A nonlinear Donsker-Varadhan Variational formula and Risk-sensitive control

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Plan of the Talk

- Donsker-Varadhan Variational formula
- Risk-sensitive control problem description.
- Main results.
- Extension of Collatz-Wielandt Formula.
- Nisio-semi group associated with risk-sensitive control.
- Outline of proof of main results.

Donsker-Varadhan variational formula

Let X be a compact metric space and $\{T_t : C(X) \to C(X) | t \geq 0\}$ be a strongly continuous semigroup satisfying

$$f \geq 0 \implies T_t f \geq 0, T_t 1 = 1$$

For $V \in C(X)$, if λ_V denote the principle eigen value of $L + V$, L is the infinitsemal generator of $\{T_t\}$, then we have

$$\lambda_V = \sup_{\mu \in \mathcal{P}(X)} \left[\int_X V(x)\mu(dx) - I(\mu) \right],$$

where

$$I(\mu) = - \inf_{u \in \mathcal{D}^+(L)} \int_X \left(\frac{Lu}{u} \right)(x)\mu(dx),$$

where $\mathcal{D}^+(L)$ is the set of all positive functions in $\mathcal{D}(L)$, the domain of L and $\mathcal{P}(X)$ is the space of all probability measures on X.

Proc. Nat. Acad. Sci. USA, Vol. 72, no.3, pp. 780-783, 1975.

An example

Let $Q \subset \mathbb{R}^d$ be an open bounded domain with a C^3 boundary ∂Q and $\bar{Q}$ denote its closure. Consider the diffusion in Q given by

$$\begin{aligned} dX(t) &= b(X(t))\,dt + \sigma(X(t))\,dW(t) - \gamma(X(t))\,d\xi(t)\,, \\ d\xi(t) &= I\{X(t) \in \partial Q\}\,d\xi(t) \end{aligned} \qquad (1)$$

for $t \geq 0$, with $X(0) = x$ and $\xi(0) = 0$, γ is co-normal, i.e.

$$\gamma(x) = a(x)n(x)^{\perp},\ n(x) \text{ is outward unit normal.}$$

By a solution of (1), a pair of processes $(X(\cdot), \xi(\cdot))$ satisfying (1) where $X(\cdot)$ is a $\bar{Q}$-valued continuous process and $\xi(\cdot)$ is a non decreasing process.

The existence of a unique strong solution holds under the usual Lipschitz conditions.

An Example

Consider $T_t : C(\bar{Q}) \to C(\bar{Q})$ defined by

$$T_t f(x) = E\left[f(X_t)\middle|X(0) = x\right], f \in C(\bar{Q}),\ t \geq 0.$$

One can also identify the principal eigen value λ_V by

$$\lambda_V = \limsup_{t\to\infty} \frac{1}{t} \log E\left[e^{\int_0^t V(X_s)ds}\middle|X(0) = x\right].$$

Risk-sensitive control problem

Consider a controlled diffusion $X(\cdot)$ taking values in a bounded domain $\bar{Q}$ satisfying

$$\begin{aligned} dX(t) &= b(X(t), v(t))\,dt + \sigma(X(t))\,dW(t) - \gamma(X(t))\,d\xi(t)\,, \\ d\xi(t) &= I\{X(t) \in \partial Q\}\,d\xi(t) \end{aligned} \tag{2}$$

for $t \geq 0$, with $X(0) = x$ and $\xi(0) = 0$.

State dynamics- continued

- **(a)** $b : \bar{Q} \times \mathcal{V} \to \mathbb{R}^d$, for a prescribed compact metric control space $\mathcal{V}$, is continuous and Lipschitz in its first argument uniformly with respect to the second,
- **(b)** $\sigma : \bar{Q} \to \mathbb{R}^{d \times d}$ is continuously differentiable, its derivatives are Hölder continuous with exponent $\beta_0 > 0$, and is uniformly non-degenerate
- **(c)** $\gamma : \mathbb{R}^d \to \mathbb{R}^d$ is co-normal,
- **(d)** $W(\cdot)$ is a d-dimensional standard Wiener process,
- **(e)** $v(\cdot)$ is a $\mathcal{V}$-valued measurable process satisfying the non-anticipativity condition: for $t > s \geq 0$, $W(t) - W(s)$ is independent of $\{v(y), W(y) : y \leq s\}$.

Risk-sensitive cost

Let $r : \bar{Q} \times \mathcal{V} \to \mathbb{R}$ be a continuous 'running cost' function which is Lipschitz in its first argument uniformly with respect to the second.
Risk-sensitive cost is

$$\limsup_{T\to\infty} \frac{1}{T} \log E\left[e^{\int_0^T r(X_t,v_t)dt}\right].$$

Risk-sensitive control problem is about minimizing the "cost" over all addmissible controls.
Risk-sensitive value ρ is defined as

$$\rho = \inf \limsup_{T\to\infty} \frac{1}{T} \log E\left[e^{\int_0^T r(X_t,v_t)dt}\right].$$

Infimum is over all admissible controls, i.e. all controls $v(\cdot)$ satisfying the condition (e).

Main Results

Define

$$\mathcal{G}f(x) \;:=\; \frac{1}{2}\mathit{trace}\left(a(x)\nabla^2 f(x)\right) + \mathcal{H}(x, f(x), \nabla f(x)),$$
$$\mathcal{H}(x, f, p) \;:=\; \min_{v\in\mathcal{V}} \left[\langle b(x,v), p\rangle + r(x,v)f\right]. \tag{3}$$

Set

$$C^2_{\gamma,+}(\bar{Q}) \;:=\; \{f \in C^2(\bar{Q}) : \; f \geq 0\,, \; \nabla f\cdot\gamma \;=\; 0 \text{ on } \partial Q\}\,.$$

Theorem

There exists a unique pair $(\rho, \varphi) \in (0,\ \infty) \times C^2_{\gamma,+}(\bar{Q})$ satisfying

$$\rho\varphi = \mathcal{G}\varphi, \quad \textit{in } Q\,, \quad \langle\nabla\varphi, \gamma\rangle = 0 \quad \textit{on } \partial Q\,, \|\varphi\|_{0;\bar{Q}} = 1\,.$$

Moreover ρ is characterized as the risk-sensitive value.

Main Results

Theorem

The scalar ρ given in previous theorem satisfies

$$\rho = \inf_{f\in C^2_{\gamma,+}(\bar{Q})} \sup_{\substack{\mu\in\mathcal{M}(\bar{Q}) \\ \int f\,d\mu=1}} \int \mathcal{G}f\,d\mu \qquad (4)$$

$$= \sup_{f\in C^2_{\gamma,+}(\bar{Q})} \inf_{\substack{\mu\in\mathcal{M}(\bar{Q}) \\ \int f\,d\mu=1}} \int \mathcal{G}f\,d\mu\,,$$

or equivalently

$$\rho = \inf_{f\in C^2_{\gamma,+}(\bar{Q}),\, f>0} \sup_{\nu\in\mathcal{P}(\bar{Q})} \int \frac{\mathcal{G}f}{f}\,d\nu \qquad (5)$$

$$= \sup_{f\in C^2_{\gamma,+}(\bar{Q}),\, f>0} \inf_{\nu\in\mathcal{P}(\bar{Q})} \int \frac{\mathcal{G}f}{f}\,d\nu\,,$$

where $\mathcal{P}(\bar{Q})$ denotes the space of probability measures on $\bar{Q}$and $\mathcal{M}(\bar{Q})$ is the space of all finite Borel measures.

Proof using nonlinear version of Collatz-Wielandt formula

The classical Collatz–Wielandt formula characterizes the principal (i.e., the Perron-Frobenius) eigenvalue κ of an irreducible non-negative matrix Q as

$$\kappa = \max_{\{x=(x_1,\ldots,x_d):x_i\geq 0\}} \min_{\{i:x_i>0\}} \left(\frac{(Qx)_i}{x_i}\right) = \min_{\{x=(x_1,\ldots,x_d):x_i>0\}} \max_{\{i:x_i>0\}} \left(\frac{(Qx)_i}{x_i}\right).$$

Collatz, L.(1942), Mathematische Zeitschrift 48(1) pp.221-226
Wielandt, H. (1950) Mathematische Zeitschrift 52(1) pp.642-648.

Extension of Collatz-Wielandt Formula

Let $\mathcal{X}$ be a real Banach space with total order cone P, i.e., $\mathcal{X} = P - P$ and $P \cap -P = \{0\}$, where 0 denotes the zero vector. Let $\dot{P} = P \setminus \{0\}$. Write $x \preceq y$ if $y - x \in P$. Define the dual cone

$$P^* = \{x \in \mathcal{X}^* : \langle x^*, x \rangle \geq 0 \ \forall x \in P\} .$$

A map $T : \mathcal{X} \to \mathcal{X}$ is said to be increasing if $x \preceq y \implies T(x) \preceq T(y)$, and strictly increasing if $x \prec y \implies T(x) \prec T(y)$.
If $\mathrm{int}(P) \neq \emptyset$, and $T : \dot{P} \to \mathrm{int}(P)$, then T is called strongly positive.
It is called positively 1-homogeneous if $T(tx) = tT(x)$ for all $t > 0$ and $x \in \mathcal{X}$.
T is super additive if

$$T(x + y) \preceq T(x) + T(y), \forall x, y \in \mathcal{X}.$$

Extesion of Collatz-Wielandt

$$
\begin{aligned}
P^* &:= \{x^* \in \mathcal{X}^* | \langle x^*, x \rangle \geq 0 \; \forall \; x \in P \\
P^*(x) &:= \; \{x^* \in P^* : \langle x^*, x \rangle > 0\}, \\
r_*(T) &:= \; \sup_{x \in \dot{P}} \; \inf_{x^* \in P^*(x)} \frac{\langle x^*, T(x) \rangle}{\langle x^*, x \rangle}, \\
r^*(T) &:= \; \inf_{x \in \dot{P}} \; \sup_{x^* \in P^*(x)} \frac{\langle x^*, T(x) \rangle}{\langle x^*, x \rangle}.
\end{aligned}
$$

Extension of Collatz-Wielandt

Theorem

(Non linear Krein Rutman theorem - K.C. Chang(2009)) Let $T : \mathcal{X} \to \mathcal{X}$ be a strictly increasing, positively 1-homogeneous compact continuous map satsifying $u \preceq MTu$ for some $u \in \dot{P}$ and $M > 0$. Then there exists an eigen pair $(\hat{\lambda}, \hat{x}) \in (0, \infty) \times P$. If futher T is strongle positive and super addtitve, then the eigen pair is unique in $(0, \infty) \times P$.

Theorem

Let $T : \mathcal{X} \to \mathcal{X}$ and $(\hat{\lambda}, \hat{x})$ be as in nonlinear Krein-Rutman theorem. Then $\hat{\lambda} = r^*(T) = r_*(T)$.

$\hat{\lambda}$ need not be principal eigen value. Example in a while.

Extension of Collatz-Wielandt

Theorem

Let $\mathcal{X}$ be a Banach space with total order cone P having non empty interior. Let $\{S_t | t \geq 0\}$ be a strongly continuous semi group of strongly positive, strictly increasing, positively 1-homogeneous, compact, continuous operators on $\mathcal{X}$. Then there extsts a unique $\rho \in \mathbb{R}$ and a unique $\hat{x} \in \mathrm{int}(P)$, with $\|\hat{x}\| = 1$ such that

$$S_t \hat{x} = e^{\rho t} \hat{x} \ \forall t \geq 0.$$

Proof.

- By theorem on slide 14, there exists a unique pair $(\lambda(t), x_t) \in (0, \infty) \times P$ with $\|x_t\| = 1$ such that
$$S_t x_t = \lambda(t) x_t.$$
- Now semi group property implies $x_t = \hat{x}$ for t which are dyadic rationals, for some $\hat{x} \in P$.
- Now strong continuity implies λ is continuous
- Semigroup property and positive 1-homogeneity implies $\lambda(t+s) = \lambda(t)\lambda(s)$.

Nisio Semi group

Define for each $t \geq 0$ the operator $S_t : C(\bar{Q}) \to C(\bar{Q})$ by

$$S_t f(x) \ := \ \inf_{v(\cdot)} \ E_x \left[e^{\int_0^t r(X(s),v(s))ds} f(X(t)) \right] , \tag{6}$$

infimum is over all admissible controls. Let

$$T_t^u f \ := \ E_x \left[e^{\int_0^t r(X^u(s),u)ds} f(X^u(t)) \right] ,$$

$X^u(\cdot)$ is the reflected diffusion (2) for the admissible control $v(\cdot) = u \in \mathcal{V}$.

The (multiplicative)Dynamic programming principle implies that $\{S_t | t \geq 0\}$ defines a semi group.

Nisio Semi group

Theorem

$\{S_t, t \geq 0\}$ satisfies the following properties:

1. *Boundedness: $\|S_t f\|_{0;\bar{Q}} \leq e^{r_{\max} t}\|f\|_{0;\bar{Q}}$. Furthermore, $e^{r_{\min} t} S_t \mathbf{1} \geq \mathbf{1}$, where $\mathbf{1}$ is the constant function $\equiv 1$, and $r_{\min} = \min_{(x,u)} \bar{r}(x, u)$.*
2. *Semigroup property: $S_0 = I$ and $S_t \circ S_s = S_{t+s}$ for $s, t \geq 0$.*
3. *Monotonicity: $f \geq$ (resp., $>$) $g \implies S_t f \geq$ (resp., $>$) $S_t g$.*
4. *Lipschitz property: $\|S_t f - S_t g\|_{0;\bar{Q}} \leq e^{r_{\max} t}\|f - g\|_{0;\bar{Q}}$.*
5. *Strong continuity: $\|S_t f - S_s f\|_{0;\bar{Q}} \to 0$ as $t \to s$.*
6. *Envelope property: $T_t^u f \geq S_t f$ for all $u \in U$, and $S_t f \geq S_t' f$ for any other $\{S_t'\}$ satisfying this along with the foregoing properties.*

Lemma

Let $\delta \in (0, \beta_0)$. For each $t > 0$, the map $S_t : C^{2+\delta}_\gamma(\bar{Q}) \to C^{2+\delta}_\gamma(\bar{Q})$ is compact.

- Proof is based on PDE theory.
- Consider the PDE

$$\frac{\partial}{\partial t}\psi(t,x) = \inf_{v \in \mathcal{V}} \left(\mathcal{L}_v \psi + r(x,v)\psi\right) \text{ in } (0,T] \times Q, \tag{7}$$

  with $\psi(0,x) = f(x) \in C^{2+\delta}_\gamma(\bar{Q}), \delta < \beta_0$.
- The PDE has a unique solution $S_t f \in C^{1+\delta/2, 2+\delta}(\bar{Q}_T)$.
- A further regularity argument implies $S_t f \in C^{2+\beta}_\gamma(\bar{Q})$ for each $t > 0$ and for all $\beta < \beta_0$.
- Now the lemma follows from the compact inclusion $C^{2+\beta}_\gamma(\bar{Q}) \hookrightarrow C^{2+\delta}_\gamma(\bar{Q})$.

Lemma

There exists a unique pair $(\rho,\varphi) \in \mathbb{R} \times C^2_{\gamma,+}(\bar{Q})$ *satisfying* $\|\varphi\|_{0;\bar{Q}} = 1$ *such that*

$$S_t\varphi \;=\; e^{\rho t}\varphi\,, \quad t \geq 0\,.$$

The pair (ρ,φ) *is a solution to the p.d.e.*

$$\rho\,\varphi(x) \;=\; \mathcal{G}\varphi(x) \quad \text{in } Q\,, \quad \langle \nabla\varphi, \gamma\rangle = 0 \quad \text{on } \partial Q\,, \tag{8}$$

where (8) *specifies* ρ *uniquely in* $\mathbb{R}$ *and* φ, *with* $\|\varphi\|_{0;\bar{Q}} = 1$, *uniquely in* $C^2_{\gamma,+}(\bar{Q})$.

Sketch of Proof

- The PDE (7) implies S_t is strongly positive.
- Now Theorem on slide 17 implies the existence of a pair $(\rho, \varphi) \in \mathbb{R} \times C^2_{\gamma,+}(\bar{Q})$ satisfying

$$S_t\varphi = e^{\rho t}\varphi, \ t \geq 0.$$

- The fact that (ρ, φ) satisfies (8) follows from Envelop property of the Nisio semi group.

An example

Define $R_t : C^{2+\delta}_\gamma(\bar{Q}) \to C^{2+\delta}_\gamma(\bar{Q})$ by $R_t f = -S_t(-f)$.
Then there exists a unique $\beta \in \mathbb{R}$ and $\psi > 0$ in $C^{2+\delta}_\gamma(\bar{Q})$ such that

$$R_t \psi \;=\; e^{\beta t} \psi \,.$$

Hence the pair $(e^{\beta t}, -\psi)$ is an eigenvalue-function pair of S_t.
Now the same arguments as in the proof of Lemma (previous slide) lead to the conclusion that (β, ψ) is the unique positive solution pair of

$$\beta\, \psi(x) \;=\; \sup_{v \in \mathcal{V}} \bigl(\mathcal{L}_v \psi(x) + r(x, v)\psi(x)\bigr) \quad \text{in } Q\,, \quad \langle \nabla\psi, \gamma \rangle = 0 \quad \text{on } \partial Q\,,$$

Hence $(\beta, -\psi)$ is the unique solution pair of (8) satisfying $-\psi < 0$.
Moreover $\rho \le \beta$ and that β is the principal eigenvalue of both operators R_t, S_t.

A representation lemma

Lemma

Let $\mathcal{M}(\bar{Q})$ denote the space of all finite Borel measures on $\bar{Q}$. Then

$$(C^2_\gamma(\bar{Q})^*_+ = \mathcal{M}(\bar{Q}).$$

Proof.

- For $\Lambda \in (C^2_\gamma(\bar{Q}))^*$, one can see that Λ is a bounded linear functional on the linear subspace $C^2_\gamma(\bar{Q})$ of $C(\bar{Q})$.
- Hahn-Banach theorem implies the extension of Λ to $C^*(\bar{Q})$.
- Riesz representation theorem implies $\mu \in \mathcal{M}(\bar{Q})$ satisfying $\Lambda(f) = \mu(f)$.
- Reverse is easy.

$\square$

Lemma

Let $\delta \in (0, \beta_0)$. *Then for any* $f \in C^{2+\delta}_{\gamma,+}(\bar{Q})$ *we have*

$$\limsup_{t \downarrow 0} \inf_{\substack{\mu \in \mathcal{M}(\bar{Q}) \\ \int f\, d\mu = 1}} \int_{\bar{Q}} \frac{S_t f(x) - f(x)}{t}\, \mu(dx) \;=\; \inf_{\substack{\mu \in \mathcal{M}(\bar{Q}) \\ \int f\, d\mu = 1}} \int_{\bar{Q}} \mathcal{G} f(x)\, \mu(dx)$$

and

$$\liminf_{t \downarrow 0} \sup_{\substack{\mu \in \mathcal{M}(\bar{Q}) \\ \int f\, d\mu = 1}} \int_{\bar{Q}} \frac{S_t f(x) - f(x)}{t}\, \mu(dx) \;=\; \sup_{\substack{\mu \in \mathcal{M}(\bar{Q}) \\ \int f\, d\mu = 1}} \int_{\bar{Q}} \mathcal{G} f(x)\, \mu(dx)\,.$$

Proof is technical.

Proof of Main result

- Using $\rho\varphi = \mathcal{G}\varphi$, we get

$$\begin{aligned}\rho &= \inf_{\substack{\mu\in\mathcal{M}(\bar{Q})\\ \int \varphi\, d\mu=1}} \int \mathcal{G}\varphi d\mu \\ &\leq \sup_{f\in C^{2+\delta}_{\gamma,+}(\bar{Q})} \inf_{\substack{\mu\in\mathcal{M}(\bar{Q})\\ \int f\, d\mu=1}} \int \mathcal{G} f d\mu\end{aligned}$$

- From Theorem on slide 14 and the representation lemma we get

$$e^{\rho t} = \sup_{g\in C^{2+\delta}_{\gamma,+}(\bar{Q})} \inf_{\substack{\mu\in\mathcal{M}(\bar{Q})\\ \int g\, d\mu=1}} \int S_t g\, d\mu\,.$$

- Using Lemma on slide 25, we get

$$\rho \geq \inf_{\substack{\mu\in\mathcal{M}(\bar{Q})\\ \int f\, d\mu=1}} \int \mathcal{G} f d\mu$$

for all $f \in C^{2+\delta}_{\gamma,+}(\bar{Q})$.

Proof continued

- Thus we have

$$\rho \geq \sup_{f \in C^{2+\delta}_{\gamma,+}(\bar{Q})} \inf_{\substack{\mu \in \mathcal{M}(\bar{Q}) \\ \int f \, d\mu = 1}} \int \mathcal{G} f d\mu.$$

- This completes

$$\rho = \sup_{f \in C^{2+\delta}_{\gamma,+}(\bar{Q})} \inf_{\substack{\mu \in \mathcal{M}(\bar{Q}) \\ \int f \, d\mu = 1}} \int \mathcal{G} f d\mu.$$

- A symmetric argument gives

$$\rho = \inf_{f \in C^{2+\delta}_{\gamma,+}(\bar{Q})} \sup_{\substack{\mu \in \mathcal{M}(\bar{Q}) \\ \int f \, d\mu = 1}} \int \mathcal{G} f d\mu.$$

Proof continued

- We get

$$\inf_{f\in C^2_{\gamma,+}(\bar{Q})} \sup_{\mu\in\mathcal{P}(\bar{Q})} \int \frac{\mathcal{G}f}{f} d\mu \le \rho \le \sup_{f\in C^2_{\gamma,+}(\bar{Q})} \inf_{\mu\in\mathcal{P}(\bar{Q})} \int \frac{\mathcal{G}f}{f} d\mu$$

- If strict inequailty holds above, then there exists $\hat{f} \in C^2_{\gamma,+}(\bar{Q})$ satisfying

$$\inf_{\nu\in\mathcal{P}(\bar{Q})} \int \frac{\mathcal{G}\hat{f}}{\hat{f}} d\nu > \rho.$$

- There exists $g \in C^{2+\delta}_{\gamma,+}(\bar{Q})$ such that

$$\min_{\bar{Q}} \frac{\mathcal{G}g}{g} > \rho.$$

- This contradicts last display on slide 25.