, \quad (9)$$

where $\Delta Y_n^{(a)} = Y_n^{(a)} - Y_{n-1}^{(a)}$; also $Z_n^{(a)} \in \overline{G}, n \geq 0$ (*Constraint*), $Y_n^{(a)} \geq Y_{n-1}^{(a)}, \;\; n \geq 1$, as vectors, and (*Minimality*)

$$\left\langle Z_n^{(a)}, \Delta Y_n^{(a)} \right\rangle = 0, \;\; n \geq 1 \quad (10)$$

$\{Y_n^{(a)}\}$ 'pushing part', $\{Z_n^{(a)}\}$ 'regulated/ reflected part' of solution

$LCP(\eta, R)$

$\xi, \zeta \in \mathbb{R}^d$ solution pair to *Linear complementarity problem* $LCP(\eta, R)$ if

$$\zeta = \eta + R\xi,\ \xi \geq 0,\ \zeta \geq 0,\ \langle \xi, \zeta \rangle = 0.$$

ξ 'pushing part', ζ 'regulated part' of solution

(H1) implies $LCP(\eta, R)$ as well as $SP(\{a + \sum_\ell U_\ell\}, R)$ are well-posed

$\{Y_n^{(a)}, n \geq 0\}, \{Z_n^{(a)}, n \geq 0\}$ solution pair to $SP(\{a + \sum_\ell U_\ell\}, R)$ if and only if for each $n = 1, 2, \cdots$ $\Delta Y_n^{(a)}, Z_n^{(a)}$ is solution pair to $LCP(Z_{n-1}^{(a)} + U_n, R)$

As before these can be solved 'path-by-path'

Regulated/ reflected random walk

$a + \sum_{\ell=1}^{n} U_\ell, n \geq 0$ random walk in $\mathbb{R}^d$ starting at a.
Hence $Z_n^{(a)}, n \geq 0$, (or the pair $Y_n^{(a)}, Z_n^{(a)}, n \geq 0$) considered corresponding regulated/ reflected random walk in the orthant $\overline{G}$.
(7) implies $\Delta Y_n^{(a)} = Y^{(a)}(T_n) - Y^{(a)}(T_{n-1})$,
$Z_n^{(a)} = Z^{(a)}(T_n),\ n \geq 1$. Also $Y^{(a)}(t) - Y^{(a)}(t-) \gg 0$ for some $t > 0$ if and only if $\Delta Y_n^{(a)} \gg 0$ for some $n \geq 1$.

$(U_k)_i =$ (premium income for Co.i during $(k-1, k]$) *minus* (claim amt. for Co.i due to $k-$th claim to network),
$(Z_k^{(a)})_i =$ current surplus for Co.i at time k, under optimality,
$(\Delta Y_k^{(a)})_i =$ marginal deficit of Co.i at time k, under optimality.

Sufficient to study 'ruin' in the context of regulated random walk

Ruin of network

'Ruin' of insurance network $= \Delta Y_n^{(a)} \gg 0$ for some $n \geq 1$.
(For vectors $\xi \gg \zeta$ denotes $(\xi)_i > (\zeta)_i, 1 \leq i \leq d$)
Ruin means every company has strictly positive deficit at the *same time* n, for some $n \geq 1$.

Ruin probability: Prob.(ruin in finite time, starting with initial capital a) $= \mathbb{P}(\Delta Y_n^{(a)} \gg 0$ for some $n \geq 1)$

To understand ruin probability, we look at the deterministic (i.e., sample path) set up first. Considering $\omega \in \Omega$ as fixed, $u_\ell, y_k^{(a)}, z_k^{(a)}, \cdots$ may be regarded as particular realization of $U_\ell(\omega), Y_k^{(a)}(\omega), Z_k^{(a)}(\omega), \cdots$

Deterministic set up

$\{y_n^{(a)}, z_n^{(a)}\}$ solution pair to $SP(\{a + \sum_\ell u_\ell\}, R)$: So $z_0^{(a)} = a, y_0^{(a)} = 0$; $z_n^{(a)} \in \overline{G}, n \geq 1$ (*constraint*); *Skorokhod equation* holds, that is,

$$z_n^{(a)} = z_{n-1}^{(a)} + u_n + R\Delta y_n^{(a)}, \ \ n \geq 1, \tag{11}$$

where $\Delta y_n^{(a)} = y_n^{(a)} - y_{n-1}^{(a)} \geq 0, n \geq 1$ componentwise; and $\left\langle z_n^{(a)}, \Delta y_n^{(a)} \right\rangle = 0, n \geq 1$ *minimality*.

'Ruin' means $\Delta y_n^{(a)} \gg 0$ for some $n \geq 1$.

Ruin in deterministic set up

Using minimality and (11) repeatedly

Lemma

Fix $n \geq 1$. *Then* $\Delta y_n^{(a)} \gg 0 \Leftrightarrow -R^{-1}u_n \gg R^{-1}z_{n-1}^{(a)} \Leftrightarrow \cdots \Leftrightarrow -\sum_{\ell=1}^{n} u_\ell \gg R^{-1}a + y_{n-1}^{(a)}$.

Corollary

Fix $n \geq 1$; *suppose* $\Delta y_n^{(a)} \gg 0$. *Then* $-R^{-1}u_n \gg 0$,
$-\sum_{\ell=k}^{n} R^{-1}u_\ell \gg 0, \quad k = n, n-1, \cdots, 2, 1$, *and*
$-\sum_{\ell=1}^{n} R^{-1}u_\ell \gg R^{-1}a$.

An elementary observation

$$z = u + R(\Delta y),\ z \geq 0,\ \Delta y \geq 0,\ \langle z, \Delta y \rangle = 0$$

is equivalent to

$$\Delta y = -R^{-1}u + R^{-1}z,\ \Delta y \geq 0,\ z \geq 0,\ \langle \Delta y, z \rangle = 0;$$

that is, $\Delta y, z$ is the solution pair to $LCP(u, R)$ if and only if $z, \Delta y$ is the solution pair to $LCP(-R^{-1}u, R^{-1})$

This leads to a sequence of LCP's, resulting in a SP with reflection matrix R^{-1}, related to the preceding SP with a time reversal over a finite time horizon

Deterministic (dual) storage network

Fix $n \geq 1$. $\{u_\ell \in \mathbb{R}^d, 1 \leq \ell \leq n\}$ as before

Set $\hat{u}_1 = -R^{-1}u_n,\ \hat{u}_2 = -R^{-1}u_{n-1}, \cdots, \hat{u}_n = R^{-1}u_1$

Put $w_0 = 0, v_0 = 0$ Define $\Delta v_k, w_k,\ 1 \leq k \leq n$ by

$$w_1 = \hat{u}_1 + R^{-1}\Delta v_1,\ w_1 \geq 0,\ \Delta v_1 \geq 0,\ \langle w_1, \Delta v_1\rangle = 0,$$

$$w_k = w_{k-1} + \hat{u}_k + R^{-1}\Delta v_k,\ w_k \geq 0,\ \Delta v_k \geq 0,\ \langle w_k, \Delta v_k\rangle = 0$$

An interpretation of storage network

$d \geq 1$ storage depots of infinite capacity, 0 initial stock; demands may be continuous, but fresh stocks and reinforcements arrive only at the end of periods $k = 1, 2 \cdots$

$(\hat{u}_k)_i =$ (fresh supply at Depot i at the end of period k) *minus* (demand at Depot i during $(k-1, k]$)

$(w_k)_i =$ current stock at Depot i at the end of period k, after taking into account all reinforcement *to* Depot i till the end of period k; so $(w_k)_i \geq 0, \forall i, k$

$(R^{-1})_{ii}(\Delta v_k)_i =$ amount of reinforcement sent to Depot i at the end of k, due to unfulfilled demand after taking into account existing stock, fresh supply and inflow to Depot i due to shortfall at other depots at the end of period k

$(R^{-1})_{ij}(\Delta v_k)_j =$ reinforcement sent to Depot i due to shortfall at Depot $j, j \neq i$ at the end of period k

Ruin and storage sequence

$\sigma_{bd} = \inf\{k \geq 1 : w_k \in \partial G\}$ = hitting time of boundary
$\vartheta_{R^{-1}a} = \inf\{k \geq 1 : w_k \gg R^{-1}a\}$ = entrance time into open upper orthant with vertex $R^{-1}a$

Theorem

Assume (H1). Fix $n \geq 1$. Then $\Delta y_n^{(a)} \gg 0$ if and only if $\vartheta_{R^{-1}a} \leq n < \sigma_{bd}$. Moreover, taking $a = 0$,
$[y_n^{(0)} : \{\Delta y_n^{(0)} \gg 0\}] = [w_n : \{\sigma_{bd} > n\}] = -\sum_{\ell=1}^{n} R^{-1}u_\ell$

$\Rightarrow$: Earlier corollary rephrased

$\Leftarrow$: If $\sigma_{bd} > n$, then during time span $\{1, 2, \cdots, n\}$, $\{w_k : 1 \leq k \leq n\}$ has no need for reinforcement at any of the d depots

Stochastic set up

$U_\ell = A_\ell c - X_\ell, \ell \geq 1$ i.i.d. $\mathbb{R}^d-$valued r.v.'s
$\hat{U}_\ell = R^{-1} U_\ell, \ell \geq 1$ also i.i.d. $\mathbb{R}^d-$valued r.v.'s Put
$W_0 = 0, V_0 = 0$ Define $W_n,\ V_n = V_0 + \sum_{k=1}^n \Delta V_k,\ n \geq 1$ by

$$W_1 = \hat{U}_1 + R^{-1}\Delta V_1,\ W_1 \geq 0,\ \Delta V_1 \geq 0,\ \langle W_1, \Delta V_1 \rangle = 0,$$

$$W_k = W_{k-1} + \hat{U}_k + R^{-1}\Delta V_k,\ W_k \geq 0,\ \Delta V_k \geq 0,\ \langle W_k, \Delta V_k \rangle = 0$$

Above process is another regulated random walk in the orthant (*storage network*)
Results from deterministic set up applied to make statements of equality in distribution
(H2),(H6) imply events making ruin (of network) possible are non-null

Ruin probability

$\varrho^{(a)}(\omega) = \inf\{k \geq 1 : \Delta Y_k^{(a)}(\omega) \gg 0\}$ = ruin time
$\sigma_{bd}(\omega) = \inf\{k \geq 1 : W_k(\omega) \in \partial G\}$ = hitting time of boundary
$\vartheta_{R^{-1}a}(\omega) = \inf\{k \geq 1 : W_k(\omega) \gg R^{-1}a\}$ = entrance time into open upper orthant with vertex $R^{-1}a$

Theorem

Assume (H1)-(H7). Let $a \in \overline{G}$. Then

$$0 < \mathbb{P}(\varrho^{(a)} < \infty) \;=\; \mathbb{P}(\vartheta_{R^{-1}a} < \sigma_{bd}) < 1. \tag{12}$$

Moreover $\mathbb{P}(\Delta Y_n^{(a)} \gg 0) > 0$ and hence $\mathbb{P}(\sigma_{bd} > n) > 0$ for any $n \geq 1$. Also

$$\lim_{|a| \to \infty, a \in G} \mathbb{P}(\varrho^{(a)} < \infty) \;=\; 0. \tag{13}$$

Ladder height r.v.'s

Take $\tau_0 \equiv 0$. For $n \geq 1$, define stopping times $\tau_n(\omega) = \inf\{k \geq \tau_{n-1}(\omega) + 1 : \Delta Y_k^{(0)}(\omega) \gg 0\}$, if the set $\{k \geq \tau_{n-1}(\omega) + 1 : \Delta Y_k^{(0)}(\omega) \gg 0\} \neq \varnothing$; put $\tau_n(\omega) = +\infty$ if there is no $k \geq \tau_{n-1}(\omega) + 1$ such that $\Delta Y_k^{(0)}(\omega) \gg 0$. For $n \geq 1$, define

$$
\begin{aligned}
L_n(\omega) &= Y^{(0)}(\tau_n, \omega) - Y^{(0)}(\tau_{n-1}, \omega), \quad \text{if} \quad \tau_n(\omega) < \infty, \\
&= 0, \quad \text{if} \quad \tau_n(\omega) = +\infty; \qquad (14) \\
L_n^+(\cdot) &= L_n(\cdot) \quad \text{restricted to} \quad \{\tau_n < \infty\}; \qquad (15)
\end{aligned}
$$

in the above note that $Y^{(0)}(\tau_0) \equiv 0$. Clearly L_n takes value in $\{0\} \cup G$, and L_n^+ in G. Call L_1^+ the *$d-$dimensional first strictly ascending ladder height* random variable, and L_k^+ the *$d-$dimensional $k-$th strictly ascending ladder height* random variable

Ladder height distribution

$$\alpha_+(B) = \mathbb{P}(L_1^+ \in B), \quad B \subseteq G, \tag{16}$$

$$\alpha_0(B) = \frac{1}{\alpha_+(G)}\alpha_+(B), \quad B \subseteq G. \tag{17}$$

α_+ is a defective distribution, while α_0 a prob. distn. both concentrated on G. Take $M_0 \equiv 0$; define

$$M_n(\omega) = \sum_{j=1}^{n} L_j(\omega), \; M(\omega) = \sum_{j=1}^{\infty} L_j(\omega)$$

note that $M_n(\omega) = Y^{(0)}(\tau_n(\omega), \omega)$, if $\tau_n(\omega) < \infty$ and $M_n(\omega) = M_{n-1}(\omega)$ if $\tau_n(\omega) = \infty$. Thanks to (NPC) (H7), M is finite with prob. 1

Pollaczek-Khinchine formula

$$\beta(\omega) = \inf\{k \geq 1 : \tau_k(\omega) = +\infty\} = \inf\{k \geq 1 : L_k(\omega) = 0\}$$

Theorem

Assume (H1)-(H7). Denote $p \triangleq \mathbb{P}(\hat{U}_1 \in G) = \mathbb{P}(-R^{-1}U_1 \in G)$; *note that* $0 < p < 1$. *Then* $(\beta - 1)$ *has Geom.*$(1-p)$ *distn.,* $\alpha_+(G) = p$, *and* M *has geometric compound distn.*

$$\nu_M(B) = (1-p)\delta_0(B) + \sum_{k=1}^{\infty}(1-p)p^k\alpha_0^{*(k)}(B), \qquad (18)$$

$B \subseteq \{0\} \cup G$. *Also ruin probability for insurance network is*

$$\begin{aligned}\mathbb{P}(\varrho^{(a)} < \infty) &= \mathbb{P}(M \gg R^{-1}a) \\ &= (1-p)\sum_{n=1}^{\infty}\alpha_+^{*(n)}(\{x \gg R^{-1}a\}),\ a \in \overline{G}\ (19)\end{aligned}$$

P-K formula: contd.

Theorem

Assume (H1)-(H7). Set $\hat{U}_k^+ = \hat{U}_k$ *restricted to* $\{\hat{U}_k \in G\}$

$$\mu_+(B) \;=\; \mathbb{P}(\hat{U}_1^+ \in B) = \mathbb{P}(\hat{U}_1 \in B), \;\; B \subseteq G \tag{20}$$

μ_+ *is a defective distn. with* $0 < p = \mu_+(G) < 1$. *Let* $\mu_0(\cdot) = \frac{1}{p}\mu_+(\cdot)$. *Define the compound geometric*

$$\nu(B) \;=\; (1-p)\delta_0(B) + \sum_{k=1}^{\infty}(1-p)p^k \mu_0^{*(k)}(B). \tag{21}$$

Then the following hold:

(i) $(\sigma_{bd} - 1)$ *has Geom.*$(1-p)$ *distn.; hence* $\sigma_{bd} =^d \beta$.

P-K formula: contd.

Theorem

(ii) ν *is a prob. measure concentrated on* $\{0\} \cup G$, *s.t.* $(\max_{k<\sigma_{bd}} W_k) =^d W(\sigma_{bd} - 1) =^d \nu$. *Also, on* $[0, \sigma_{bd})$, W_n *converges in distribution to* $W(\sigma_{bd} - 1)$.

(iii) $M =^d \max_{k<\sigma_{bd}} W_k$, *and hence* $\nu_M = \nu$.

(iv) $\hat{U}_1^+ =^d L_1^+$; *so* μ_+ *is the d−diml ladder height distn. i.e.,*

$$\mathbb{P}(L_1^+ \in B) = \mathbb{P}(-R^{-1}(cA_1 - X_1) \in B), \; B \subseteq G. \quad (22)$$

(v) For $a \in \overline{G}$,

$$\mathbb{P}(\varrho^{(a)} < \infty) = \sum_{k=1}^{\infty} (1-p)\mu_+^{*(k)}(\{x \gg R^{-1}a\}). \quad (23)$$