ULTRADISCRETE SYSTEMS

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PLAN OF MY LECTURE

- CELLULAR AUTOMATON
- ULTRADISCRETISATION: DEFINITION AND SIMPLE EXAMPLES
- SOLITON: A BRIEF REVIEW
- BOX-BALL SYSTEM: A TYPICAL AND MOST INVESTIGATED ULTRADISCRETE SYSTEM
- FURTHER TOPICS:
    i. ULTRADISCRETISATION WITH PARITY VARIABLES
    ii. ULTRADISCRETISATION FOR NON-INTEGRABLE SYSTEMS

CELLULAR AUTOMATON

DISCRETE DYNAMICAL SYSTEM OF CELLS WHICH TAKE ONLY FINITE NUMBER OF STATES

WHAT IS A CELLULAR AUTOMATON (CA)

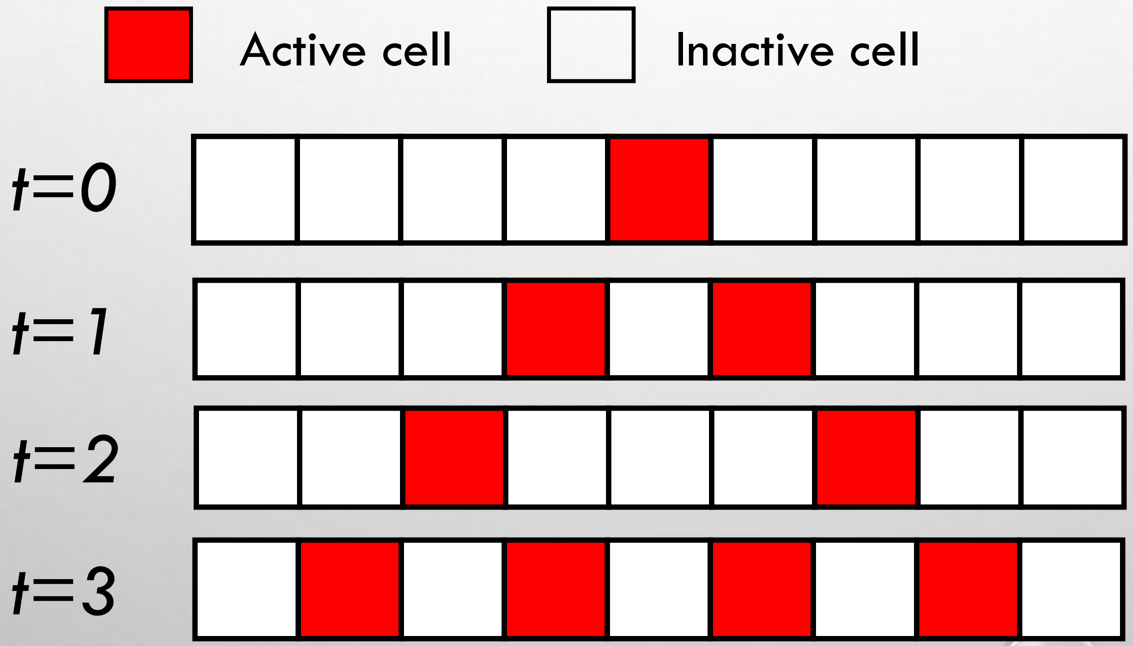
- AUTOMATON CONSTITUTED BY CELLS --- A DISCRETE DYNAMICAL SYSTEM OF CELLS WHICH TAKE FINITE NUMBER OF STATES

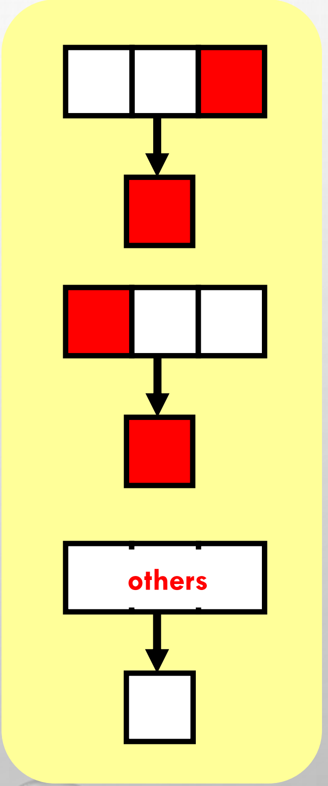
- ✓ A class of spatially and temporally discrete systems characterised by local interaction; state of a cell at the next time step is determined by the present states of itself and its adjacent cells.
- ✓ Used as mathematical models for complex phenomena such as crystal growth, spatiotemporal pattern formation in chemical reaction, biology, self-organization in networks, fluid and chemical turbulence, traffic jams, and so on.
- ✓ Von Neumann used CA to construct a mathematical model of self-reproducing essential for life (1948).
- ✓ A typical example is the Game of Life by Conway (1970).

AN EXAMPLE (ELEMENTARY CA)

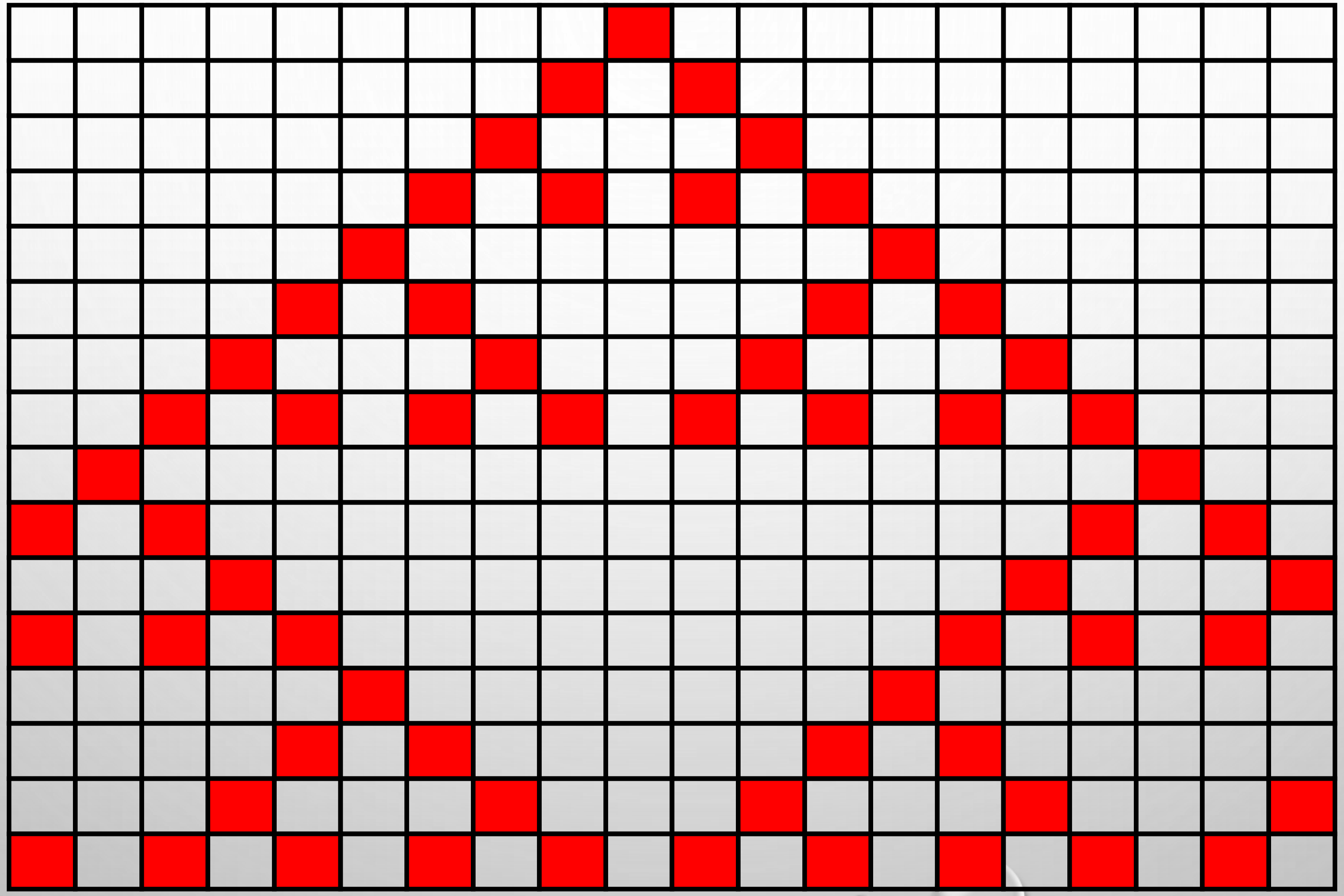
Elementary Cellular Automaton (ECA) takes only two states at each cell and time evolution of the cell is determined by the two nearest neighbor cells and the cell itself. There are $2^8=256$ distinct time evolution rules for ECA.

example) rule 90 ECA (a 1D 2states CA)

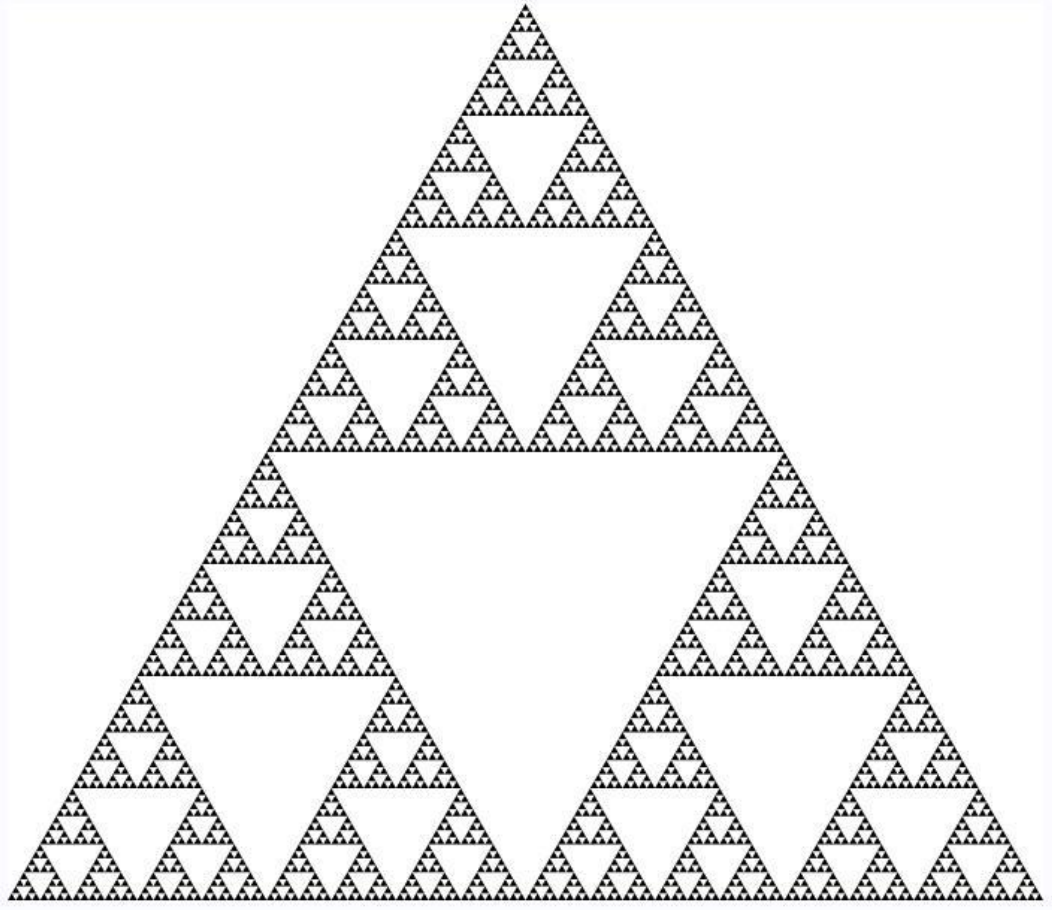




A larger system

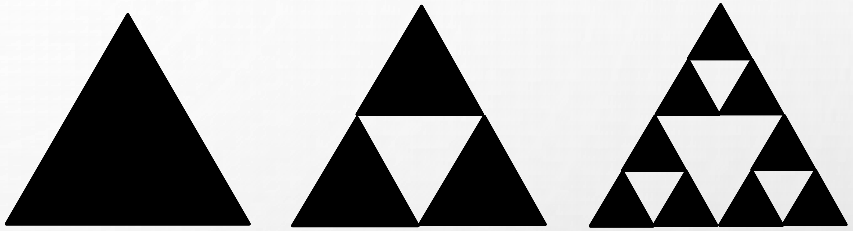


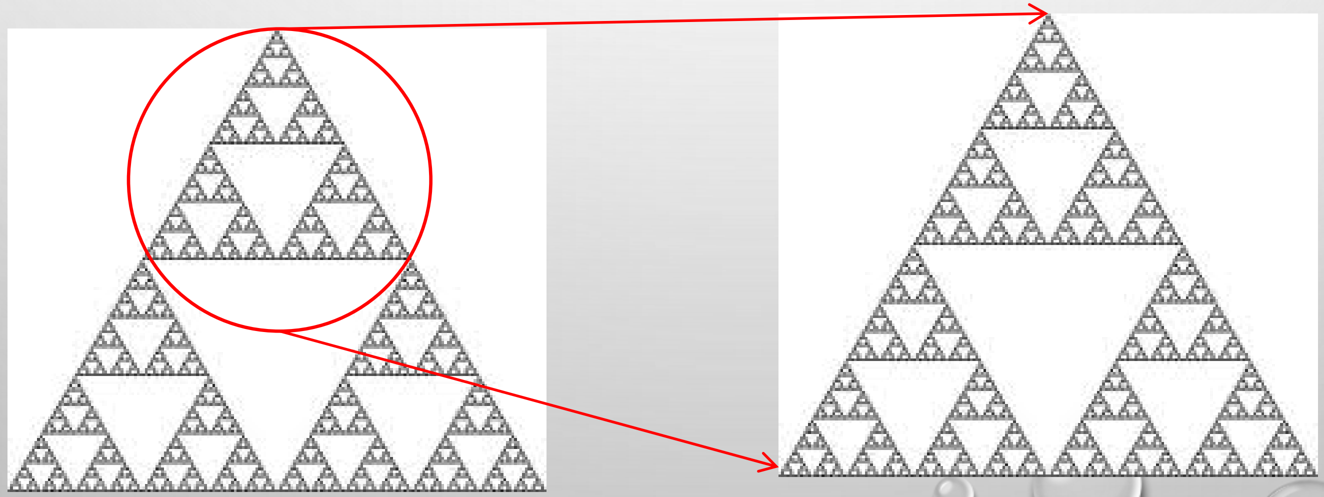
CF.) SIERPINSKII TRIANGLE



CONSTRUCTION OF SIERPINSKII TRIANGLE AND ITS SELF-SIMILARITY

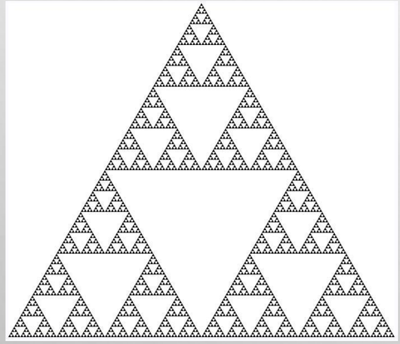
1. Start with an equilateral triangle.
2. Subdivide it into four smaller congruent equilateral triangles and remove the central one.
3. Repeat step 2 with each of the remaining smaller triangles



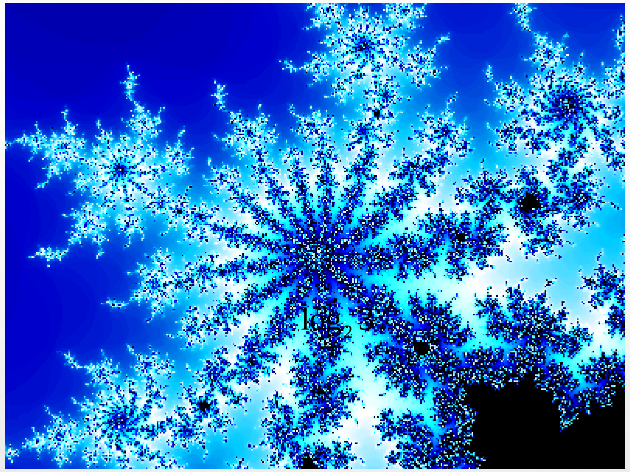


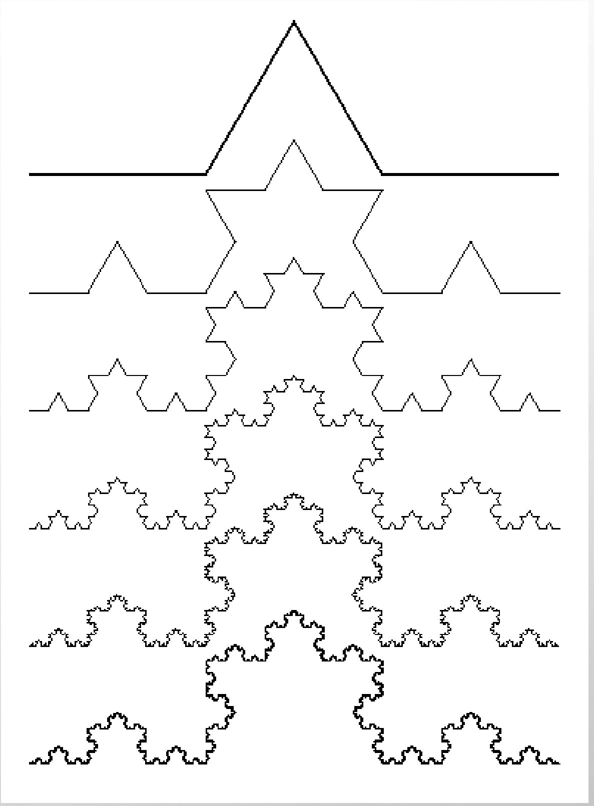
FRACTAL

Fractal is a mathematical set that typically displays self-similar patterns.
Fractals are distinguished from regular geometric figures by their fractal dimensional scaling.



$$D_F = \log_2 3$$





$$D_F = \log_3 4$$

NOTE: CA CAN BE A MATHEMATICAL MODEL OF COMPLEX PHENOMENA WITH SIMPLE TIME EVOLUTION RULE.

CF.) $u_n^{t+1} = \left| u_{n-1}^t - u_{n+1}^t \right|$ DESCRIBES TIME EVOLUTION OF THIS CA.

∵) $u_n^t \in \{0,1\} \quad \rightarrow \quad \begin{cases} u_n^{t+1} = 1 \quad \cdots \quad \text{if} \quad u_{n-1}^t + u_{n+1}^t = 1 \\ u_n^{t+1} = 0 \quad \cdots \quad \text{otherwise} \end{cases}$

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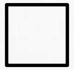
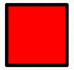
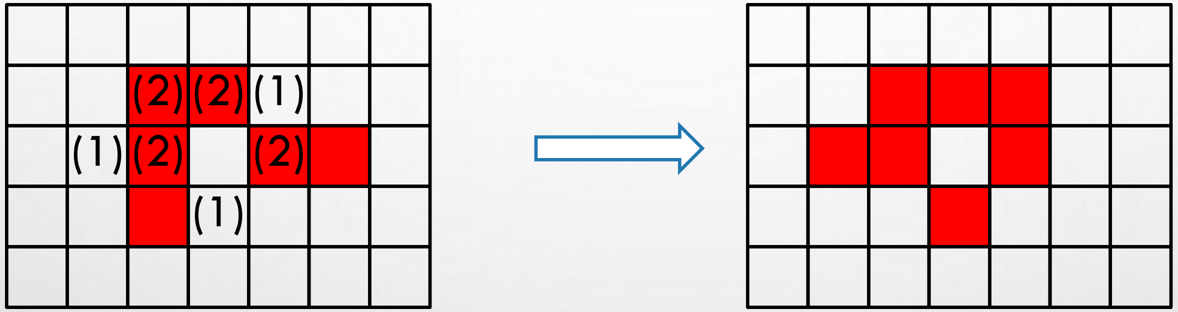
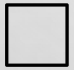
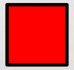
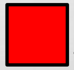
THE GAME OF LIFE

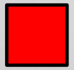
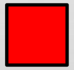
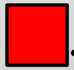
- THE GAME OF LIFE CONSISTS OF A 2D ARRAY OF SQUARE CELLS, EACH OF WHICH IS IN ONE OF TWO POSSIBLE STATES, ALIVE OR DEAD.
- EVERY CELL INTERACTS WITH ITS EIGHT NEIGHBOURS, WHICH ARE THE CELLS THAT ARE HORIZONTALLY, VERTICALLY, OR DIAGONALLY ADJACENT.
- ANY LIVE CELL WITH FEWER THAN TWO LIVE NEIGHBOURS DIES.
- ANY LIVE CELL WITH TWO OR THREE LIVE NEIGHBOURS LIVES ON TO THE NEXT GENERATION.
- ANY LIVE CELL WITH MORE THAN THREE LIVE NEIGHBOURS DIES.
- ANY DEAD CELL WITH EXACTLY THREE LIVE NEIGHBOURS BECOMES A LIVE CELL.

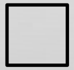
TIME EVOLUTION OF THE GAME OF LIFE

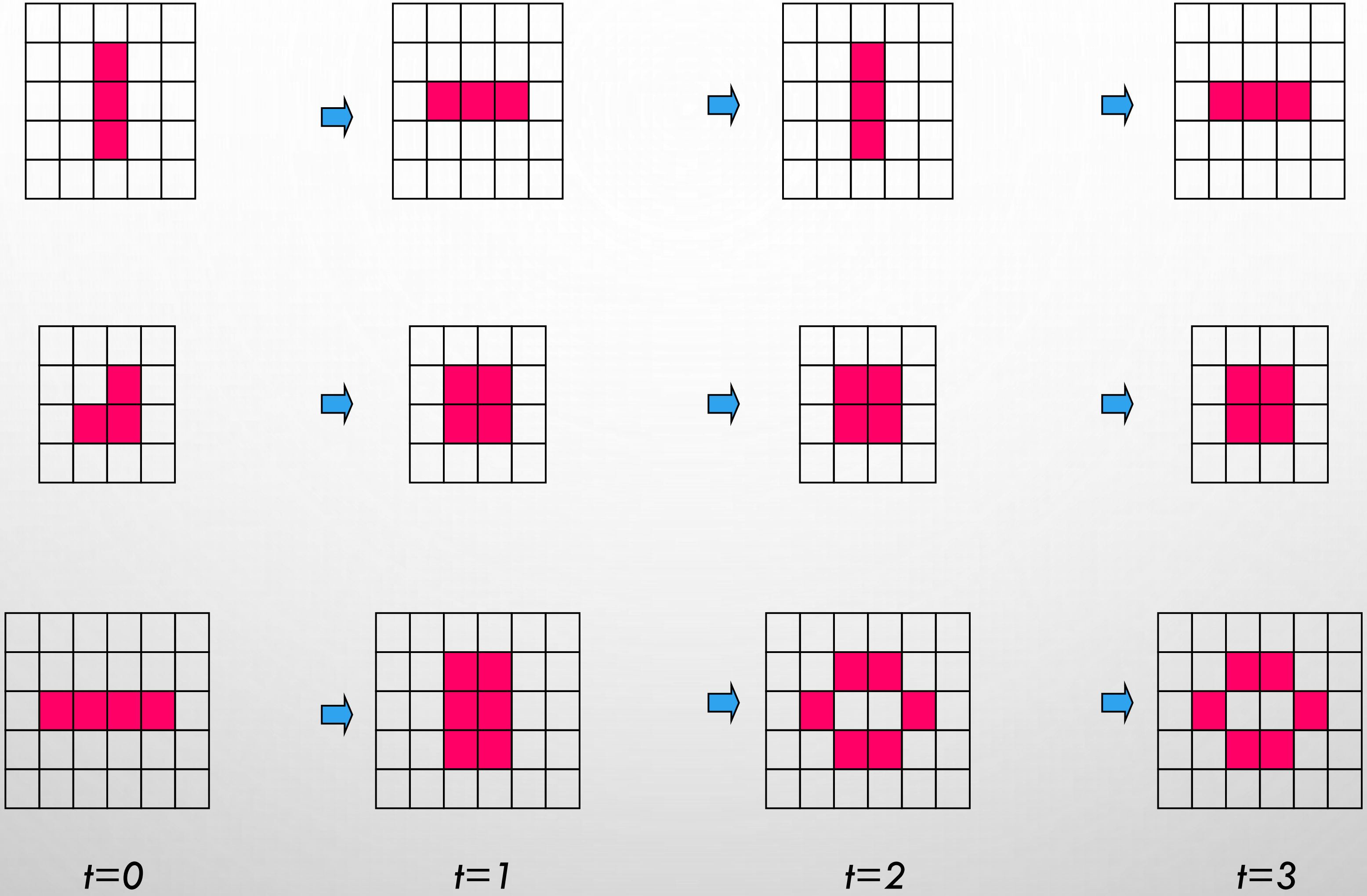
 : dead cell  : live cell



(1) For  , if there are three  it becomes .

(2) For  , if there are two or three , it remains .

(3) Otherwise, the cells is in  .


t=0
t=1
t=2
t=3

ULTRADISCRETIZATION

CA AS LIMIT OF CONTINUOUS SYSTEMS

ULTRADISCRETISATION
(Discretisation of dependent variables)

Discretisation of independent variables

$\longrightarrow$ Partial difference equations

Discretisation of dependent variables

$\longrightarrow$ Equation on a finite number of integers

$\longleftrightarrow$ Piecewise linear equation

FORMAL DEFINITION OF ULTRADISCRETE SYSTEM

- DEF : [ULTRADISCRETE SYSTEM (UDS) $\left(\hat{F}(\varepsilon), L_\varepsilon, \hat{P}\right)$]

  UDS IS THE TRIPLE OF

  1) DISCRETE EQUATION WITH A PARAMETER $\hat{F}(\varepsilon)$,

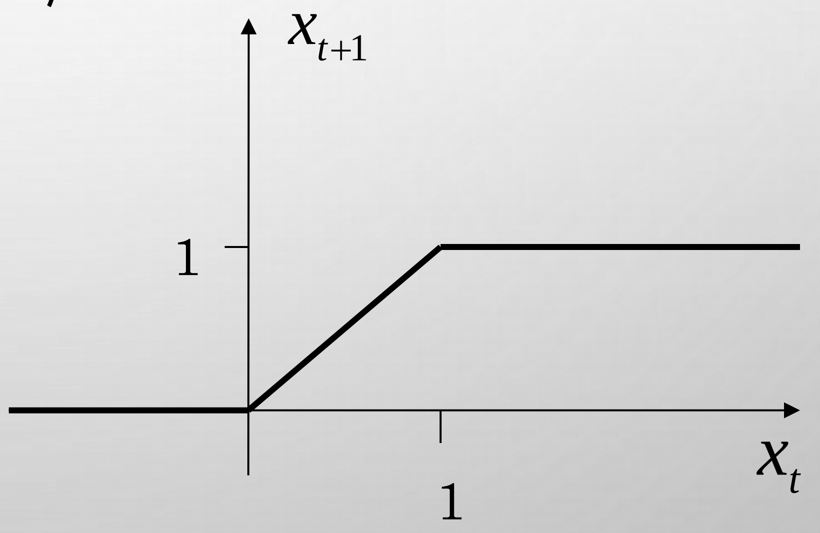
  2) LIMITING PROCEDURE W.R.T. THE PARAMETER L_ε ,

  3) PIECEWISE LINEAR EQUATION (= CA) $\hat{P}$.

- EX.) PIECEWISE LINEAR EQS.

(1) $x_{t+1} = \max[x_t, 0] - \max[x_t - 1, 0]$

(2) $u_n^{t+1} = \left|u_{n-1}^t - u_{n+1}^t\right|$



SIMPLE EXAMPLE 1
TWO TERM RECURRENCE RELATION $x_{n+1} = ax_n + b$

- $x_{n+1} = ax_n + b \quad \rightarrow \left(\hat{F}(\varepsilon)\right) \ : \ x_{n+1}(\varepsilon) = a(\varepsilon)x_n(\varepsilon) + b(\varepsilon), \ \ a(\varepsilon) = e^{A/\varepsilon}, b(\varepsilon) = e^{B/\varepsilon}$
- $(L_\varepsilon): \ \lim_{\varepsilon \to +0} \varepsilon \log x_{n+1}(\varepsilon) = \lim_{\varepsilon \to +0} \varepsilon \log[a(\varepsilon)x_n(\varepsilon) + b(\varepsilon)]$
- $\left(\hat{P}\right) \ : \quad X_{n+1} = \max[A + X_n, B] \qquad X_n := \lim_{\varepsilon \to +0} \varepsilon \log x_n(\varepsilon)$
- USEFUL IDENTITY : $\lim_{\varepsilon \to +0} \varepsilon \log\left[e^{\frac{\alpha}{\varepsilon}} + e^{\frac{\beta}{\varepsilon}}\right] = \max[\alpha, \beta]$
- NOTE : $x_n(\varepsilon) \sim e^{X_n/\varepsilon}$

SIMPLE EXAMPLE 2: A NONLINEAR MAPPING: $x_{n+1} = \frac{x_n + a}{x_n x_{n-1}}$

- $x_{n+1} = \frac{x_n+a}{x_n x_{n-1}} \quad \rightarrow \left(\hat{F}(\varepsilon)\right) \;:\; x_{n+1}(\varepsilon) = \frac{x_n(\varepsilon)+a(\varepsilon)}{x_n(\varepsilon)x_{n-1}(\varepsilon)}, \qquad a(\varepsilon) = e^{A/\varepsilon}$
- $(L_\varepsilon) \;:\; \lim_{\varepsilon \to +0} \varepsilon \log[x_{n+1}(\varepsilon)] = \lim_{\varepsilon \to +0} \varepsilon \log\left[\frac{x_n(\varepsilon)+a(\varepsilon)}{x_n(\varepsilon)x_{n-1}(\varepsilon)}\right]$
- $(\hat{P}) \;:\; X_{n+1} = \max[X_n, A\,] - X_n - X_{n-1}, \qquad X_n := \lim_{\varepsilon \to +0} \varepsilon \log x_n(\varepsilon)$
- NOTE: $\alpha + \beta \to \max[\alpha, \beta]$, $\alpha\beta \to \alpha + \beta$, $\frac{\alpha}{\beta} \to \alpha - \beta$ but $\alpha - \beta \to$ impossible

MERITS OF ULTRADISCRETIZATION

- CONSTRUCT A CA THAT INHERITS PROPERTIES OF A CONTINUOUS SYSTEM.
- SOLUTIONS, CONSERVED QUANTITIES ETC. OF THE DISCRETE EQUATION TURN TO THOSE OF THE CA. IN FACT, IF $x_n(\varepsilon)$ IS A SOLUTION TO $\hat{F}(\varepsilon)$, $X_n := \lim_{\varepsilon \to +0} \varepsilon \log x_n(\varepsilon)$ IS A SOLUTION TO $\hat{P}$.
- NEW SOLUTIONS MAY BE FOUND AND ANALYZED WITH COMBINATORIAL METHODS ETC. (CF. ULTRADISCRETE SINE-GORDON EQ. [WILLOX-NAKATA-GRAMMATICOS-RAMANI2012])

EXAMPLE: ULTRADISCRETISATION OF AN SIR MODEL

SIR MODEL: AN SIR MODEL IS AN EPIDEMIOLOGICAL MODEL THAT COMPUTES THE THEORETICAL NUMBER OF PEOPLE INFECTED WITH A CONTAGIOUS ILLNESS IN A CLOSED POPULATION OVER TIME.

(*S*: SUSCEPTIBLE, *I*: INFECTED, *R*: REMOVED)

$$\begin{cases} \dot{S}(t) = -\alpha\, S(t) I(t) \\ \dot{I}(t) = \alpha\, S(t) I(t) - \beta\, I(t) \\ \dot{R}(t) = \beta\, I(t) \end{cases}$$

$\alpha,\ \beta$ ARE POSITIVE CONSTANTS.

R(T) IS NOT NECESSARY. (TWO VARIABLE MODEL)

DISCRETIZATION AND ULTRADISCRETIZATION OF SIR MODEL

[WILLOX-GRAMMATICOS-CARSTEA-RAMANI. 2003]

- DISCRETE SIR MODEL

$$\begin{cases} \dfrac{S_n}{S_{n-1}} = \dfrac{1+\lambda I_n}{1+I_n} \\ \dfrac{I_{n+1}}{I_n} = \dfrac{\lambda + S_n}{1+\lambda S_n} \end{cases} \Longleftrightarrow \begin{cases} S_n - S_{n-1} = \lambda S_{n-1} I_n - S_n I_n \\ I_{n+1} - I_n = S_n I_n - \lambda S_n I_{n+1} - (1-\lambda) I_n \end{cases}$$

- INTRODUCE A PARAMETER (ε):

$$S_n = S_n(\varepsilon) =: \mathrm{e}^{U_n/\varepsilon}, \quad I_n = I_n(\varepsilon) =: \mathrm{e}^{V_n/\varepsilon}, \quad \lambda = \lambda(\varepsilon) =: \mathrm{e}^{k/\varepsilon}$$

$$\Rightarrow \begin{cases} \mathrm{e}^{(U_n - U_{n-1})/\varepsilon} = \dfrac{1+\mathrm{e}^{(V_n+k)/\varepsilon}}{1+\mathrm{e}^{V_n/\varepsilon}} \\ \mathrm{e}^{(V_{n+1} - V_n)/\varepsilon} = \dfrac{\mathrm{e}^{k/\varepsilon} + \mathrm{e}^{U_n/\varepsilon}}{1+\mathrm{e}^{(U_n+k)/\varepsilon}} \end{cases} \leftrightarrow \hat{F}(\varepsilon)$$

Discrete equation with a parameter

LIMITING PROCEDURE: (L_ε)

$$\lim_{\varepsilon\to+0}\varepsilon\log\left[e^{(U_n-U_{n-1})/\varepsilon}\right]=U_n-U_{n-1},$$

$$\lim_{\varepsilon\to+0}\varepsilon\log\left[\frac{1+e^{(V_n+k)/\varepsilon}}{1+e^{V_n/\varepsilon}}\right]=\max[0,V_n+k]-\max[0,V_n]$$

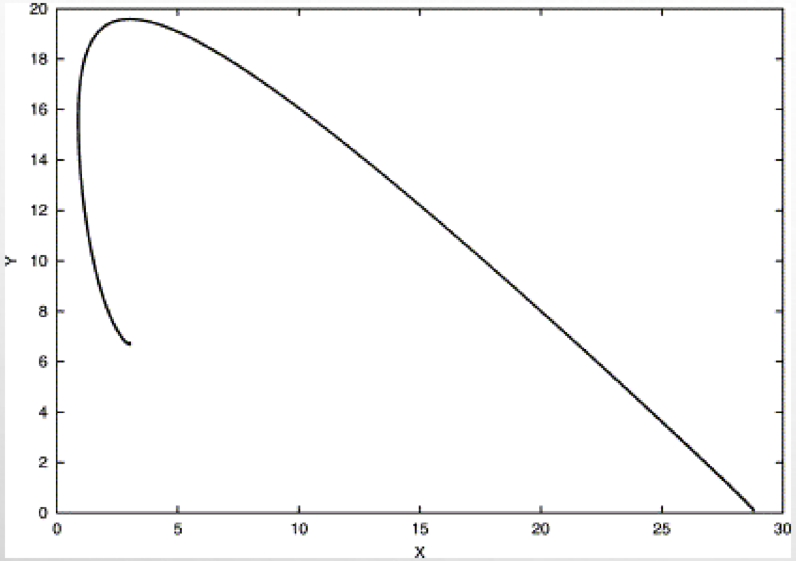
NOTE) $$\lim_{\varepsilon\to+0}\varepsilon\log\left[e^{a/\varepsilon}+e^{b/\varepsilon}\right]=\max[a,b]$$

ULTRADISCRETE SIR EQ.:

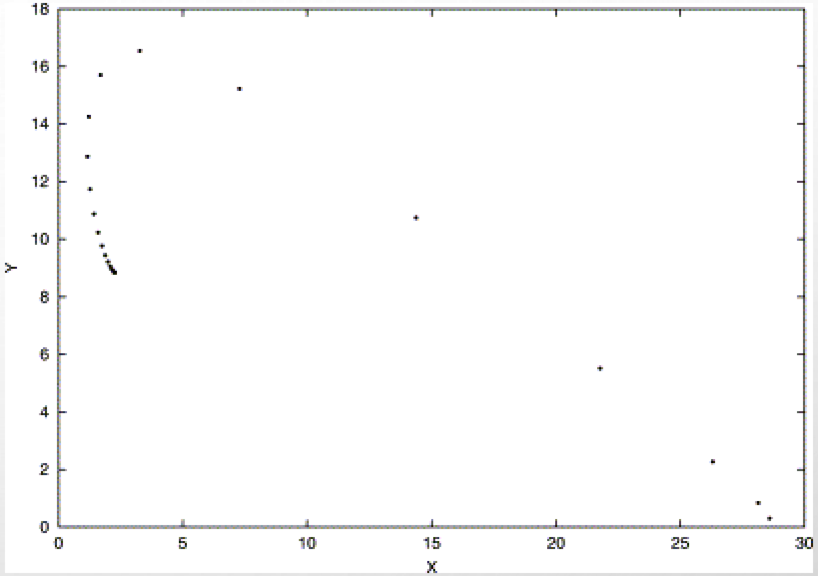
$$\begin{cases}U_n=U_{n-1}+\max[0,V_n+k]-\max[0,V_n]\\ V_{n+1}=V_n+\max[k,U_n]-\max[0,U_n+k]\end{cases}\quad\cdots(\hat{P})$$

Piecewise linear equation

SIR AND DISCRETE SIR MODEL

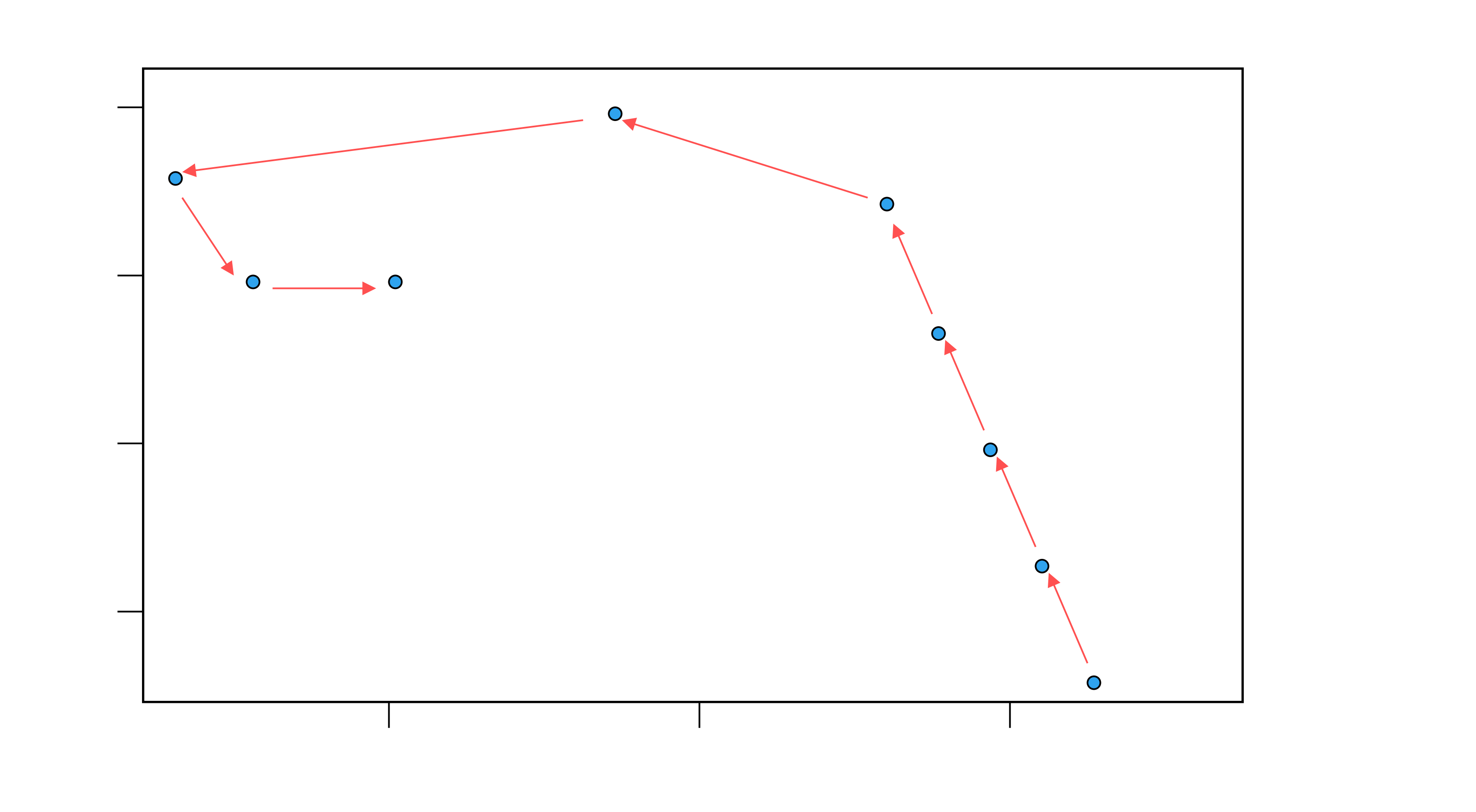


SIR model



d-SIR model

U-SIR MODEL

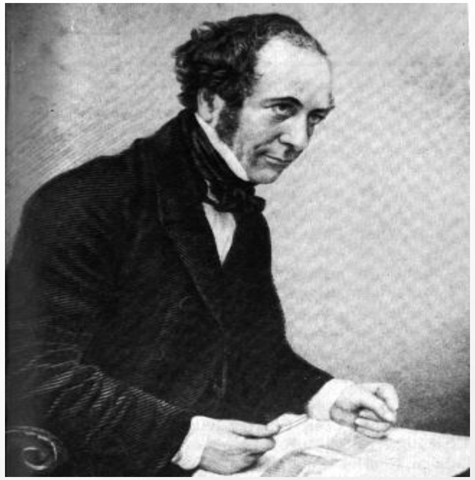


SOLITON

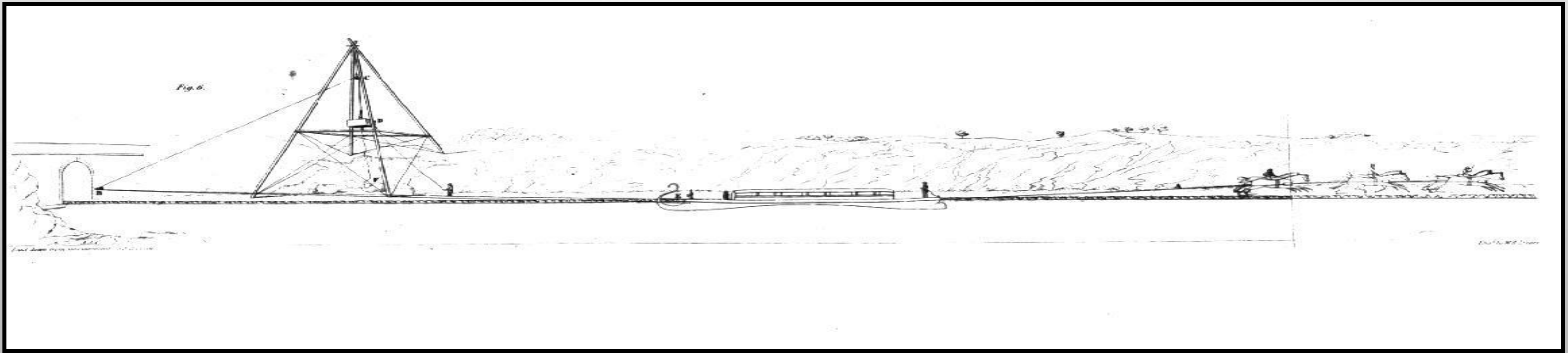
SOLITON = “SOLITARY WAVE” + “ON”

SOLITON

1834 : J. Scott Russell observed a stable solitary wave



John Scott Russell
(1808-1882)



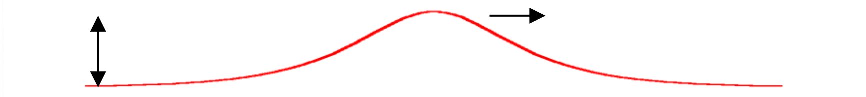
EXPERIMENTS

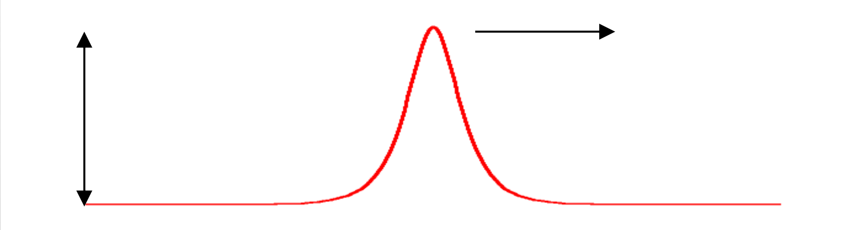
KDV EQUATION

1895 : Korteweg & de Vries derived a shallow water wave equation (KdV eq.) which justifies Scott-Russel's observation.

$$u_t + 6uu_x + u_{xxx} = 0$$

KdV eq.

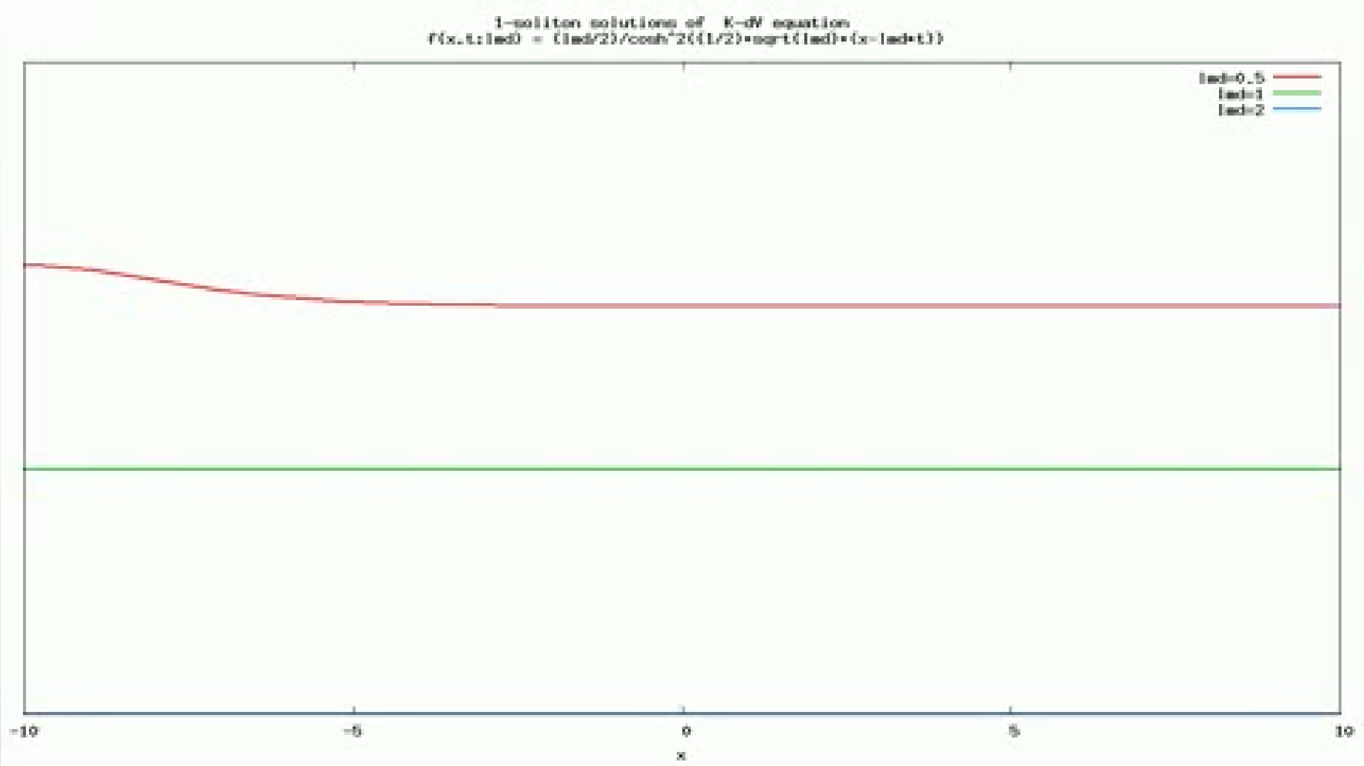




amplitude ~ velocity

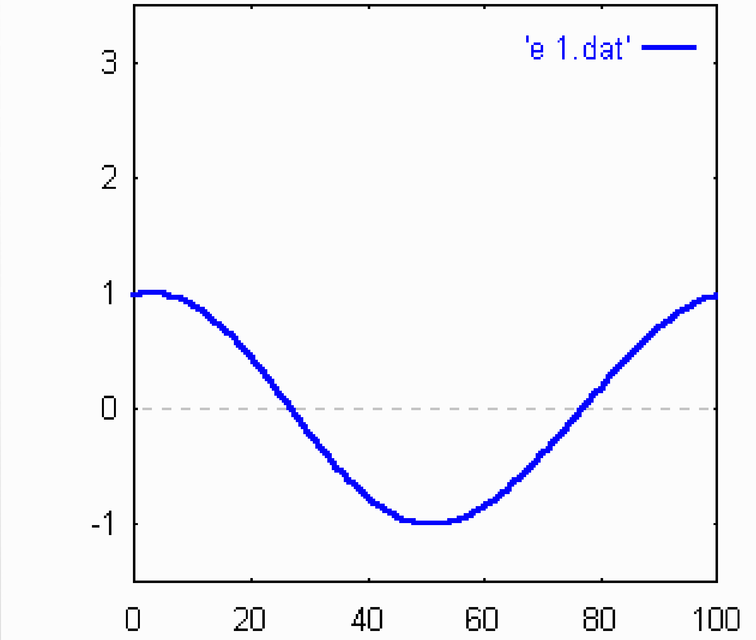
1965 : Zabsky & Kruskal rediscovered KdV eq. , observed particle behaviors and named the solitary wave a *soliton*.

1 SOLITON



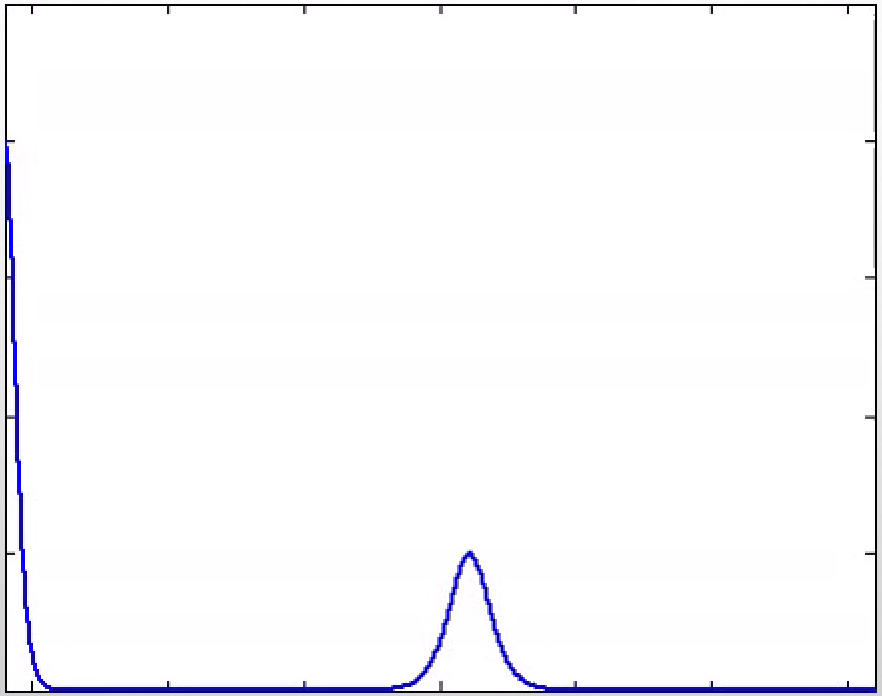
1-soliton solutions of K-dV equation
f(x,t;lmd) = (lmd/2)/cosh^2((1/2)*sqrt(lmd)*(x-lmd*t))
lmd=0.5
lmd=1
lmd=2
-10
-5
0
5
10
x

NUMERICAL SIMULATION OF KDV EQ.

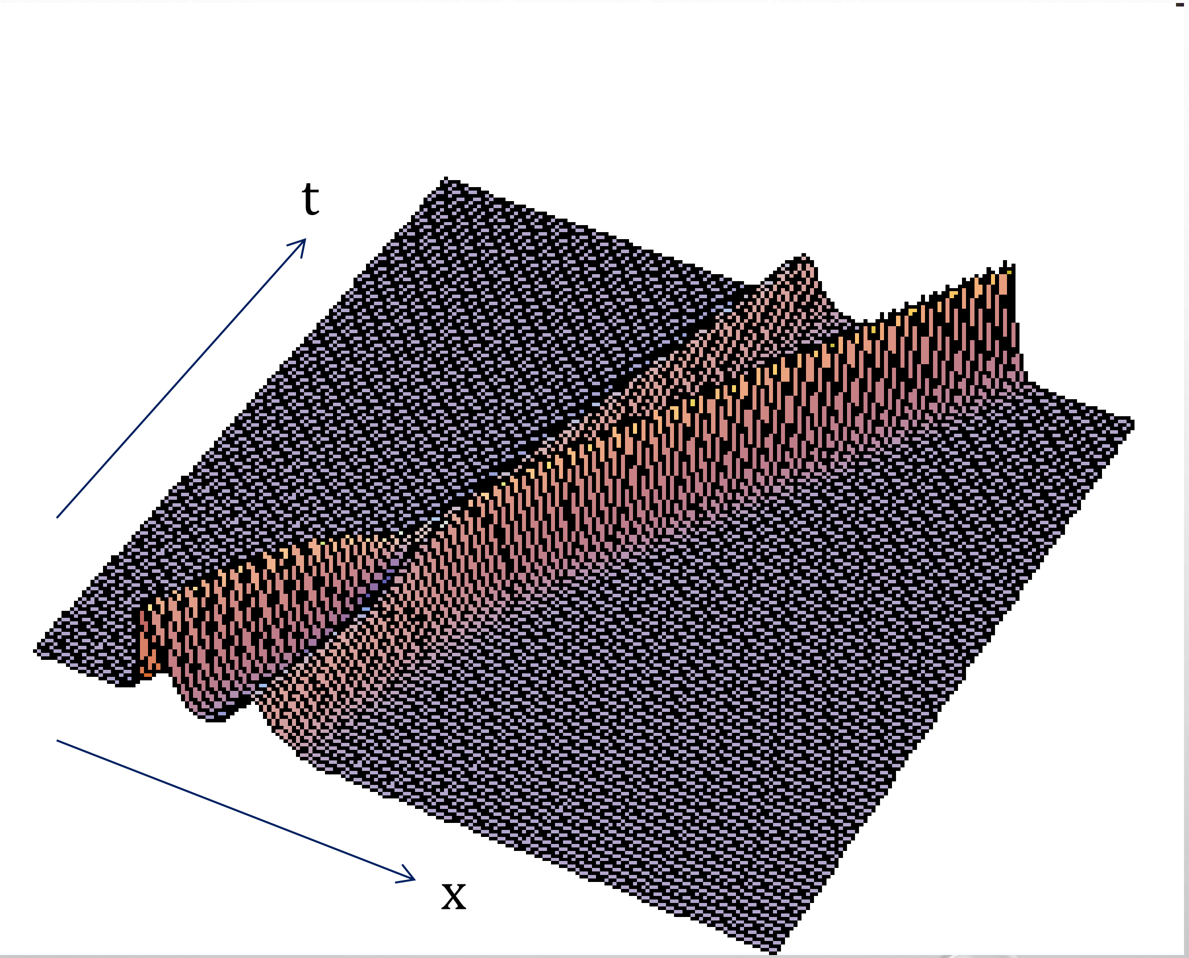


Periodic boundary condition

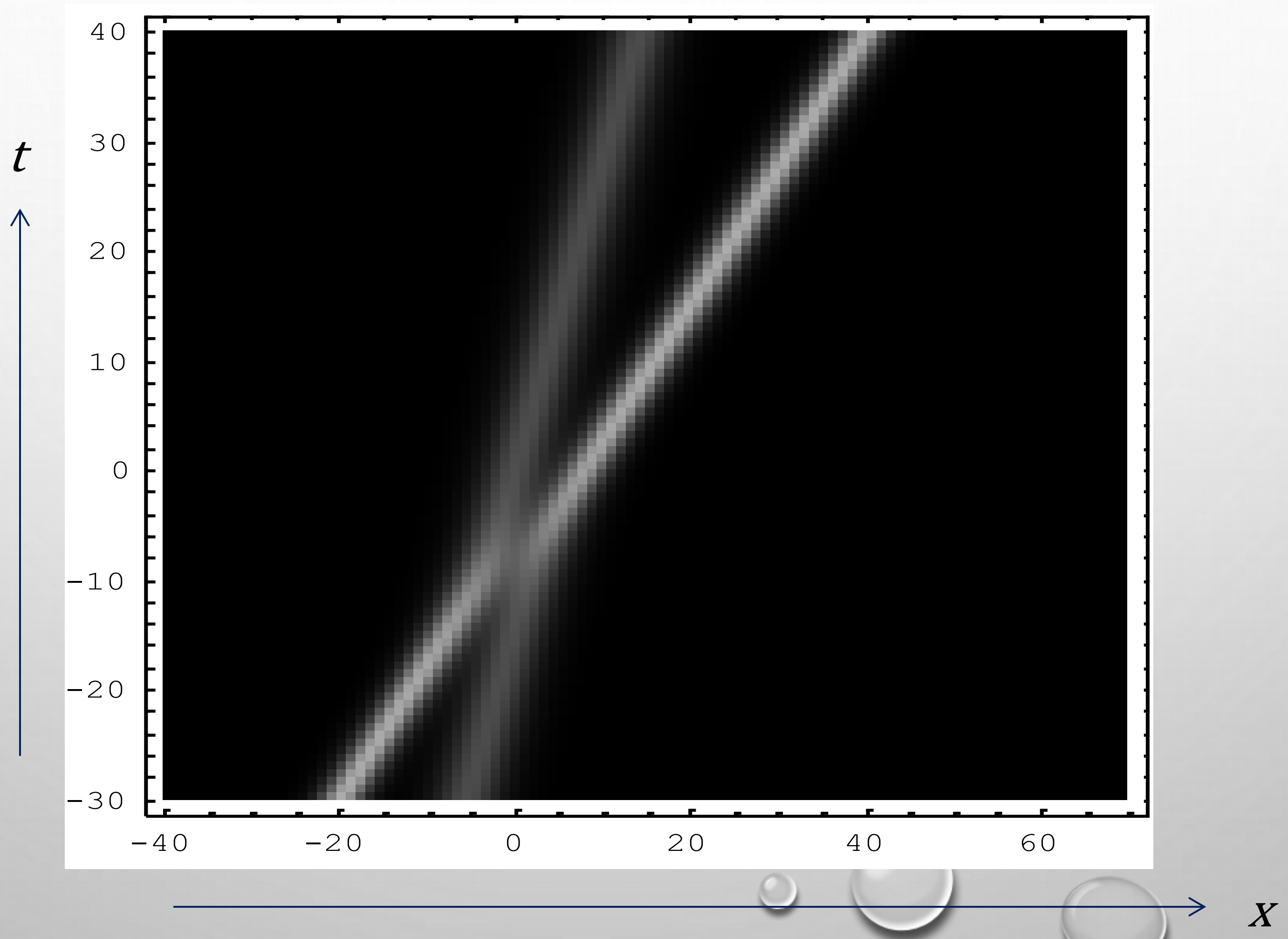
2-SOLITON



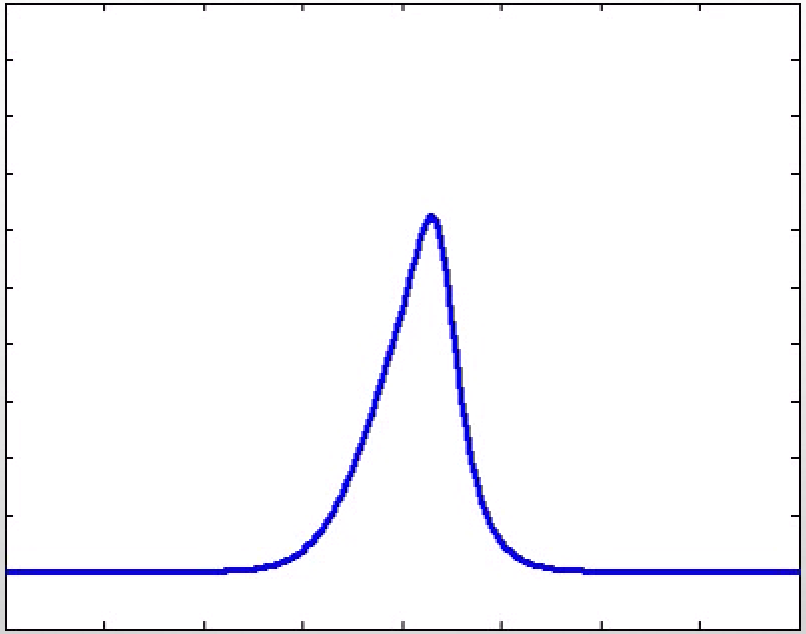
TWO SOLITON SCATTERINGS



ORBITS OF TWO SOLITONS



3-SOLITON



【SUMMARY OF SOLITONS】

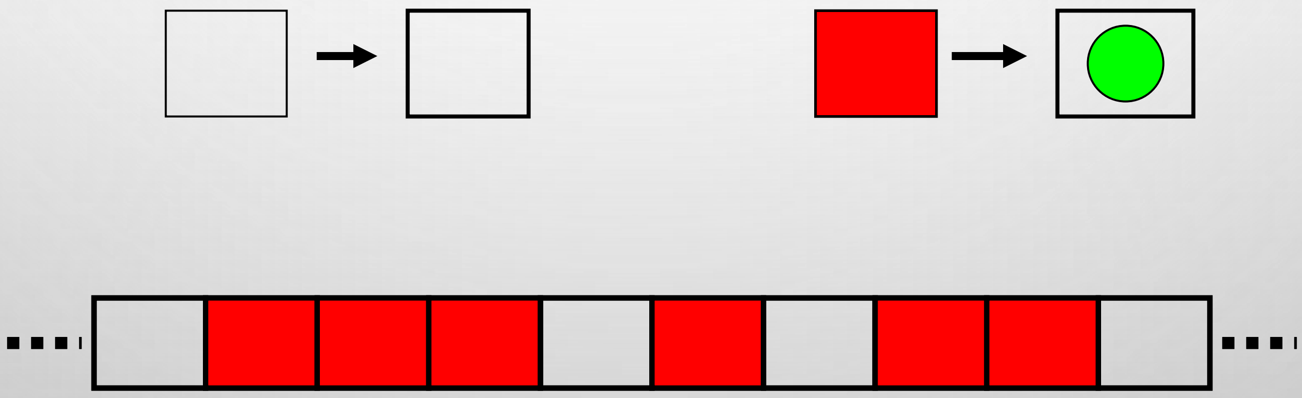
- SOLUTION TO NONLINEAR PDE.
- AMPLITUDE ~VELOCITY
- SCATTERING LIKE PARTICLES
- PHASE CHANGE AFTER COLLISION

BOX-BALL SYSTEM

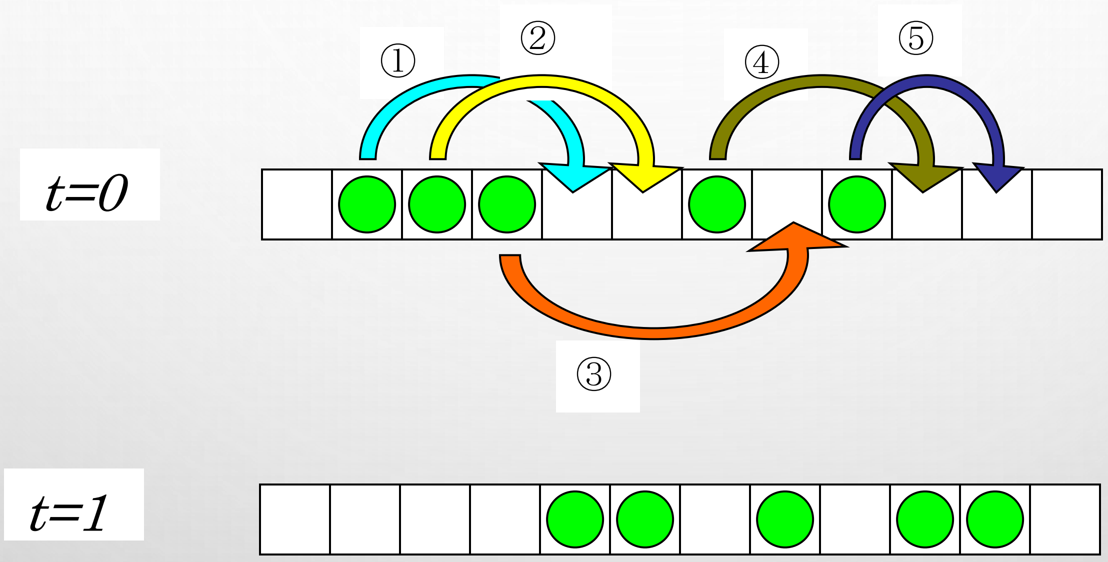
(A SOLITON CELLULAR AUTOMATON)

BOX-BALL SYSTEM (BBS)

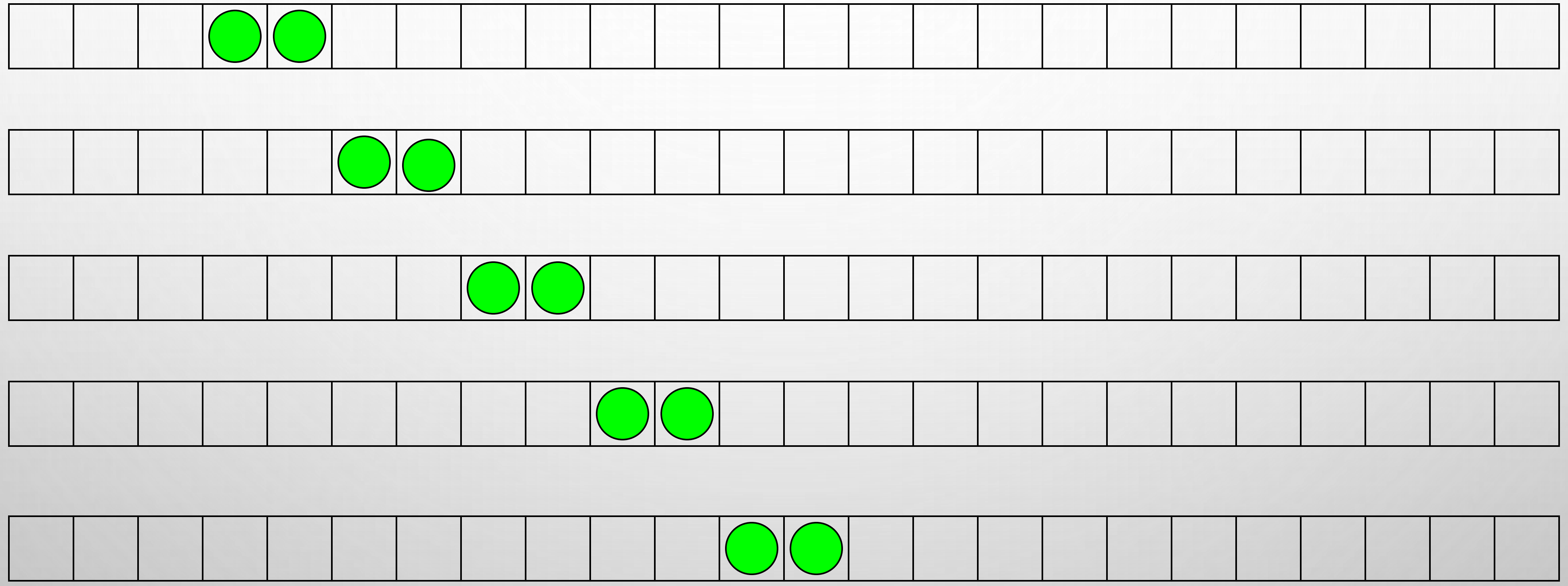
- BBS IS A REINTERPRETATION OF A SOLITON CA PROPOSED BY TAKAHASHI-SATSUMA (1990).



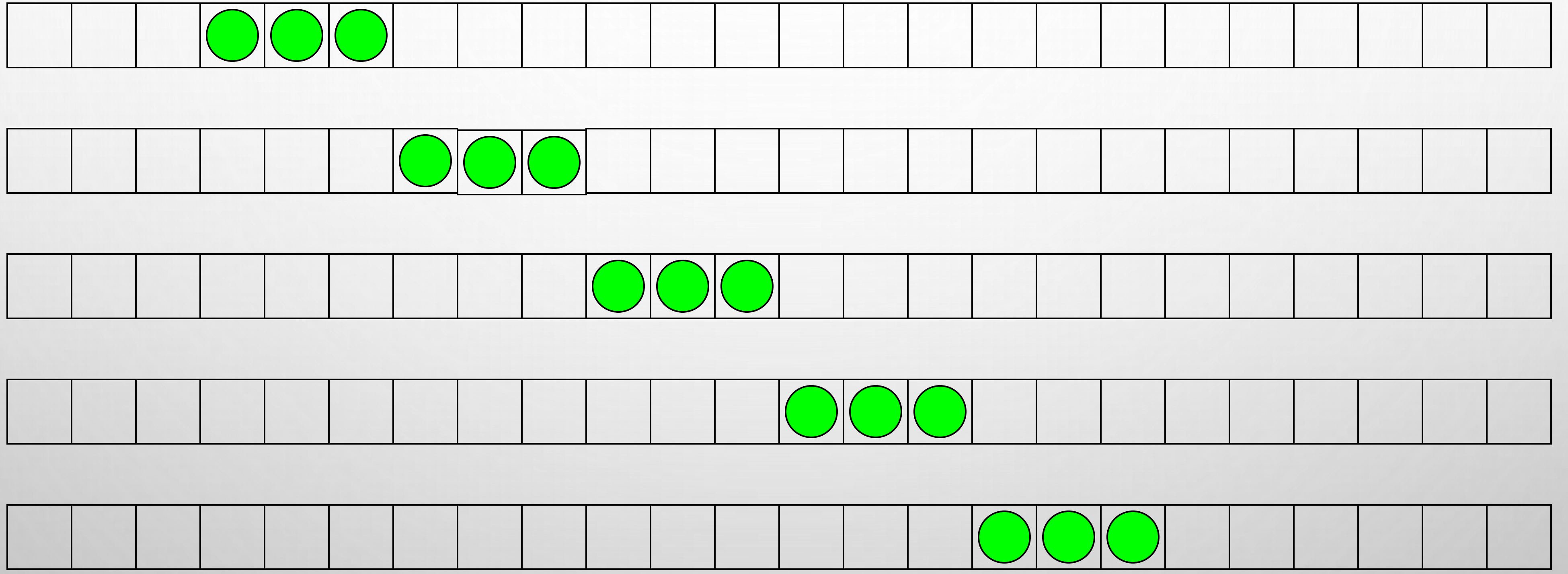
TIME EVOLUTION RULE FOR BBS



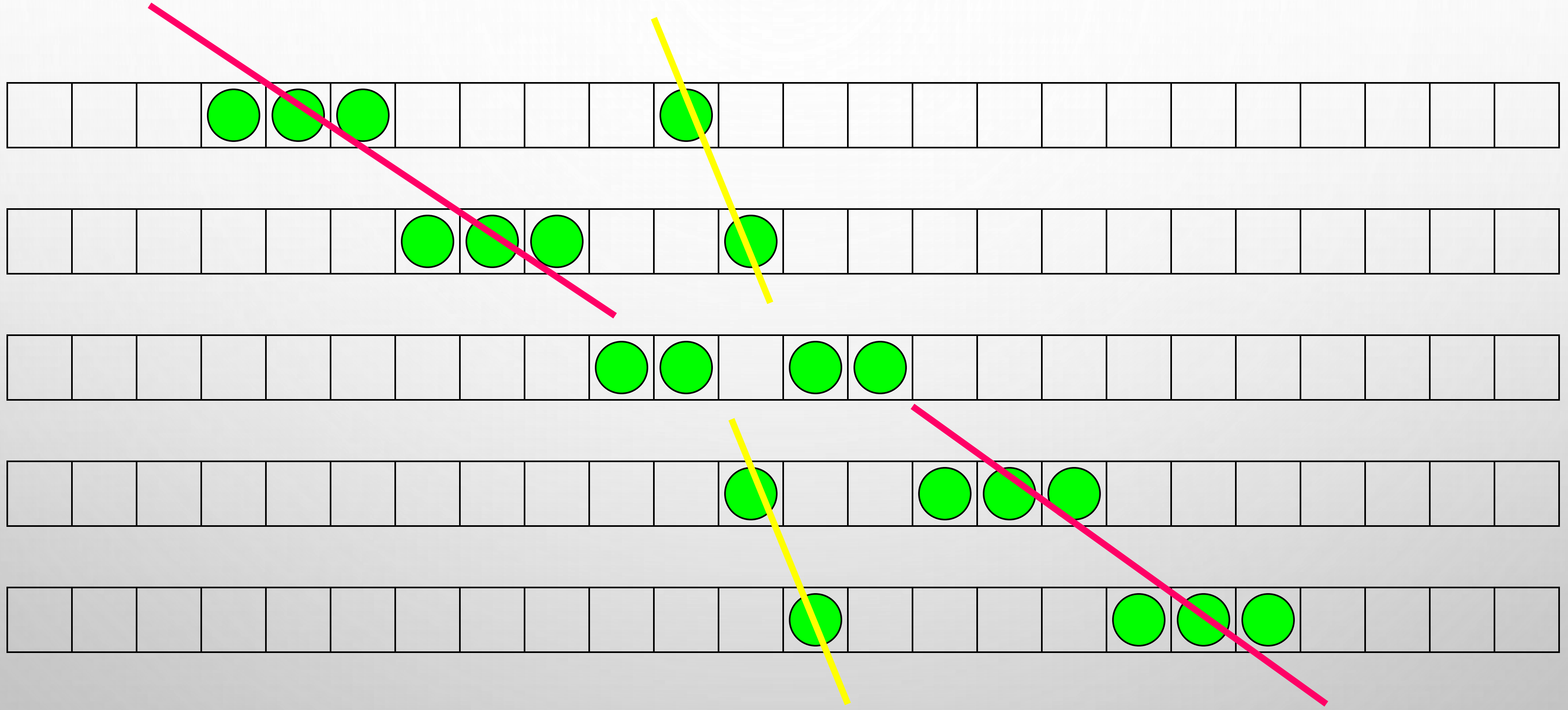
EXAMPLE 1



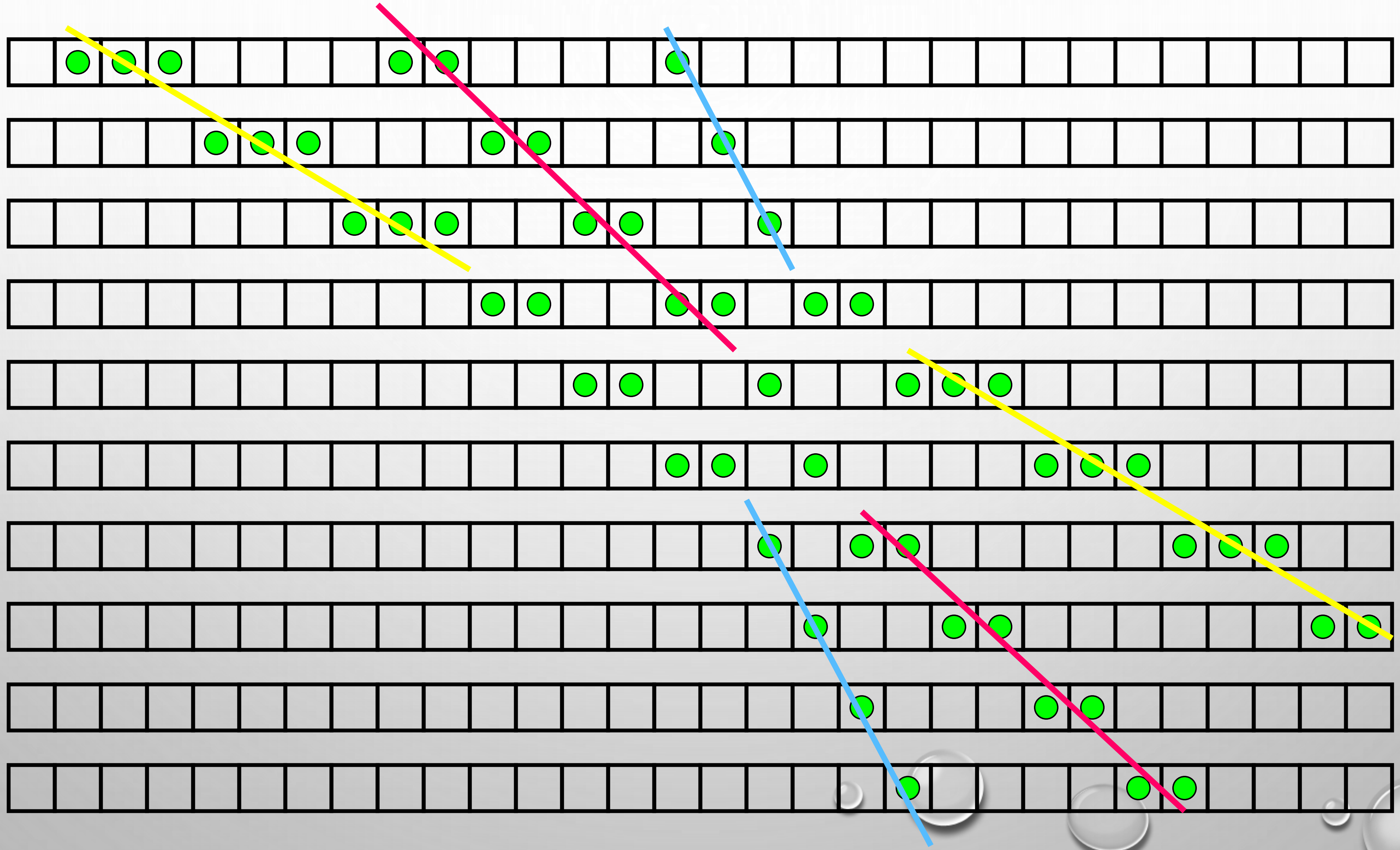
EXAMPLE 2



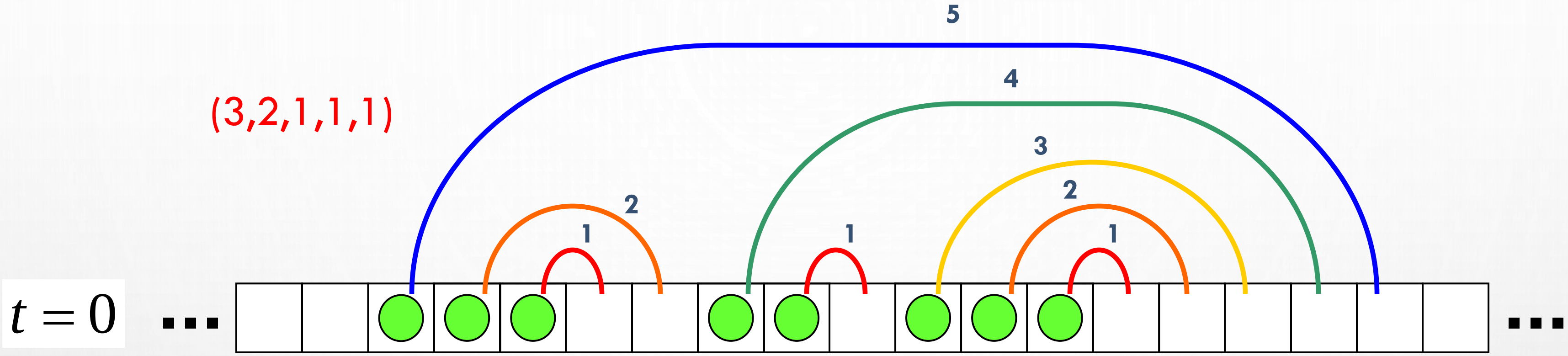
Example 3



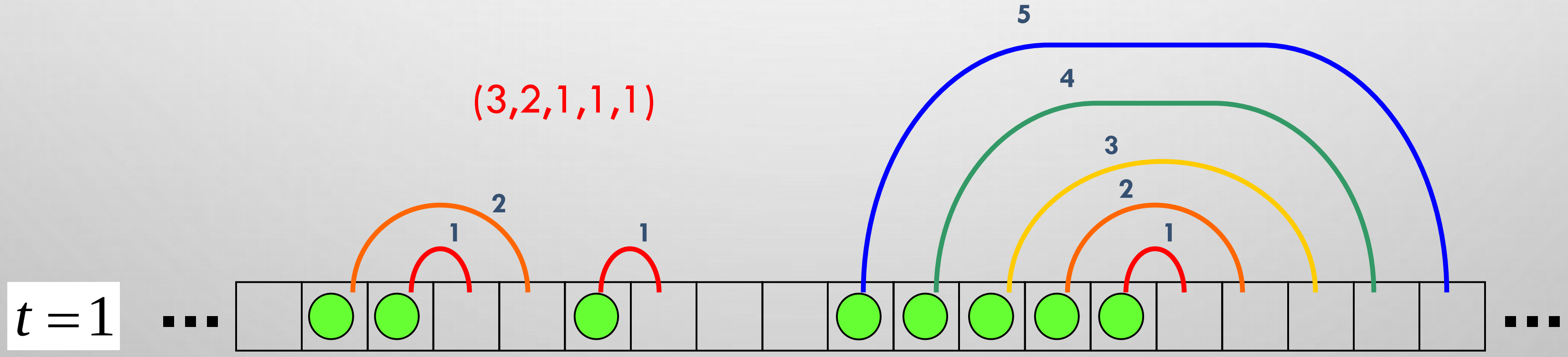
A COMPLICATED EXAMPLE



CONSERVED QUANTITIES OF BBS



p_j : #lines with index j $\Rightarrow$ ($p_1, p_2, p_3, p_4, \ldots\ldots$) is conserved



PROPERTIES OF BBS

- EVERY STATE CONSISTS OF *KDV SOLITONS* .
- EVERY STATE HAS *SUFFICIENT NUMBER OF CONSERVED QUANTITIES* AS AN INTEGRABLE DYNAMICAL SYSTEM.

Why ?

1. BBS is constructed by ultradiscretization from KdV eq.
2. It is also regarded as crystallization of a solvable lattice model.

FROM KDV EQ. TO BBS

KdV eq.

↓ Discretization of independent variables

d-KdV eq.

↓ Discretization of *dependent* variables (ultradiscretisation)

ud-KdV eq.

|||

evolution eq. for BBS

difference equation

↓ *limit*

Piecewise linear equation

FROM d-KDV eq. TO BBS (1)

- EQUATION FOR BBS: ${u_n}^t = \{0,1\}$ $\cdots$ NUMBER OF BALLS OF n-TH BOX AT TIME t

$$u_n^{t+1} = \begin{cases} 1 & \text{If } u_n^t = 0 \text{ and } \sum_{k=-\infty}^{n-1} u_k^t > \sum_{k=-\infty}^{n-1} u_k^t \\ 0 & \text{otherwise} \end{cases}$$

$$\leftrightarrow \quad u_n^{t+1} = \max\left[1 - u_n^t, \sum_{k=-\infty}^{n-1} (u_k^t - u_k^{t+1})\right]$$

- *dKDVeq.* $$\frac{1}{w_{n+1}^{t+1}} - \frac{1}{w_n^t} + \frac{\delta}{1+\delta}\{w_n^{t+1} - w_{n+1}^t\} = 0$$

- BOUNDARY CONDITION: $\lim\limits_{n\to-\infty} u_n^t = 0, \qquad \lim\limits_{n\to-\infty} w_n^t = 1$

FROM d-KDV eq. TO BBS (2)

- dKDV eq. $\rightarrow \frac{1}{w_{n+1}^{t+1}} - \frac{\delta w_{n+1}^{t}}{1+\delta} = \frac{1}{w_n^t} - \frac{\delta w_n^{t+1}}{1+\delta} \rightarrow \frac{1+\delta-\delta w_{n+1}^{t} w_{n+1}^{t+1}}{w_{n+1}^{t+1}} = \frac{1+\delta-\delta w_n^t w_n^{t+1}}{w_n^t}$
- $\rightarrow \frac{w_{n+1}^{t+1}}{w_n^t} = \frac{1+\delta-\delta w_{n+1}^{t} w_{n+1}^{t+1}}{1+\delta-\delta w_n^t w_n^{t+1}} \rightarrow \prod_{k=-\infty}^{n-1} \frac{w_{k+1}^{t+1}}{w_k^t} = \prod_{k=-\infty}^{n-1} \frac{1+\delta-\delta w_{k+1}^{t} w_{k+1}^{t+1}}{1+\delta-\delta w_k^t w_k^{t+1}} = 1+\delta-\delta w_n^t w_n^{t+1}$
- $\rightarrow w_n^{t+1} \prod_{k=-\infty}^{n-1} \frac{w_k^{t+1}}{w_k^t} = 1+\delta-\delta w_n^t w_n^{t+1} \rightarrow w_n^{t+1}\left(\prod_{k=-\infty}^{n-1} \frac{w_k^{t+1}}{w_k^t} + \delta w_n^t \right) = 1+\delta$
- $\rightarrow \frac{1+\delta}{w_n^{t+1}} = \prod_{k=-\infty}^{n-1} \frac{w_k^{t+1}}{w_k^t} + \delta w_n^t \rightarrow \delta = e^{-\frac{1}{\varepsilon}},\ w_n^t = e^{u_n^t/\varepsilon} \rightarrow \lim_{\varepsilon\rightarrow+0} \varepsilon\log$ (both sides)
- $\rightarrow -u_n^{t+1} = \max[u_n^t - 1,\ \sum_{k=-\infty}^{n-1}(u_k^{t+1} - u_k^t)] \rightarrow u_n^{t+1} = \min[1-u_n^t,\ \sum_{k=-\infty}^{n-1}(u_k^t - u_k^{t+1})]$

KDV EQ. (BILINEAR FORM)

$$\tau_{xt}\tau - \tau_x\tau_t + \tau_{xxxx}\tau - 4\tau_{xxx}\tau_x + 3(\tau_{xx})^2 = 0$$

$$(u = 2\partial_x^2 \log\tau)$$

d-KdV eq. (bilinear form)

$$\tau_{n+1}^{t+1}\tau_n^{t-1} = (1-\delta)\tau_{n+1}^t\tau_n^t + \delta\tau_{n+1}^{t-1}\tau_n^{t+1}$$

$$\delta = e^{-1/\varepsilon}$$

$$\lim_{\varepsilon\to+0}\varepsilon\log\tau_n^t(\varepsilon) =: \rho_n^t$$

ud-KdV eq. (bilinear form)

$$\rho_{n+1}^{t+1} + \rho_n^{t-1} = \max[\rho_{n+1}^t + \rho_n^t, \rho_{n+1}^{t-1} + \rho_n^{t+1} - 1]$$

$$u_n^t := (\rho_n^t - \rho_{n+1}^t) - (\rho_n^{t-1} - \rho_{n+1}^{t-1})$$

time evolution eq. for BBS

$$u_n^t = \min\left[1 - u_n^{t-1}, \sum_{k=-\infty}^{n-1} u_k^{t-1} - \sum_{k=-\infty}^{n-1} u_k^t\right]$$

$u_n^t \in \{0,1\}$: #ball in box n at time t

KEY IDENTITIES AND FACT

$$\lim_{\varepsilon \to +0} \varepsilon \log\left[e^{a/\varepsilon} \times e^{b/\varepsilon}\right] = a + b$$

$$\lim_{\varepsilon \to +0} \varepsilon \log\left[e^{a/\varepsilon} + e^{b/\varepsilon}\right] = \max[a, b]$$

$$\lim_{\varepsilon \to +0} -\varepsilon \log\left[e^{-a/\varepsilon} + e^{-b/\varepsilon}\right] = \min[a, b]$$

FACT: If $\tau(\varepsilon)$ is a one parameter family of solutions to d-KdV eq. and if the limit $\lim_{\varepsilon \to +0} \varepsilon \log \tau(\varepsilon) =: \rho$ exists, then ρ is a solution to ud-KdV eq.

N SOLITON SOLUTION OF D-KDV EQ.

$$\tau_n^t = \sum_{J\subseteq\{1,2,\dots,N\}} \prod_{i\in J} C_i \left(\frac{1-\delta-p_i}{p_i}\right)^t \left(\frac{\delta+p_i}{1-p_i}\right)^n \prod_{\substack{i,j\in J\\ i<j}} \left(\frac{p_i-p_j}{p_i+p_j-1+\delta}\right)^2$$

$$C_i = -\exp\left[\frac{\theta_i}{\varepsilon}\right],\ \delta = \exp\left[-\frac{1}{\varepsilon}\right],\ p_i = \exp\left[-\frac{P_i}{\varepsilon}\right]$$

N soliton solution of ud-KdV eq.

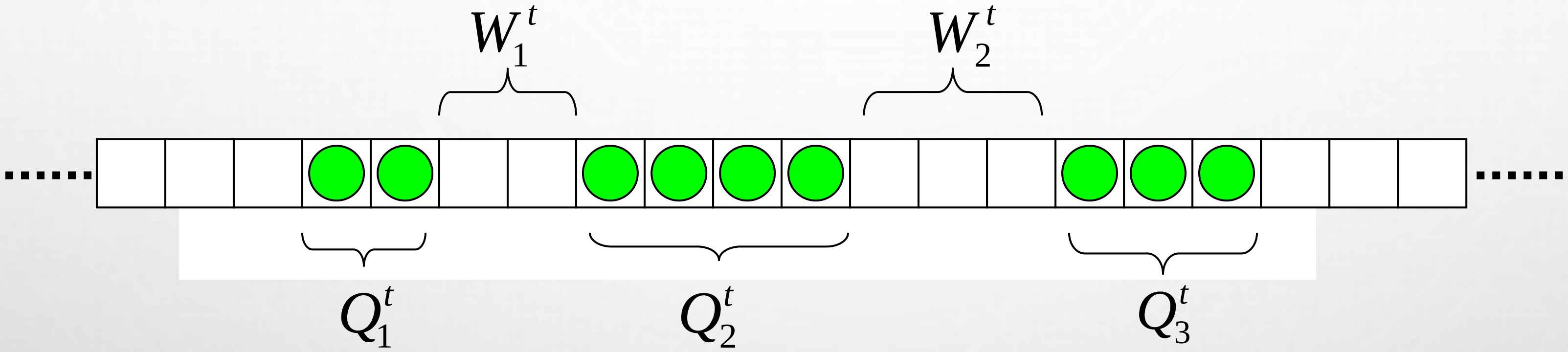
$$\rho_n^t = \lim_{\varepsilon\to+0} \varepsilon \log \tau_n^t(\varepsilon) = \max_{J\subseteq\{1,2,\dots,N\}} \left[\sum_{i\in J}(\theta_i + tP_i - n) - 2\sum_{\substack{i,j\in J\\ i<j}} \min[P_i, P_j]\right]$$

Theorem [Mada-Idzumi-T (2008)]

Any state of BBS is given by an N soliton solution of ud-KdV eq.

RELATION TO D-TODA EQUATION

Introduce new variables $\{W_n^t\},\{Q_n^t\}$.



$$\begin{cases} Q_n^{t+1} = \min\left[W_n^t, X_n^t + Q_n^t\right] \\ W_n^{t+1} = Q_{n+1}^t + W_n^t - Q_n^{t+1} \end{cases}$$

…time evolution equation

$$X_n^t := \sum_{i=1}^{n} Q_i^t - \sum_{i=1}^{n-1} Q_i^{t+1} \qquad (W_0^t = W_N^t = +\infty)$$

From d-Toda molecule equation to BBS

d-Toda eq.

$$\begin{cases} I_n^{t+1} = I_n^t + V_n^t - V_{n-1}^{t+1} \\ V_n^{t+1} = \dfrac{I_{n+1}^t V_n^t}{I_n^{t+1}} \end{cases}$$

$$(V_0^t = V_N^t = 0)$$

⟺

d-Toda eq. (Ⅱ)

$$\begin{cases} I_n^{t+1} = V_n^t + Y_n^t \\ V_n^{t+1} = \dfrac{I_{n+1}^t V_n^t}{I_n^{t+1}} \end{cases} \quad (V_0^t = V_N^t = 0)$$

$$Y_n^t := \frac{\Pi_{i=1}^n I_i^t}{\Pi_{i=1}^{n-1} I_i^{t+1}}$$

$$I_n^t = \exp\left[-Q_n^t/\varepsilon\right],\ V_n^t = \exp\left[-W_n^t/\varepsilon\right]$$

ultradiscretization

$\varepsilon \downarrow 0$

ud-Toda eq.

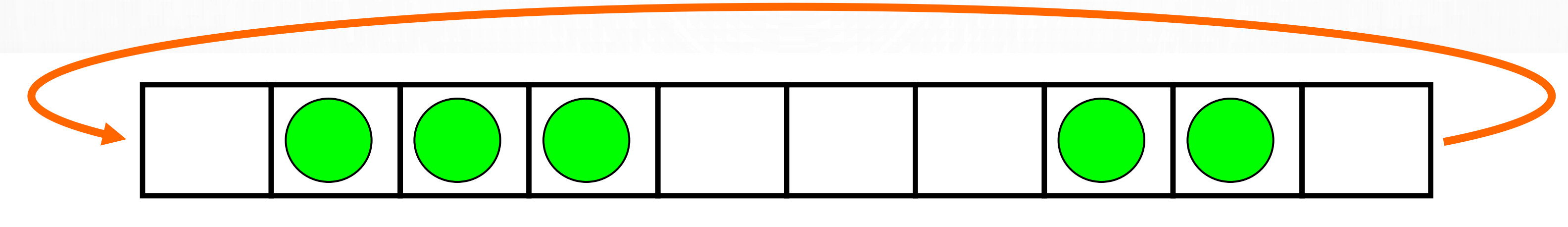
$$\begin{cases} Q_n^{t+1} = \min\left[W_n^t, X_n^t + Q_n^t\right] \\ W_n^{t+1} = Q_{n+1}^t + W_n^t - Q_n^{t+1} \end{cases} \quad (W_0^t = W_N^t = +\infty)$$

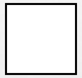
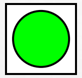
$$X_n^t := \sum_{i=1}^{n} Q_i^t - \sum_{i=1}^{n-1} Q_i^{t+1}$$

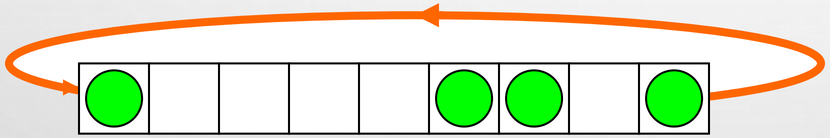
Eq. for BBS = ud-Toda eq.

PERIODIC BOX-BALL SYSTEM

PERIODIC BOX-BALL SYSTEM (PBBS)



-  = 0,  = 1

 ⟷ 100001101

$$\Omega_{M;N} := \{\text{states with } M \text{ balls and } N \text{ boxes}\}$$
$$= \{0{,}1 \text{ sequences with } M \text{ 1s and } (N\text{-}M) \text{ 0s}\}$$

CHARACTERISTIC PROPERTY OF PBBS

- A REVERSIBLE DYNAMICAL SYSTEM OF FINITE NUMBER OF STATES

$\Rightarrow$ ALL ORBITS ARE CYCLIC.

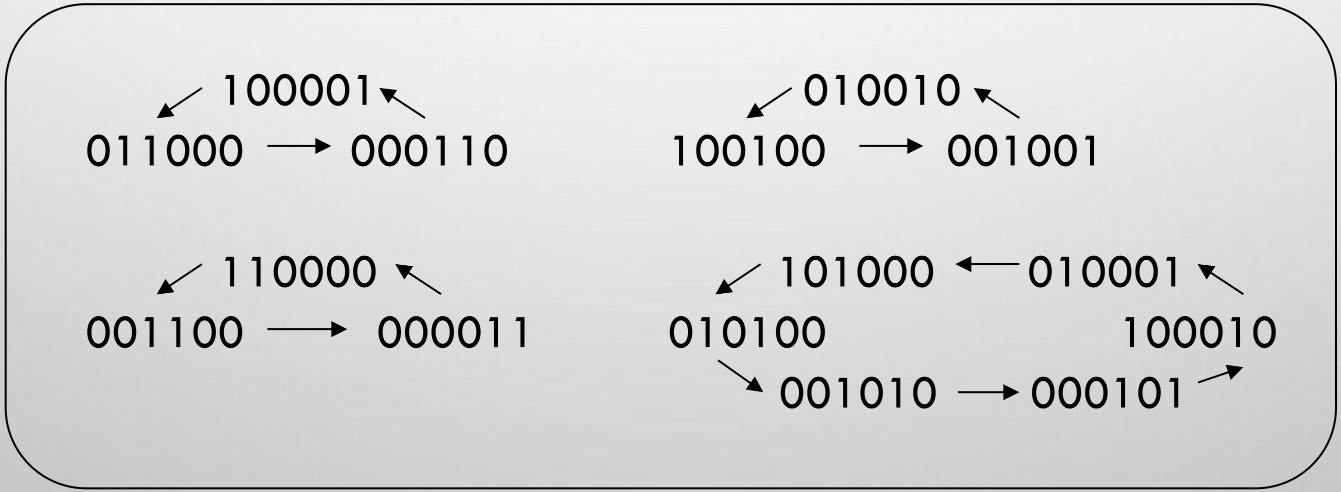
Def. [Fundamental cycle]

The length of the orbit (number of the states in the orbit) to which a state belongs is called the *fundamental cycle* of the state.

EX) #BOXES : 6, #BALLS : 2 ⇒ #STATES : 15

$$\Omega_{2;6} = \begin{Bmatrix} 110000, 101000, 100100, 100010, 100001 \\ 011000, 010100, 010010, 010001, 001100 \\ 001010, 001001, 000110, 000101, 000011 \end{Bmatrix}$$

decomposition



STUDY ON PBBS

⇒ clarify the characteristic properties of these orbits

(1) ASYMPTOTIC BEHAVIORS OF THE DISTRIBUTION OF FUNDAMENTAL CYCLES IN THERMODYNAMIC LIMIT ($N \to \infty$, M/N: FIX).

(2) RELATION BETWEEN THE ORBITS AND OTHER MATHEMATICAL AND PHYSICAL OBJECTS

- RIEMANN HYPOTHESIS
- BETHE ANSATZ AND STIRNG HYPOTHESIS

ASYMPTOTIC BEHAVIORS OF FUNDAMENTAL CYCLES

- CF) CONTINUOUS DYNAMICAL SYSTEM

ERGODIC ⇒ AN ORBIT PASSES THROUGH ALL OVER THE PHASE SPACE

INTEGRABLE ⇒ AN ORBIT IS CONFINED TO LOW DIMENSIONAL (N-DIMENSIONAL) SUBSPACE IN THE ($2N$-DIMENSIONAL) PHASE SPACE

- How about PBBS ? (PBBS…completely discrete)

The ratio of length of an orbit to the total number of the states may give some information.

NOTE: CONSERVED QUANTITIES OF (P)BBS ARE CHARACTERIZED BY A YOUNG DIAGRAM.

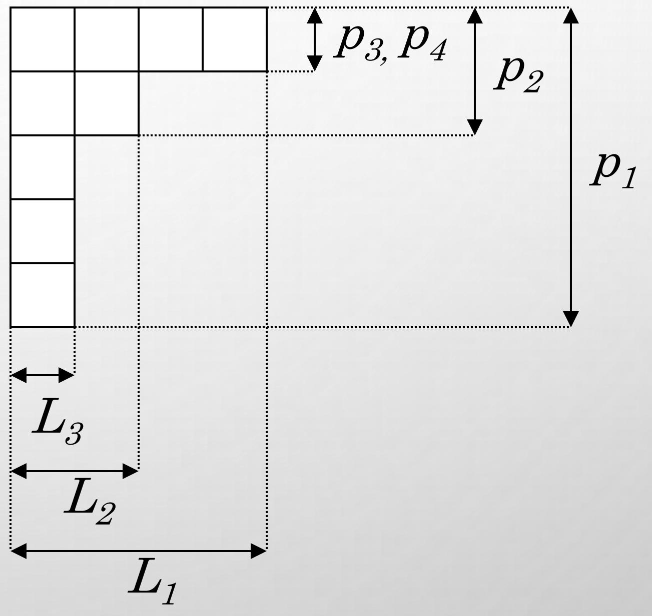
$(p_1, p_2, p_3, p_4, \dots)$ is a non-increasing positive integer sequence.

$p_k \equiv$ length of the k-th column

L_j : length of the j-th largest row

n_j : # rows with length L_j

$\{L_j, n_j\}$ is another expression of the conserved quantities



Theorem 1 [Yoshihara-Yura-T]

For a state with no internal symmetry, its fundamental cycle T is given by

$$T = \text{L.C.M.}\left(\frac{N_s N_{s-1}}{\ell_s \ell_0}, \frac{N_{s-1} N_{s-2}}{\ell_{s-1}\ell_0}, \cdots, \frac{N_1 N_0}{\ell_1 \ell_0}, 1\right)$$

(In general, it is a divisor of T.)

Here, N: #boxes, M: #balls, and

$$\text{L.C.M.}\left[2^{a1}3^{a2}5^{a3}\ldots, 2^{b1}3^{b2}5^{b3}\ldots\right] := 2^{\max[a1,b1]}3^{\max[a2,b2]}5^{\max[a3,b3]}\cdots$$

$$\ell_j = L_{j+1} - L_j, \quad \ell_0 = N_0 = N - 2M, \;\; N_j = \ell_0 + \sum_{k=1}^{j} 2n_k (L_k - L_{j+1})$$

where $\{L_j, n_j\}_{j=1}^{s}$ are its conserved quantities.

Theorem 2 [Mada-T]

- FOR GIVEN N (AND M), THE MAXIMUM VALUE OF THE FUNDAMENTAL CYCLE IS ESTIMATED AS $\log T_{\max} \approx 2\sqrt{N}$
- HOWEVER, ALMOST ALL STATES SATISFY $\log T \leq (\log N)^2$

Cf.) volume of the phase space ~ e^N (N :#boxes)

maximum value of fundamental cycle ~ $e^{\sqrt{N}}$

almost all fundamental cycles $\lesssim e^{(\log N)^2}$

⇒ at least non-ergodic

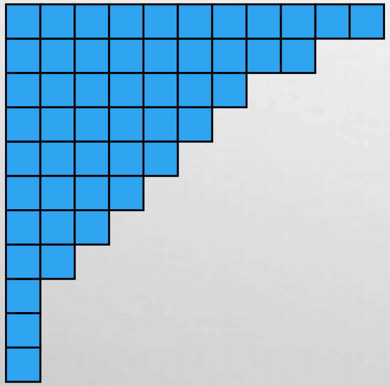
Theorem 3 [T-Mada]

THE FOLLOWING ESTIMATE FOR THE STATE WHICH HAS THE CONSERVED QUANTITIES CHARACTERIZED BY THE QUASI-TRIANGULAR YOUNG DIAGRAM:

$$\log T_{qt}(N) = 2\sqrt{N} + O\left(N^{1/4}\log^2 N\right) \qquad N \to \infty$$

IS EQUIVALENT TO THE RIEMANN HYPOTHESIS.

Cf.1)



quasi-triangular
Young diagram

Cf.2) Riemann hypothesis:

All nontrivial zeros of the zeta function:

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$$

exist on

$$\Re_e[s] = \frac{1}{2}$$

ULTRADISCRETIZATION WITH PARITY VARIABLES

[Mimura N, Isojima S, Murata M and Satsuma J 2009]

REVIEW OF UD WITH A SIMPLE EXAMPLE

$x_{n+1}=ax_n+b \qquad a,b>0 \quad \rightarrow \qquad x_n=a^n x_0+\left(\frac{1-a^n}{1-a}\right)b \qquad (a\neq 1)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = x_0+nb \qquad\qquad (a=1)$

Ultradiscretization : $\quad x_n=e^{X_n/\varepsilon}, a=e^{A/\varepsilon}, b=e^{B/\varepsilon}$

Eq. $\quad \rightarrow\ X_{n+1}=\max[A+X_n,\ B] \quad \cdots(*)$

Solution: $\quad X_n=\lim_{\varepsilon\to+0}\varepsilon\log\left[e^{nA/\varepsilon}e^{X_0/\varepsilon}+\left(\frac{1-e^{nA/\varepsilon}}{1-e^{A/\varepsilon}}\right)e^{B/\varepsilon}\right]$

$$=\begin{cases}\max[nA+X_0,\ (n-1)A+B] & (A>0)\\ \max[nA+X_0,B] & (A<0)\end{cases}$$

We can obtain the solution of (*) from the solution of the discrete equation!

NEGATIVE SIGN PROBLEM

$$x_{n+1} = (-1)ax_n + b \qquad a, b > 0 \quad \rightarrow \qquad x_n = (-a)^n x_0 + \left(\frac{1-(-a)^n}{1+a}\right) b$$

How to ultradiscretize?

Example) $x_{n+1} + ax_n = b \qquad \rightarrow \qquad \max[X_{n+1}, A + X_n] = B$

But, we cannot obtain the solution, in fact,

$$X_n = \lim_{\varepsilon \to +0} \varepsilon \log\left[(-1)^n e^{nA/\varepsilon} e^{X_0/\varepsilon} + \left(\frac{1-(-1)^n e^{nA/\varepsilon}}{1+e^{A/\varepsilon}}\right) e^{B/\varepsilon}\right] \quad ???$$

does not make sense.

INTRODUCE PARITY VARIABLE

$$x_n = \omega_n |x_n| \quad (\omega_n \in \{\pm 1\}) \quad \to \quad |x_n| = e^{X_n/\varepsilon}, \;\; \omega_n = \omega_n(\varepsilon) = \theta(\omega_n) - \theta(-\omega_n)$$

$$\theta(\omega_n) := \begin{cases} 1 & \cdots \;\; \omega_n = 1 \\ 0 & \cdots \;\; \omega_n = -1 \end{cases}$$

From $x_{n+1} = -a x_n + b,$ we have

$$(\theta(\omega_{n+1}) - \theta(-\omega_{n+1}))\, e^{X_{n+1}/\varepsilon} = -e^{A/\varepsilon}(\theta(\omega_n) - \theta(-\omega_n))e^{X_n/\varepsilon} + e^{B/\varepsilon}$$

$$\therefore \quad \theta(\omega_{n+1})\, e^{X_{n+1}/\varepsilon} + \theta(\omega_n) e^{A/\varepsilon} e^{X_n/\varepsilon}$$

$$= \theta(-\omega_{n+1}) e^{X_{n+1}/\varepsilon} + \theta(-\omega_n) e^{A/\varepsilon} e^{X_n/\varepsilon} + e^{B/\varepsilon}$$

Using the identity:

$$\lim_{\varepsilon \to +0} \varepsilon \log[\, \theta(\omega) e^{\alpha/\varepsilon} + e^{\beta/\varepsilon}\,] = \max[\alpha + \vartheta(\omega), \;\; \beta]$$

where $\vartheta(\omega) := \begin{cases} 0 & \cdots \;\; \omega = 1 \\ -\infty & \cdots \;\; \omega = -1 \end{cases}$

ULTRADISCRETE EQUATION WITH PARITY VARIABLE

$$\max[X_{n+1}+\vartheta(\omega_{n+1}), X_n+A+\vartheta(\omega_n)] \quad = \max[X_{n+1}+\vartheta(-\omega_{n+1}), X_n+A+\vartheta(-\omega_n), B]$$

ultradiscrete equation with parity variable

Solution:

$$x_n = (-a)^n\, x_0 + \left(\frac{1-(-a)^n}{1+a}\right) b = (-1)^n \left[a^n\left(x_0 - \frac{b}{1+a}\right) + \frac{(-1)^n}{1+a} b\right]$$

$$\rightarrow \quad \omega_n e^{X_n/\varepsilon} = (-1)^n \left[e^{nA/\varepsilon}\left(\omega_0 e^{X_0/\varepsilon} - \frac{e^{B/\varepsilon}}{1+e^{A/\varepsilon}}\right) + \frac{(-1)^n}{1+e^{A/\varepsilon}} e^{B/\varepsilon}\right] \cdots (\aleph)$$

Example: $\omega_0 = 1,\ X_0 > B > 0 > A$

$$(\aleph) \sim (-1)^n e^{(nA+X_0)/\varepsilon} \cdots n < \frac{X_0-B}{-A} \quad \text{and} \quad (\aleph) \sim e^{B/\varepsilon} \cdots n > \frac{X_0-B}{-A}$$

$$\therefore \quad \omega_n = \begin{cases} (-1)^n & \cdots \quad n < \frac{X_0-B}{-A} \\ 1 & \cdots \quad n > \frac{X_0-B}{-A} \end{cases}, \quad X_n = \begin{cases} nA + X_0 & \cdots \quad n < \frac{X_0-B}{-A} \\ B & \cdots \quad n > \frac{X_0-B}{-A} \end{cases}$$

solution to UDE

NOTE: TIME EVOLUTION OF ULTRADISCRETE EQUATION WITH PARITY VARIABLE

$$\omega_0 = 1 \;\; X_0 = 3 > B = 1 > 0 > A = -1$$

$$\max[X_{n+1} + \vartheta(\omega_{n+1}), X_n - 1 + \vartheta(\omega_n)] \quad = \max[X_{n+1} + \vartheta(-\omega_{n+1}), X_n - 1 + \vartheta(-\omega_n), 1]$$

For n=1,

$$\max[X_1 + \vartheta(\omega_1), X_0 - 1 + \vartheta(\omega_0)] = \max[X_1 + \vartheta(-\omega_1), X_0 - 1 + \vartheta(-\omega_0), 1]$$

$$\therefore \max[X_1 + \vartheta(\omega_1), 3 - 1 + \vartheta(1)] = \max[X_1 + \vartheta(-\omega_1), 2 - 1 + \vartheta(-1), 1]$$

$$\therefore \max[X_1 + \vartheta(\omega_1), 2] = \max[X_1 + \vartheta(-\omega_1), -\infty, 1]$$

$$\therefore \max[X_1 + \vartheta(-\omega_1), 2] = X_1 + \vartheta(-\omega_1) \quad \rightarrow \;\; \omega_1 = -1, X_1 = 2$$

For n=2,

$$\max[X_2 + \vartheta(\omega_2), X_1 - 1 + \vartheta(\omega_1)] = \max[X_2 + \vartheta(-\omega_2), X_1 - 1 + \vartheta(-\omega_1), 1]$$

$$\therefore \max[X_2 + \vartheta(\omega_2), 2 - 1 - \infty] = \max[X_2 + \vartheta(-\omega_2), 2 - 1 + 0, 1]$$

$$\therefore X_2 + \vartheta(\omega_2) = \max[X_2 + \vartheta(-\omega_2), 1] \quad \rightarrow \quad \omega_2 = 1, X_2 = 1$$

Thus we find that initial value problem is naturally solved with this equation. The solution coincides with the ultradiscrete limit of that of the discrete equation.

APPLICATION TO ULTRADISCRETE AIRY FUNCTIONS

The q-difference Airy equation is given as

$$w(qx) - xw(x) + w(q^{-1}x) = 0, \qquad (0 < q < 1).$$

The two independent solutions are Ai and Bi q-difference functions:

$$qAi(x) := \sum_{n=0}^{\infty} \frac{\sqrt{2}\sin\left\{\frac{\pi}{4}\left(2n+\frac{\log x}{\log q}+1\right)\right\} q^{\frac{n(n+1)}{2}}}{(q^2;q^2)_n} x^n, \qquad qBi(x) := \sum_{n=0}^{\infty} \frac{\sqrt{2}\sin\left\{\frac{\pi}{4}\left(2n+\frac{\log x}{\log q}+3\right)\right\} q^{\frac{n(n+1)}{2}}}{(q^2;q^2)_n} x^n,$$

where $(q^2;q^2)_n = \prod_{j=1}^{n}\left(1-q^{2j}\right)$.

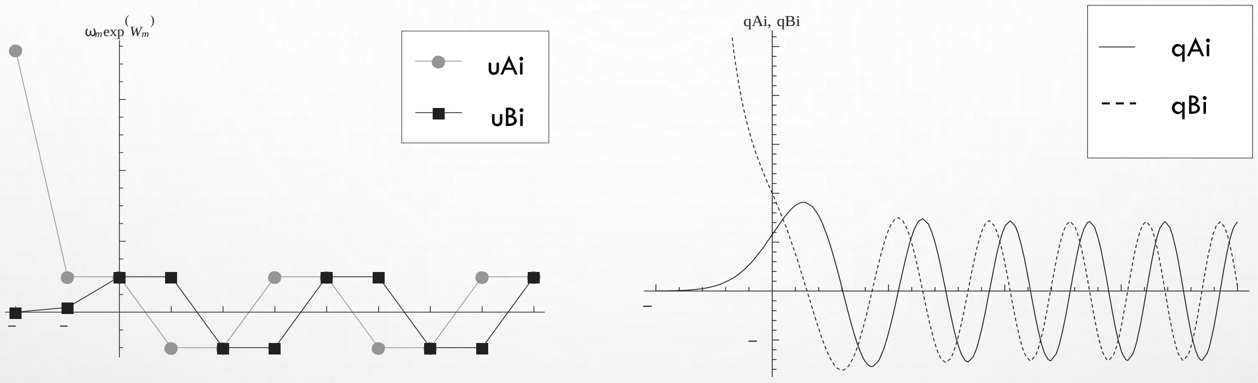
Putting $x = q^m$, $q = e^{Q/\varepsilon}$, $w(q^m) = \omega_m e^{W_m/\varepsilon}$, and taking ultradiscrete limit, we have

$$\max[\vartheta(\omega_{m+1}) + W_{m+1}, \vartheta(-\omega_m) + mQ + W_m, \vartheta(\omega_{m-1}) + W_{m-1}]$$
$$= \max[\vartheta(-\omega_{m+1}) + W_{m+1}, \vartheta(\omega_m) + mQ + W_m, \vartheta(-\omega_{m-1}) + W_{m-1}]$$

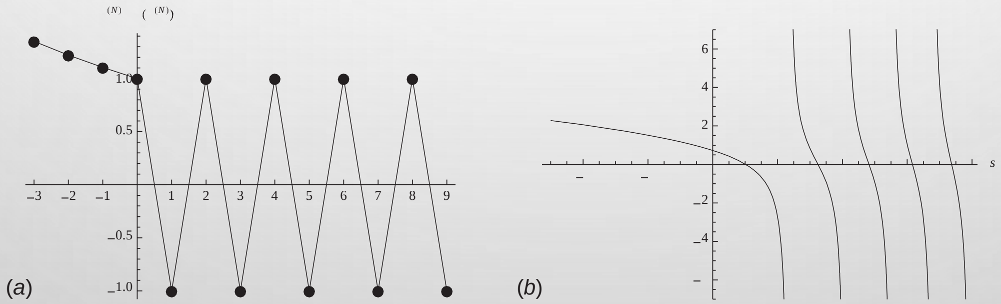
We can use ultradiscretization for q-Airy functions and obtain u-Airy functions.

Special solution to u-Pinlevé equation is obtained in a similar manner. [Isojima-Satsuma-T 2012]

ULTRADISCRETE AIRY FUNCTIONS AND SOLUTIONS TO ULTRADISCRETE PAINLEVÉ 2 EQUATION



Ultradiscrete Airy functions and q-Airy functions



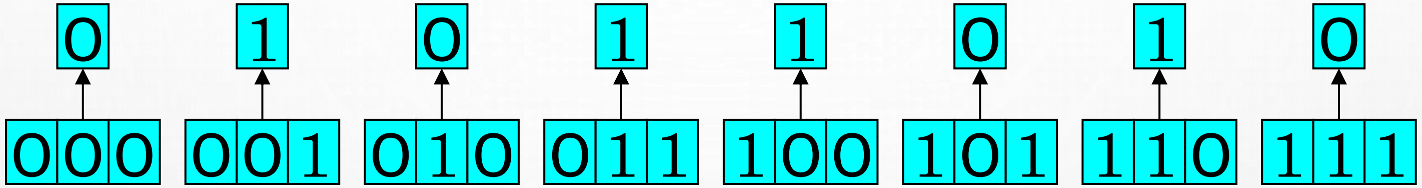
(a) a solution of u-PII eq. and (b) corresponding solution of PII eq.

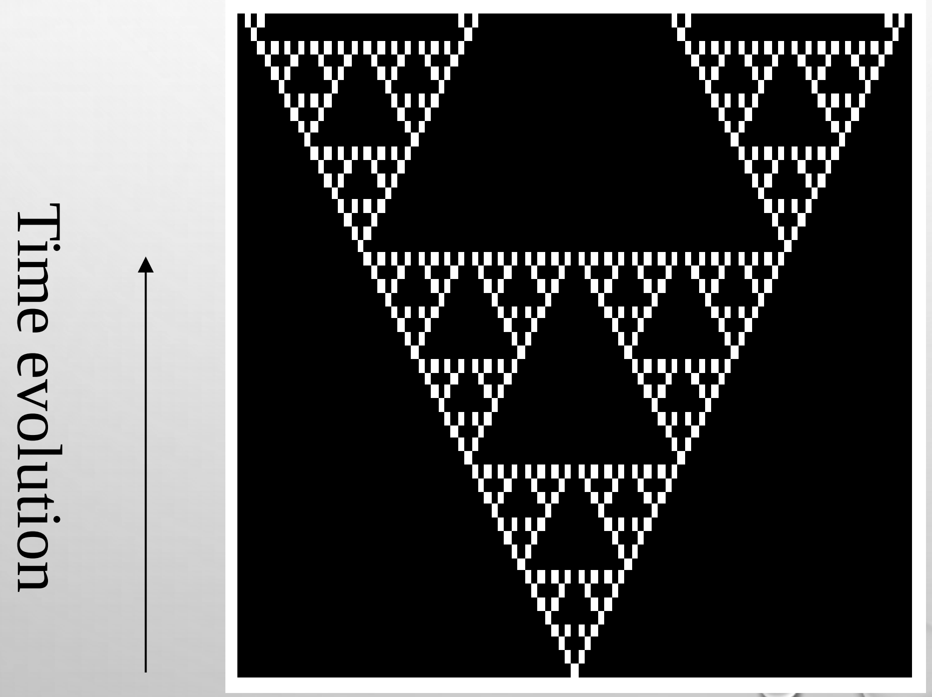
INVERSE ULTRADISCRETIZATION AND ITS APPLICATION TO BZ REACTION

WITH HIROSHI TANAKA[A], AKINOBU NISHIYAMA[B]

SHIMANE UNIVERSITY[A], UNIVERSITY OF TOKYO[B]

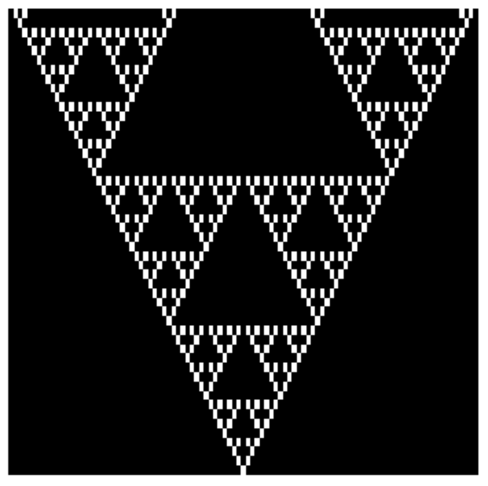
ELEMENTARY CA（RULE90）



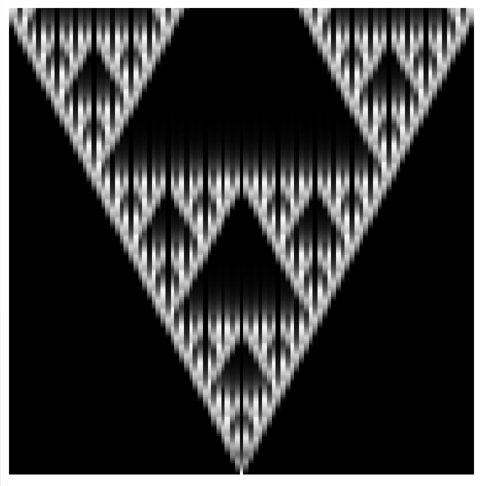


INVERSE ULTRADISCRETIZATION

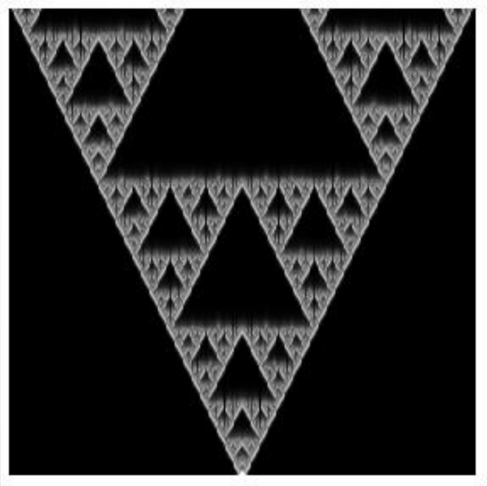
time continuous space continuous



CA

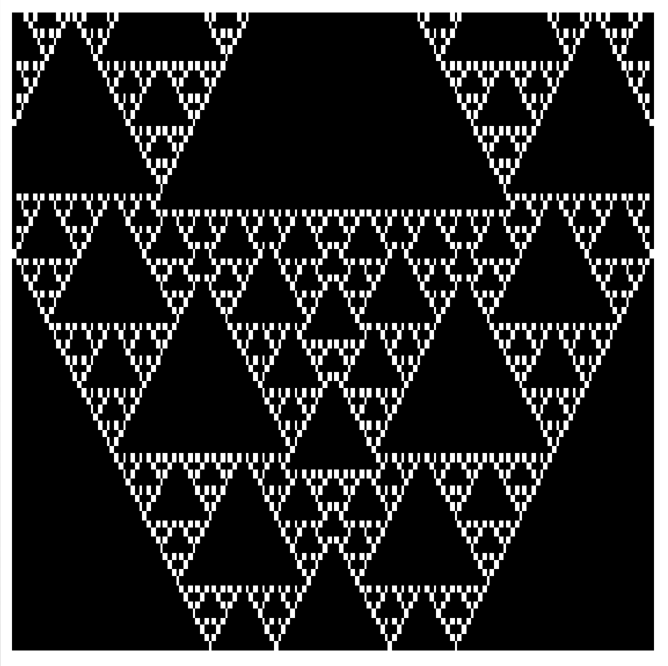


Semi-discrete eq.



PDE

ANOTHER INITIAL CONDITION



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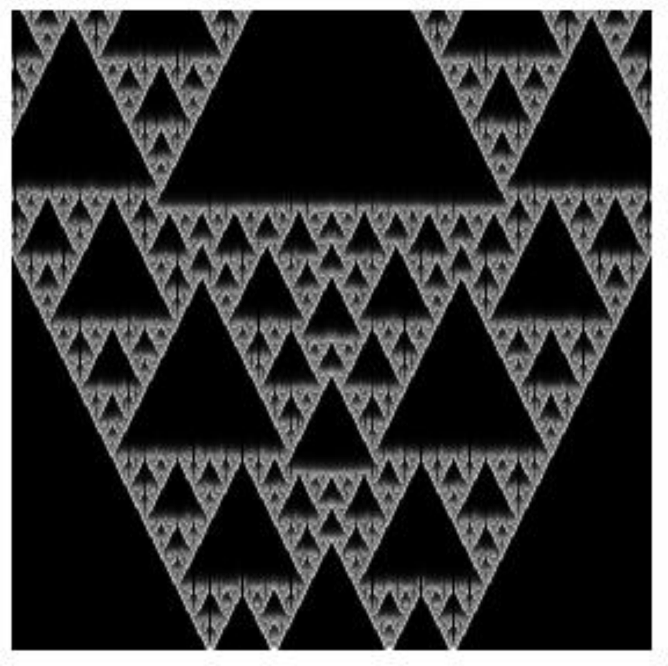
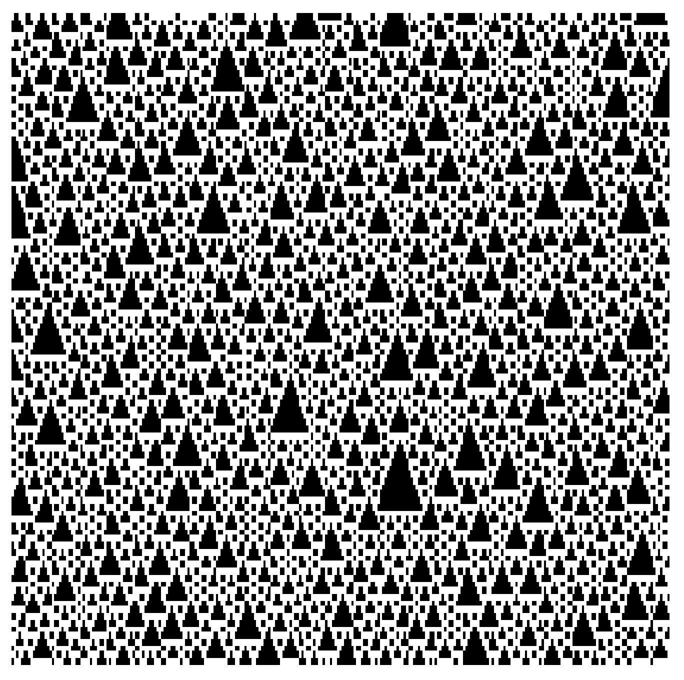
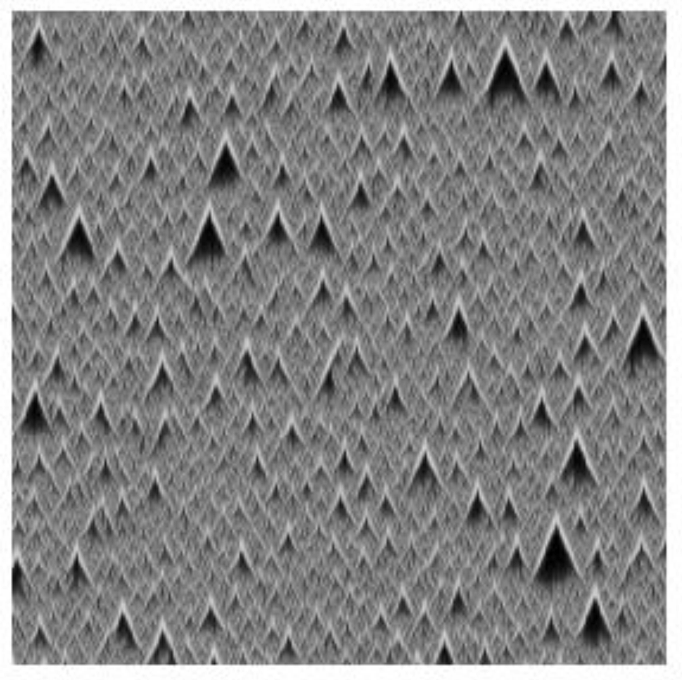


PDE

RULE122



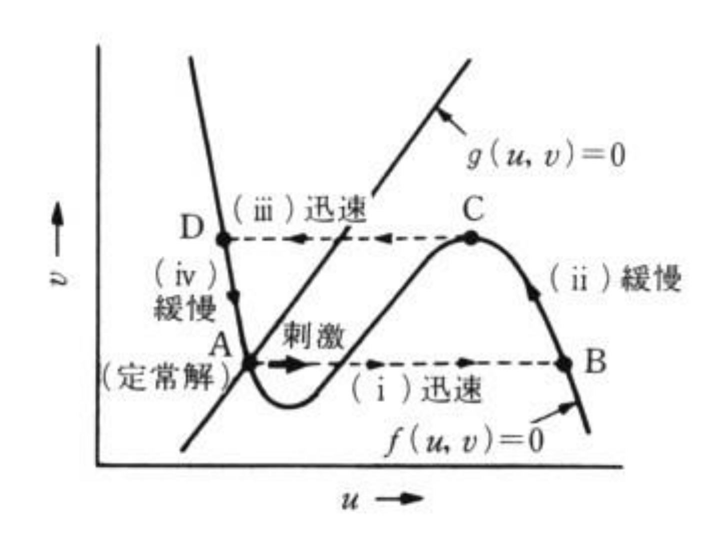
CA



PDE

REACTION-DIFFUSION EQUATION FOR BZ-REACTION

$$\begin{cases} \dfrac{\partial u}{\partial t} = f(u,v) + D_1 \nabla u \\ \dfrac{\partial v}{\partial t} = g(u,v) + D_1 \nabla v \end{cases}$$



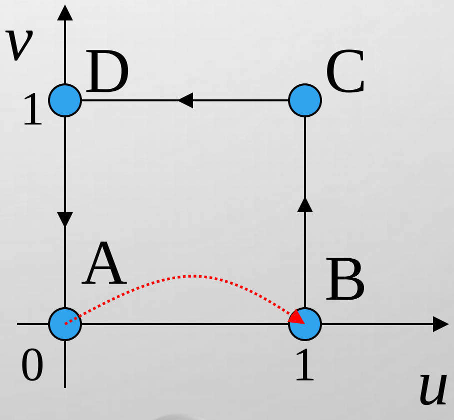
Kato et al. “cellular automaton method”
Kitamori co. ltd. (1995)

CA MODEL FOR BZ-REACTION

$$x = u^t_{m-1,n-1} + \cdots + u^t_{m+1,n+1}$$

$(u^t_{m,n}, v^t_{m,n})$	$(u^{t+1}_{m,n}, v^{t+1}_{m,n})$
A:(0, 0)	B:(1, 0) $(x \geq 2)$ A:(0, 0) $(x < 2)$
B:(1, 0)	C:(1, 1)
C:(1, 1)	D:(0, 1)
D:(0, 1)	A:(0, 0)

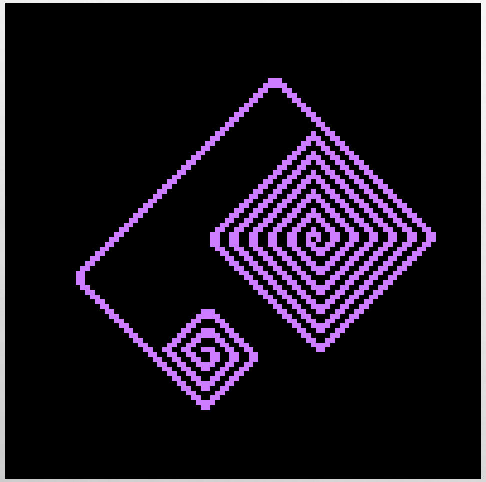
$u^t_{m-1,n-1}$	$u^t_{m-1,n}$	$u^t_{m-1,n+1}$
$u^t_{m,n-1}$	$u^t_{m,n}$	$u^t_{m,n+1}$
$u^t_{m+1,n-1}$	$u^t_{m+1,n}$	$u^t_{m+1,n+1}$



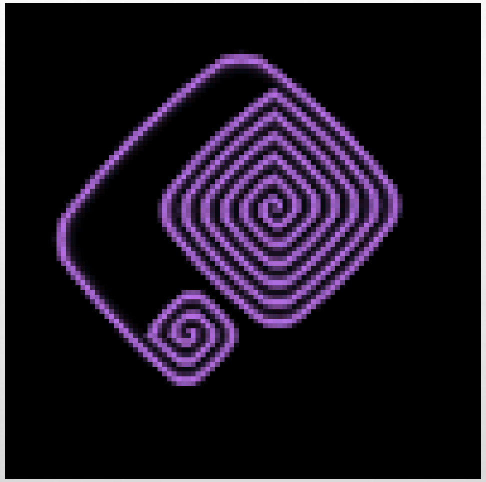
N.Doba, Master thesis, Univ. of Tokyo (2001)

EXAMPLE

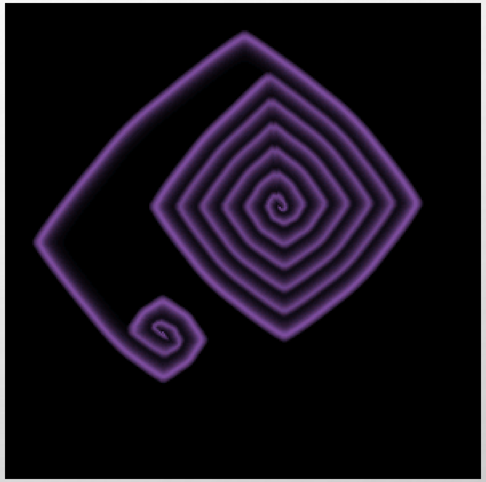
CA



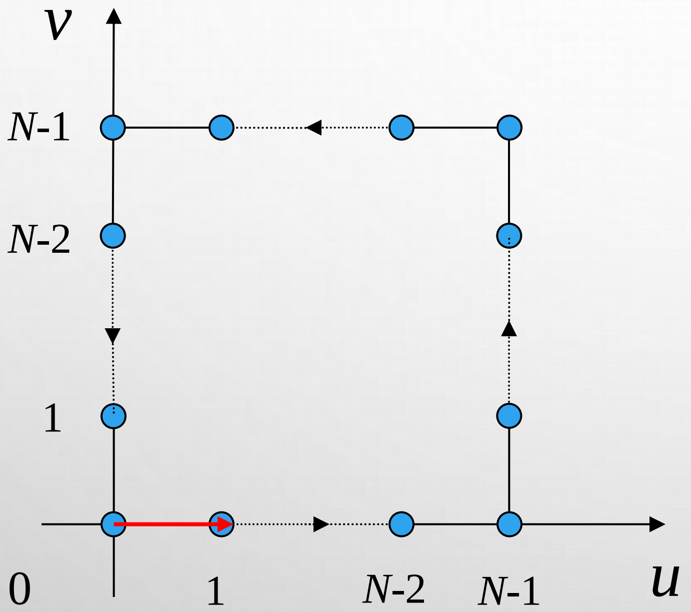
Semi-discrete eq.

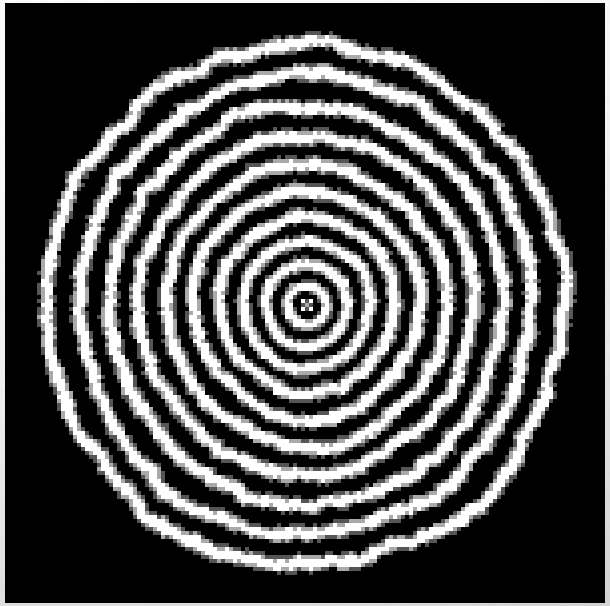


PDE

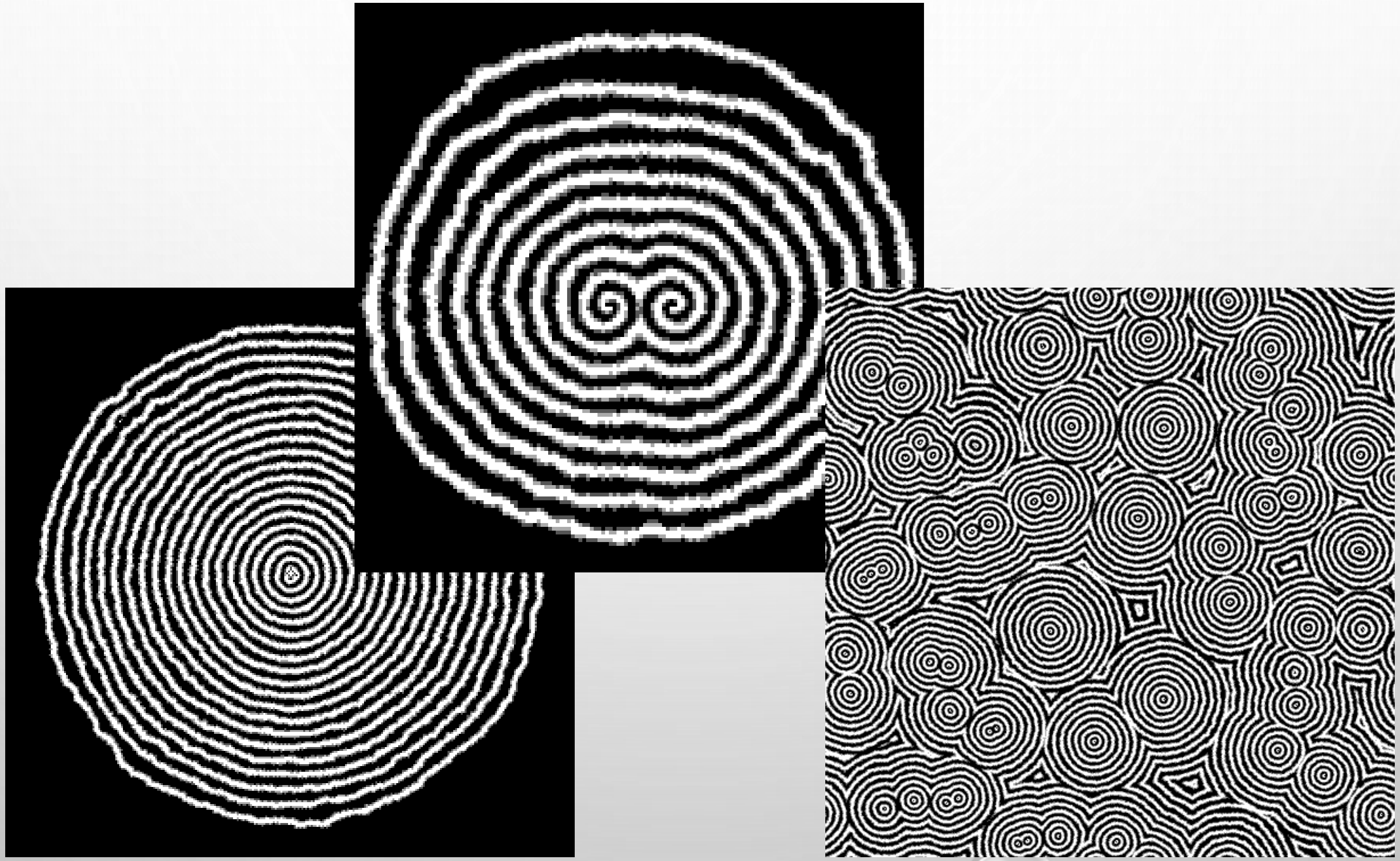


INTRODUCING PROVABILITY



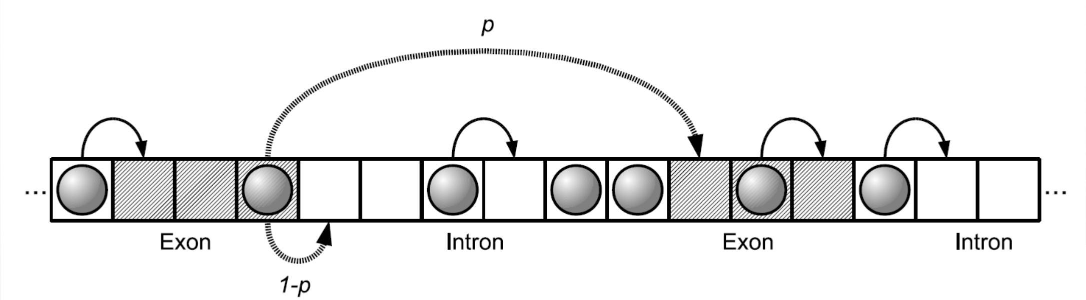


ISOTROPIC CA PATTERN FOR BZ-REACTION

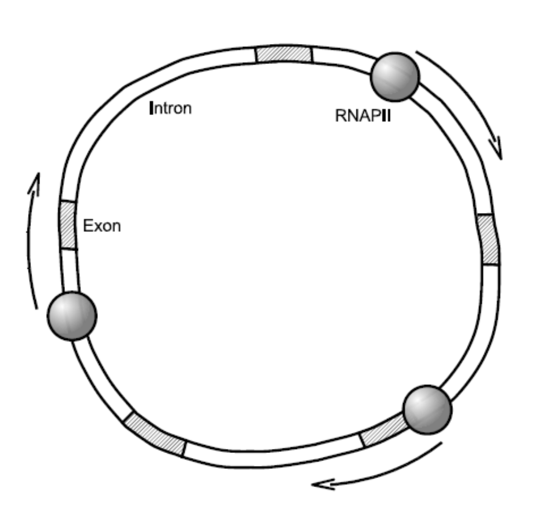


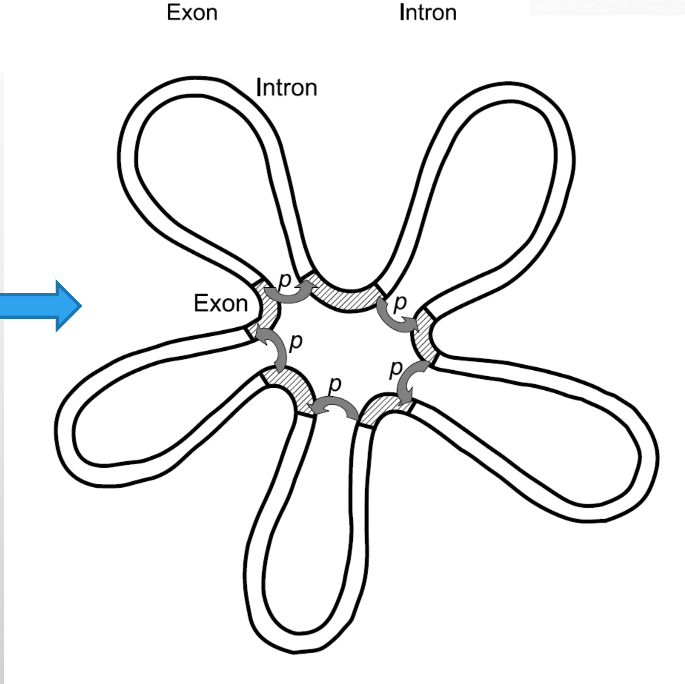
A CA model for RNA transcription

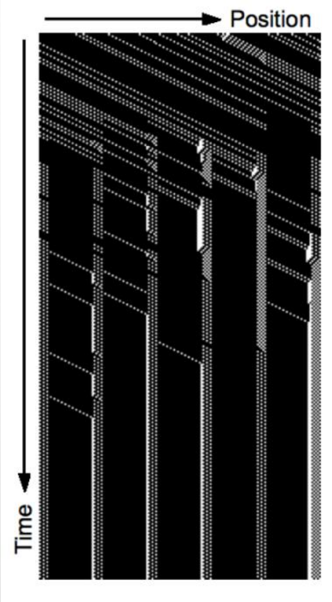
Nishiyama, Ohta, Tokihiro, Tsuboi and Ihara



・Application of the CA model for traffic flow. We extend the ASEP (Asymmetric simple exclusion process).







With the movement of RNAPII molecules, the DNA chain is transformed so that the spatial distance between the exons becomes shorter. This is an CA model for traffic flow with short-cut.

CF.) OUTLINE OF THE PROOF OF THEOREM 3

- MONGOLDT'S FORMULAE FOR CHEBYSHEV FUNCTION:

$$\psi(x) := \log\left(\mathrm{L.C.M.}(2,3,4,5,\ldots,x) \right)$$

$$= x - \sum_{\rho:\text{zeros of } \zeta(s)} \frac{x^{\rho}}{\rho} - \log 2\pi$$

$$\psi(x) = x + O(x^{1/2}\log^2 x)$$

- RIEMANN HYPOTHESIS $\leftrightarrow$

$$\left|\log T_{qt}(N) - 2\psi(\sqrt{N})\right| \le \log N$$

- THEOREM1 $\rightarrow$

PBBS AND A SOLVABLE LATTICE MODEL

- A GENERALIZED 6 VERTEX MODEL

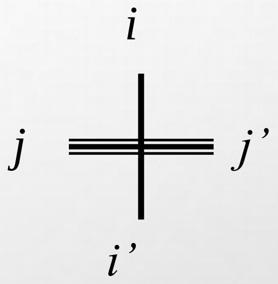
Boltzmann weight: $R^{i,j}_{j',i'} := e^{-\varepsilon^{i,j}_{j',i'}/k_B T}$

local energy of a vertex

$R^{i,j}_{j',i'} = R^{i,j}_{j',i'}(x;q)$

x : spectral parameter

q : deformation parameter (~ temperature)



vertical link…2 states

horizontal link …$l+1$ states

explicit form)

$$R^{1,k}_{k,1}(x;q) = \frac{q^{l-k}x - q^{k+1}x^{-1}}{x - q^{l+1}x^{-1}}, \quad R^{1,k-1}_{k,0}(x;q) = \frac{\sqrt{\left(1-q^{2k}\right)\left(1-q^{2(l-k+1)}\right)}}{x - q^{l+1}x^{-1}}, \quad \text{etc.}$$

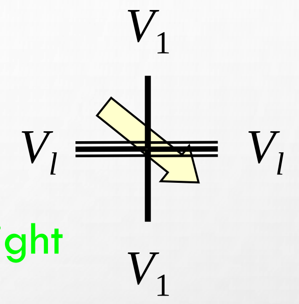
R MATRIX AND TRANSFER MATRIX

R matrix; $R_l[x,q]$: linear map $V_1 \otimes V_l \longrightarrow V_l \otimes V_1$

$V_1 = \text{span}\{e_0, e_1\} \cong \mathbb{C}^2, \; V_l = \text{span}\{f_0, f_1, \cdots, f_l\} \cong \mathbb{C}^{l+1}$

$$R_l[x;q](e_i \otimes f_j) := \sum_{i'\in\{0,1\}} \sum_{j'\in\{0,1,\ldots,l\}} R^{i,j}_{j',i'}(x;q)\,(f_{j'} \otimes e_{i'})$$

Boltzmann weight



NOTE: R-matrices satisfy Yang-Baxter relation.

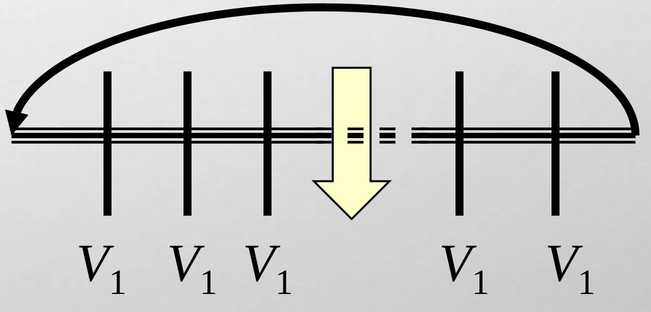
transfer matrix; $\hat{t}_l[x;q]$: linear map $V_1^{\otimes N} \longrightarrow V_1^{\otimes N}$

$$t_l[x;q](e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_N})$$

$$= \sum_{i'_1,\ldots,i'_N \in\{0,1\}} t^{i'_1 i'_2 \cdots i'_N}_{i_1 i_2 \cdots i_N}(x;q)(e_{i'_1} \otimes e_{i'_2} \otimes \cdots \otimes e_{i'_N})$$

$$t^{i'_1 i'_2 \cdots i'_N}_{i_1 i_2 \cdots i_N}(x;q) :=$$

$$\sum_{j_1,\ldots,j_N \in\{0,1,\ldots,l\}} R^{i_1,j_1}_{j_2,i'_1}(x;q) R^{i_2,j_2}_{j_3,i'_2}(x;q) \cdots R^{i_N,j_N}_{j_1,i'_N}(x;q)$$



PROPOSITION

If we identify $i_1 i_2 \cdots i_N \in \Omega_{M;N}$ with $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_N} \in V_1^{\otimes N}$,

$\hat{t}_l := \lim_{x\to 1} \lim_{q\to 0} \hat{t}_l[x;q]$ determines the map : $\hat{t}_l : \Omega_{M;N} \to \Omega_{M;N}$,

which coincides with the time evolution of PBBS for $l \geq M$.

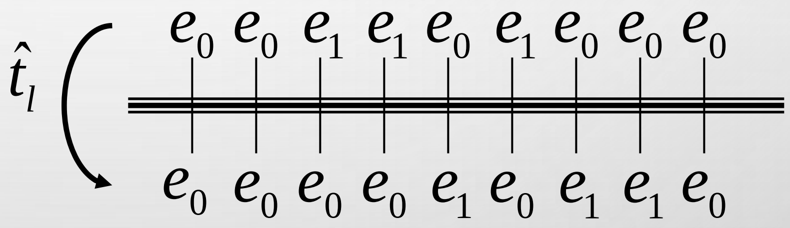
rem.) $\Omega_{M;N} = \{0,1 \text{sequence with } M \text{ 1s and } (N\text{-}M) \text{ 0s}\}$

PBBS

0 0 1 1 0 1 0 0 0

0 0 0 0 1 0 1 1 0

generalized 6 vertex model



Evolution pattern ↔ ground state

initial state ↔ boundary condition

NOTE: we can define many integrable CAs from other vertex models.

RELATION BETWEEN THE EIGENVALUES AND EIGENVECTORS OF THE TRANSFER MATRIX AND ORBITS OF THE PBBS

- $\hat{t}_l$ decomposes $\Omega_{M;N}$ into periodic orbits: $\Omega_{M;N} = \coprod \Omega^{(\nu)}$

Proposition

$\hat{t}_l$ is diagonalized on each trajectory $\Omega^{(\nu)}$ with

eigenvalues: $\Lambda_k := \exp\left[2\pi\sqrt{-1}\,\dfrac{k}{T^{(\nu)}}\right] \quad (k = 0, 1, 2, \cdots, T^{(\nu)} - 1)$

Here $T^{(\nu)} = |\Omega^{(\nu)}|$ is the fundamental cycle for $\Omega^{(\nu)}$.

$\because$) For $\Omega^{(\nu)} = \{\psi_1, \psi_2, \cdots, \psi_{T^{(\nu)}}\}$, let $\hat{t}\psi_n = \psi_{n+1}$ $(\psi_{T^{(\nu)}+1} \equiv \psi_1)$

then $\hat{t}\phi_k = \Lambda_k \phi_k \quad (k = 1, 2, \cdots, T^{(\nu)})$,

where $\Lambda_k = \mathrm{e}^{2\pi\sqrt{-1}k/T^{(\nu)}}$ and $\phi_k = \sum_{n=1}^{T^{(\nu)}} \mathrm{e}^{-2\pi\sqrt{-1}kn/T^{(\nu)}} \psi_n$

→ Eigenvalues determine fundamental cycles.

BETHE ANSATZ EQUATION AND STRING HYPOTHESIS

BETHE ANSATZ:

Let $\left|n_1, n_2, \cdots, n_M\right\rangle := e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_N} \in V_{M;N} \subset V_1^{\otimes N}$

where $i_k = 1$ if $k \in \{n_1, n_2, \cdots, n_M\}$, and $i_k = 0$ otherwise.

AN EIGENVECTOR OF $\hat{t}_l[x;q]$ IS GIVEN BY A SUPERPOSITION OF EXPONENTIALS:

$$\left|\varphi\right\rangle = \sum_{1 \le n_1 < n_2 < \cdots < n_M \le N} a(n_1, n_2, \cdots, n_M) \left|n_1, n_2, \cdots, n_M\right\rangle$$

where $a(n_1, n_2, \cdots, n_M) = \sum_{\sigma} A_\sigma \exp\left[\sum_{k=1}^{M} \sqrt{-1}\, p_{\sigma(k)} n_k\right]$

$(p_1, p_2, \cdots, p_M) \in \mathbb{C}^M$, σ : permutation of $(1, 2, \cdots, M)$

Proposition (Bethe)

$|\varphi\rangle$ is an eigenvector if $(p_1, p_2, \cdots, p_M)$ satisfies :

Bethe ansatz eq.

$$\left(\frac{q^{-1}x_k - q x_k^{-1}}{x_k - x_k^{-1}} \right)^N = \prod_{\substack{j=1 \\ j \neq k}}^{M} \frac{q^{-1}x_k x_j^{-1} - q x_k^{-1} x_j}{q\ x_k x_j^{-1} - q^{-1} x_k^{-1} x_j} \qquad (k = 1, 2, \cdots, M)$$

where $x_k^2 := \dfrac{e^{\sqrt{-1}p_k} - q}{e^{\sqrt{-1}p_k} - q^{-1}}$ and $\hat{t}_l[x; q]|\varphi\rangle = \Lambda[x; \{x_k\}; q]|\varphi\rangle.$

$$\Lambda[x; \{x_k\}; q] = \sum_{m=1}^{l} \left(\frac{q^m x - q^{l-m+1} x^{-1}}{x - q^{l+1} x^{-1}} \right)^N \prod_{j=1}^{M} r_m[x; x_j; q]$$

some rational function

STRING HYPOTHESIS

- LET Y BE A YOUNG DIAGRAM REPRESENTING A PARTITION OF M:

$$Y \cong (\underbrace{m_1, m_1, \cdots, m_1}_{K_1}, \underbrace{m_2, m_2, \cdots, m_2}_{K_2}, \cdots, \underbrace{m_s, m_s, \cdots, m_s}_{K_s})$$

$$\left(m_1 < m_2 < \cdots < m_s, \quad K_i > 0 \;\; (i = 1, 2, \cdots, s), \;\; M = \sum_{i=1}^{s} K_i m_i \right)$$

then, any solution to Bathe ansatz eq. is expressed as

$$\{ x_k \}_{k=1}^{M} = \{ x_{i\alpha\beta} \}_{i=1, \alpha=1, \beta=1}^{s \quad K_i \quad m_i}$$

$$x_{i\alpha\beta}{}^2 = q^{m_i - 2\beta + 2}\left(z_{i\alpha}^0 + O(q) \right) \quad \left(1 \le i \le s, 1 \le \alpha \le K_i, 1 \le \beta \le m_i \right)$$

2 KINDS OF SUBSPACES DETERMINED BY A YOUNG DIAGRAM Y

Let $V_{M;N} := \text{span}\{ |n_1, n_2, \cdots, n_M\rangle \in V_1^{\otimes N} \}$.

We have two decompositions:

$$V_{M;N} \supseteq \bigoplus_Y V_{M;N}^Y \quad \text{and} \quad V_{M;N} = \bigoplus_Y \text{span}\,\Omega_{M;N}^Y$$

$V_{M;N}^Y$: subspace of $V_{M;N}$ determined from a Young diagram Y using string hypothesis

$\text{span}\,\Omega_{M;N}^Y$: subspace of $\text{span}\,\Omega_{M;N} (\cong V_M)$, where $\Omega_{M;N}^Y$ is a set of states of PBBS with conserved quantities Y

$$V_{M;N}^Y := \lim_{q\to 0} V_{M;N}^Y(q)$$

$$V_{M;N}^Y(q) := \text{span}\left\{ \left|\varphi_Y^{(\mu)}(q)\right\rangle \in V_{M;N} \;\middle|\; \hat{t}_l[x;q]\left|\varphi_Y^{(\mu)}(q)\right\rangle = \Lambda_Y^{(\mu)}[l;x;q]\left|\varphi_Y^{(\mu)}(q)\right\rangle \right\}$$

$\Lambda_Y^{(\mu)}[l;x;q]$: eigenvalue of $\hat{t}_l[x;q]$ determined from Y

THEOREM

① If the solution determined by string hypothesis is a solution of BAE,

$$V_{M;N}^{Y} \subseteq \mathrm{span}\,\Omega_{M;N}^{Y}$$

② Furthermore if all the solutions of BAE are obtained from string hypothesis,

$$V_{M;N}^{Y} = \mathrm{span}\,\Omega_{M;N}^{Y}$$

where

$$\Omega_{M;N}^{Y} := \left\{ \text{states of BBS with } Y \left(\in \Omega_{M;N} \right) \right\}$$

NOTE) When an eigenvector $\left| \varphi_Y^{(\mu)} \right\rangle \in V_M^Y$ is expanded as

$$\left| \varphi_Y^{(\mu)} \right\rangle = \sum_{\vec{i} \equiv (i_1, i_2, \cdots, i_N)} C_{\vec{i}} \left(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_N} \right)$$

then, $C_{\vec{i}} \neq 0 \Rightarrow i_1 i_2 \cdots i_N \in \Omega_{M;N}$ is characterized by Y

COROLLARY

$$\forall \mu, \quad \left(\Lambda_Y^{(\mu)}\right)^{T_Y} = 1$$

(T_Y is the fundamental cycle of a state with Y)

Summary

The eigenvalues and eigenvectors of $\hat{t}$ determined by a partition Y using string hypothesis correspond to the orbits of PBBS with the conserved quantities represented by the same partition Y .

GEOMETRICAL ASPECTS OF PBBS

PERIODIC DISCRETE TODA EQUATION,

ASSOCIATED HYPERELLIPTIC CURVE,

CONSERVED QUANTITIES,

AND INITIAL VALUE PROBLEM

PERIODIC DISCRETE TODA EQUATION (PD-TODA EQ.) LAX FORM, CONSERVED QUANTITIES

pd-Toda eq.

$$\begin{cases} I_n^{t+1} = I_n^t + V_n^t - V_{n-1}^{t+1} \\ V_n^{t+1} = \dfrac{I_{n+1}^t V_n^t}{I_n^{t+1}} \end{cases}$$

$$(V_{N+i}^t = V_i^t,\ I_{N+i}^t = I_i^t)$$

$\Rightarrow$

Lax-form

$$M_{t+1}(y)R_{t+1}(y) = R_t(y)M_t(y) \ \text{ or } \ L_{t+1}(y) = R_t(y)L_t(y)R_t^{-1}(y)$$

$$M_t(y) := \begin{pmatrix} 1 & & & y^{-1}V_N^t \\ V_1^t & 1 & & \\ & \ddots & \ddots & \\ & & V_{N-1}^t & 1 \end{pmatrix}, \quad R_t(y) := \begin{pmatrix} I_1^t & 1 & & \\ & I_2^t & \ddots & \\ & & \ddots & 1 \\ y & & & I_N^t \end{pmatrix},$$

$$L_t(y) := M_t(y)R_t(y)$$

Conserved quantity:

$$\Phi(x,y) := y\det\left(xE - L_t(y)\right) = y^2 + \Delta(x)y + m^2$$

$$\Delta(x) := x^N + c_{N-1}x^{N-1} + \cdots + c_1 x + c_0$$

$$m^2 := \prod_{i=1}^N V_i^t I_i^t = \prod_{i=1}^N V_i^0 I_i^0$$

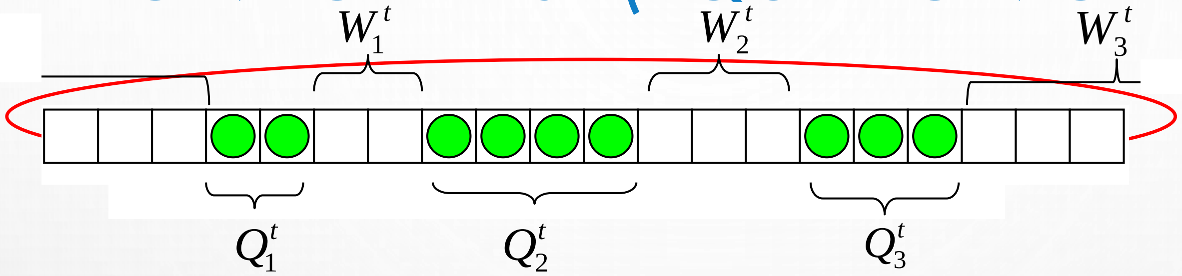
$\Rightarrow$

$\Phi(x,y) = 0 \cdots$ hyperelliptic curve $(g = N\text{-}1)$,

ramification points: $x = \lambda_0^\pm, \lambda_1^\pm, \cdots, \lambda_g^\pm$

(roots of $\Delta(x)^2 - 4m^2 = 0$)

RELATION TO PBBS (EQUATION OF MOTION)



pd-Toda eq. (Ⅱ)

$$\begin{cases} I_n^{t+1} = V_n^t + Y_n^t \\ V_n^{t+1} = \dfrac{I_{n+1}^t V_n^t}{I_n^{t+1}} \end{cases} \qquad \begin{pmatrix} V_i^t = V_{i+N}^t \\ I_i^t = I_{i+N}^t \end{pmatrix}$$

$$Y_n^t := \frac{(1 - Z_{n;N}^t) I_n^t}{1 + \Sigma_{i=1}^{N-1} Z_{n;i}^{t+1}}$$

$$Z_{n;i}^t := \frac{\Pi_{k=1}^i V_{i-k}^t}{\Pi_{k=1}^i I_{i-k}^t}$$

ud-Toda eq. = PBBS eq.

$$\begin{cases} Q_n^{t+1} = \min\left[W_n^t, X_n^t + Q_n^t\right] \\ W_n^{t+1} = Q_{n+1}^t + W_n^t - Q_n^{t+1} \end{cases} \qquad \begin{pmatrix} Q_i^t = Q_{i+N}^t \\ W_i^t = W_{i+N}^t \end{pmatrix}$$

$$X_n^t := \max_{0 \le k \le N-1} \left[\sum_{i=1}^{k} (Q_{n-i}^t - W_{n-i}^t) \right]$$

$\varepsilon \downarrow 0$

$$I_n^t(\varepsilon) = \exp\left[-Q_n^t/\varepsilon\right], \; V_n^t(\varepsilon) = \exp\left[-W_n^t/\varepsilon\right]$$

ultradiscretization

RELATION TO PBBS (CONSERVED QUANTITIES)

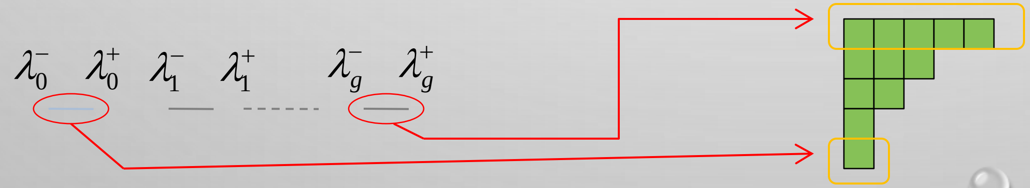
Rem) Conserved quantities are expressed by a Young diagram with N rows.

Theorem

Define $\{I_n^0(\varepsilon), V_n^0(\varepsilon)\}_{n=1}^N$ such that

$$-\lim_{\varepsilon\to+0}\varepsilon\log I_n^0(\varepsilon)=Q_n^0,\quad -\lim_{\varepsilon\to+0}\varepsilon\log V_n^0(\varepsilon)=W_n^0,$$

then $\Lambda_n := -\lim_{\varepsilon\to+0}\varepsilon\log\lambda_n^{\pm}(\varepsilon)$ is the length of the $(N\text{-}n)$th row of the Young diagram associated with the state of PBBS with $\{Q_n^0, W_n^0\}_{n=1}^N$.



INITIAL VALUE PROBLEM OF PD-TODA EQ. AND PBBS

FACT (van Moerbeke-Mumford [1979], Kimijima-T [2002]):

P(d)-Toda eq. is linearised on Jacobian variety associated with the hyperelliptic curve $C : \Phi(x,y)=0$

$$\left(\{I_i^t\}_{i=1}^{N(\equiv g+1)}, \{V_i^t\}_{i=1}^{N}\right) \xrightarrow{\text{Lax form}} \left(\mathrm{P}_1^t, \mathrm{P}_2^t, \cdots, \mathrm{P}_g^t\right) \in C \xrightarrow{\text{Abel map}} \vec{v}t + \vec{v}_0 \in J_C \cong \mathbb{C}^g/\mathbb{Z}^g + B\mathbb{Z}^g$$

Lax form:

$$(xE - L_t(y))\Psi(x,y) = 0,$$
$$\Psi(x,y) = {}^t(\psi_1, \psi_2, \cdots, \psi_N)$$
$$(\mathrm{P}_1, \mathrm{P}_2, \cdots, \mathrm{P}_g) : \psi_N(x,y) = 0$$

Abel map:

$$\int_{\mathrm{P}_0}^{\mathrm{P}} \vec{\omega}$$

Jacobian

A solution of the initial value problem of PBBS (Kimijima-T[2002],Iwao-T[2007])

$$\left(\{I_n^0\}, \{V_n^0\}\right) \longrightarrow \left(\{I_n^t\}, \{V_n^t\}\right)$$

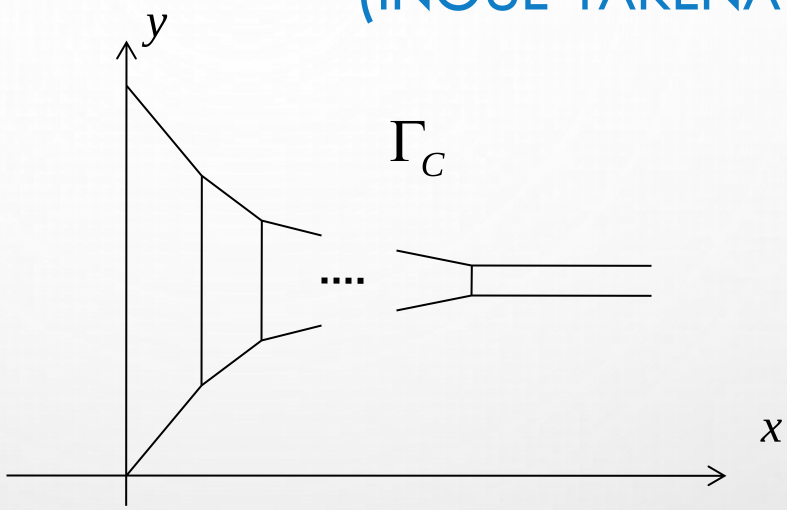
Inverse UD: $\left(\{Q_n^0\}, \{W_n^0\}\right) \to \left(\{I_n^0\}, \{V_n^0\}\right)$

UD: $\left(\{I_n^t\}, \{V_n^t\}\right) \to \left(\{Q_n^t\}, \{W_n^t\}\right)$

CF) DIRECT SOLUTION WITH TROPICAL SPECTRAL CURVE (INOUE-TAKENAWA [2007])

Tropical hyperelliptic curve

(Mikhalkin-Zharkov[2006])



$$\left(\{Q_i^t\}_{i=1}^N \{W_i^t\}_{i=1}^N\right) \xrightarrow{\text{tropical Lax form}} \left(P_1^t, P_2^t, \cdots, P_g^t\right) \in \Gamma_C \xrightarrow{\text{tropical Abel map}} \vec{v}t + \vec{v}_0 \in J_\Gamma \cong \mathbb{R}^g / K\mathbb{Z}^g$$

tropical Jacobian

Inoue-Takenawa proved the above construction is valid up to $g=3$.