Some New Results on the Geometry of Random Fields

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- What if the needles were welded together? Will the mean of the total number of intersections change? No!

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A little different setup: Graff(2, 1) – the space of lines.

Theorem (Sylvester (1890))

Consider a piecewise smooth curve C of length L. Then,

$$\int_{Graff(2,1)} \#(C \cap \ell)\, d\lambda_{2,1}(\ell) \quad \propto \quad L$$

where $\lambda_{2,1}$ is the rigid motions invariant measure on Graff(2, 1). Also, notice that if $C = \partial K$ for some compact $K \subset \mathbb{R}^2$ then writing D for the set of all straight lines that meet K

$$\int_{Graff(2,1)} \#(C \cap \ell)\, d\lambda_{2,1}(\ell) \quad = \quad 2\lambda_{2,1}(D)$$

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What about generalizations of this to higher dimensions?

Crofton's formula

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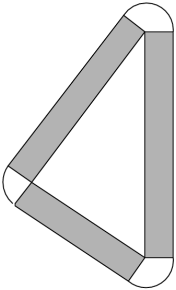
- How does one characterize LKCs? $\longrightarrow$ A tube formula

Tube formula

- Let A be a *smooth* subset of $\mathbb{R}^n$, with λ_n as the n-dimensional Euclidean measure.
- Let

$$\text{Tube}(A, \rho) = \{x \in \mathbb{R}^n : \text{ dist}(x, A) \leq \rho\}$$

Then Weyl's tube formula is given by:



$$\lambda_n(\text{Tube}(A, \rho)) = \sum_{i=0}^{\dim(A)} \omega_{n-i} \rho^{n-i} \mathcal{L}_i(A),$$

$(\mathcal{L}_i(A))_{i=1}^{\dim(A)}$ = Lipschitz–Killing curvatures, and ω_{n-i} is the volume of a unit ball in $\mathbb{R}^{n-i}$.

Lipschitz–Killing curvatures (LKCs): examples

- A box B with dimensions (a, b, c): $\mathcal{L}_0(B) = 1$, $\mathcal{L}_1(B) = (a + b + c)$, $\mathcal{L}_2(B) = (ab + bc + ac)$, $\mathcal{L}_3(B) = abc$.
- A ball $B_n(r)$ of radius r in $\mathbb{R}^n$:

$$\mathcal{L}_j(B_n(r)) = r^j \begin{pmatrix} n \\ j \end{pmatrix} \frac{\omega_n}{\omega_{n-j}}$$

- A sphere $S^{n-1}(r)$ of radius r in $\mathbb{R}^n$:

$$\mathcal{L}_j(S^{n-1}(r)) = 2r^j \begin{pmatrix} n \\ j \end{pmatrix} \frac{\omega_n}{\omega_{n-j}},$$

  for even values of $(n - j - 1)$, and 0 otherwise.
- For a unit codimensional manifold, every alternate $\mathcal{L}_i$ vanishes.

LKCs: properties

- For an m-dimensional subset $A \subset \mathbb{R}^n$, $\mathcal{L}_0(A)$ is its Euler–Poincaré characteristic, and $\mathcal{L}_m(A)$ is its m-dimensional volume.
- $\mathcal{L}_i$, of say a set A, is an intrinsic, integral geometric characteristics of the set.
- LKCs for a smooth Riemannian manifold M can be defined as

$$\mathcal{L}_k(M) = \int_M \mathrm{Tr}\left(R^{\frac{n-k}{2}}\right) Vol_g$$

whenever $\frac{n-k}{2}$ is an integer, and it is zero otherwise.
- Scaling: $\mathcal{L}_k(\lambda A) = \lambda^k \mathcal{L}_k(A)$.

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- Equip Graff(n, k) with a measure $\lambda_{n,k}$, which is invariant under the set of rigid motions $E(n)$.
- This measure can be factored as ν_k^n on Gr(n, k) and Lebesgue measure on $\mathbb{R}^n$, and can be normalized so that $\nu_k^n(\mathrm{Gr}(n, k)) = \begin{bmatrix} n \\ k \end{bmatrix} = \begin{pmatrix} n \\ k \end{pmatrix} \frac{\omega_n}{\omega_k \omega_{n-k}}$.

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- Let $M \subset \mathbb{R}^n$, *nice* and compact, then we have

$$\int_{\mathrm{Graff}(n,n-k)} \mathcal{L}_j(M \cap V)\, d\lambda_{n-k}^n(V) = \left[\begin{array}{c} k+j \\ j \end{array} \right] \mathcal{L}_{k+j}(M).$$

A Kinematic Formula

A kinematic formula

- Consider two piecewise smooth curves C_1 and C_2 in $\mathbb{R}^2$.
- Let G_2 group of rigid motions, which is equivalent to $\mathbb{R}^2 \times O(2)$, and is equipped with the obvious product measure ν.
- Let $\phi(C_1 \bigcap gC_2)$ be the Euler–Poincaré characteristic, which, in this simple case is equivalent to the number of points of intersection of the curves C_1 and gC_2.

Theorem (Kinematic formula for curves (1912))

$$\int_{G_2} \phi(C_1 \cap gC_2)\, d\nu(g) = 4\mathcal{L}_1(C_1)\mathcal{L}_1(C_2)$$

Euclidean Kinematic Fundamental Formula (KFF)

- G_n: isometry group on $\mathbb{R}^n$; isomorphic to $\mathbb{R}^n \times O(n)$.

- ν_n: a normalized measure on G_n, such that $\nu_n(g_n \in G_n : g_n x \in A) = \mathcal{H}_n(A)$, for any $x \in \mathbb{R}^n$ and $A \in \mathcal{B}(\mathbb{R}^n)$.

- Then for *smooth* M_1 and M_2, we have

$$\int_{G_n} \mathcal{L}_i(M_1 \cap g_n M_2) \, d\nu_n(g_n) = \sum_{j=0}^{n-i} \frac{s_{i+1} s_{n+1}}{s_{i+j+1} s_{n-j+1}} \mathcal{L}_{i+j}(M_1) \mathcal{L}_{n-j}(M_2)$$

Gaussian Kinematic Fundamental Formula

Gaussian geometric characteristics via a Gaussian tube formula

Gaussian Minkowski functionals (GMFs): $\mathcal{M}_j^{\gamma_n}$

- Let A be *smooth* subset of $\mathbb{R}^n$, with $\gamma_n(dx) = (2\pi)^{-n/2} e^{-\|x\|^2/2} dx$, then the GMFs can be defined as

$$\gamma_n(\mathrm{Tube}(A, \rho)) = \sum_{j=0}^{\infty} \frac{\rho^j}{j!} \mathcal{M}_j^{\gamma_n}(A),$$

  where $\mathrm{Tube}(A, \rho)$ is a tube of radius ρ around A.

- One can also define the GMFs as integral of some Hermite polynomials with respect to the measures induced by $\mathcal{L}_i$'s, called the generalized curvature measures.

A Gaussian Kinematic Formula (GKF)

- Let M be an m-dimensional smooth manifold.
- Let $y_1, \ldots, y_k$ be i.i.d. Gaussian random fields on M.
- Let $F : \mathbb{R}^k \to \mathbb{R}$ be twice differentiable, and define $f = F(y_1, y_2, \ldots, y_k)$. Then [Taylor (2006)]

$$\mathbb{E}\left(\mathcal{L}_0\left(M \cap f^{-1}[u,\infty)\right)\right) = \sum_{j=0}^{n} c_j \; \mathcal{L}_j^y(M) \; \mathcal{M}_j^{\gamma_k}\left(F^{-1}[u,\infty)\right)$$

where $\mathcal{L}_j^y(\cdot)$ are the LKCs defined w.r.t. the induced metric given by

$$g^y(X,Y) = \mathbb{E}\left(Xy_1 \cdot Yy_1\right)$$

The metric induced by any y_i is the same due to i.i.d. nature of y_i's

Define

$$\mu_k = \#\{x \in M : f(x) \geq u, \;\; \nabla f(x) = 0, \;\; \text{index}\left(\nabla^2 f(x)\right) = k\}$$

Then,

$$\mathbb{E}\left(\mathcal{L}_0\left(M \cap f^{-1}[u, \infty)\right)\right) = \mathbb{E}\left(\sum_{k=0}^{m} (-1)^k \mu_k\right)$$

A sneak peek into the proof of GKF

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& \mathbb{E}\left(\mathcal{L}_0\left(M \cap f^{-1}[u, \infty)\right)\right) = \mathbb{E}\left(\sum_{k=0}^{m} (-1)^k \mu_k\right) \\
= \;\; & \int_M \mathbb{E}\left\{ \text{Tr}\left(-\nabla^2 f(x)\right)^m 1_{(f(x) \geq u)} \middle| \nabla f(x) = 0 \right\} p_{\nabla f(x)}(0) \; dx \\
= \;\; & \int_M \mathbb{E}\left[1_{(f(x) \geq u)} \mathbb{E}\left\{ \text{Tr}\left(-\nabla^2 f(x)\right)^m \middle| f(x), \nabla f(x) = 0 \right\}\right] \\
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- In general, if W is a $(1,1)$ Gaussian form with mean and covariance are μ and C respectively, then

$$\mathbb{E}[W^k] = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \frac{k!}{(k-2j)!j!2^j} \; \mu^{k-2j} \; C^j .$$

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- The conditional covariance $= -(I^2 + 2R)\|\nabla F\|^2$, where R is the Riemannian curvature tensor with respect to the induced metric.
- Then need to go from conditioning on $(y, \nabla y)$ to conditioning on $(f, \nabla f)$, which involves another Gaussian computation (majorly technical).

Sneak peek contd...

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- In this case, the *majorly technical* step is not needed.
- Then notice that we are left with integrals of trace of polynomials of R, which can readily be identified with LKCs, and the rest matches with the GMFs in each term of the polynomial.

Testing the Limits of Gaussian Kinematic Fundamental Formula

The non IID case

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- We assume y_1 and y_2 are isotropic independent Gaussian fields, but they are not identically distributed.
- We still have:

$$\mathbb{E}\left(\mathcal{L}_0\left(M \cap f^{-1}[u,\infty)\right)\right)$$
$$= \int_{\mathbb{S}^2} \mathbb{E}\left[1_{(f(x)\geq u)}\mathbb{E}\left\{\operatorname{Tr}\left(-\nabla^2 f(x)\right)^2 \middle| f(x), \nabla f(x) = 0\right\}\right]$$
$$\times\ p_{\nabla f(x)}(0)\ dx$$

The non IID case

$\{\nabla^2 f \,|\, y, \nabla y\}$ is still a Gaussian $(1,1)$ form, with mean

$$y^* \nabla^2 F(y) - \left(\sum_{k=1}^{2} \lambda_{2,k} y_k \frac{\partial F(y)}{\partial y_k} \right) I_2,$$

and covariance

$$-\sum_{k=1}^{2} \lambda_{2,k}^2 \left(\frac{\partial F(y)}{\partial y_k} \right)^2 I_2^2 - 2 \sum_{k=1}^{2} \lambda_{2,k} \left(\frac{\partial F(y)}{\partial y_k} \right)^2 R$$

where R is the Riemannian curvature tensor w.r.t. the spherical metric.

A partial result

$$\mathbb{E}\left(\mathcal{L}_0(M \cap f^{-1}[u,\infty))\right)$$

$$= \left(\sum_{\nu=1}^{k} \frac{1}{\lambda_{2,\nu}} \mathbb{E}\left[1_{(f>u)}\left(\frac{\partial F(y)}{\partial y_\nu}\right)^2\right]\right) p_{\nabla f}(0) 4\pi \mathcal{L}_0(M)$$

$$+\frac{1}{2}\sum_{i,j=1}^{2} \mathbb{E}\Big[1_{(f>0)}\Big(\mu^2(y,\nabla y)(E_i,E_j,E_i,E_j)$$

$$-S^T_{\nabla F}(E_i,E_i)\Sigma_{M,(y,\nabla y)}\Sigma^{-1}_{(y,\nabla y)}\Sigma_{(y,\nabla y),M}S_{\nabla F}(E_j,E_j)$$

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&+\frac{1}{2}\sum_{i,j=1}^{2} \mathbb{E}\Big[1_{(f>0)}\Big(\mu^2(y,\nabla y)(E_i,E_j,E_i,E_j) \\
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Good news: we still have a breakup of the two spaces (by possibly a bit of cheating) Bad news: we are yet to figure out meaning of the coefficients of the LKCs.

Thank you!