DISCRETE INTEGRABILITY BLENDING NEVANLINNA THEORY AND SINGULARITY CONFINEMENT

Thamizharasi Tamizhmani
principal

Tagore Arts college
Govt.of Pondicherry ,India

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INTEGRABILITY

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An ODE is said to be integrable if there exists a **well-defined solution about all movable singularities**

- No unified classical definition-only working definition
- Integrability- more constrained (richer) than solvability

Example

$$(1 + w^2)w'' + (1 - 2w)(w')^2 = 0, w = tan[\alpha + ln(z - z_0)]$$

- Integrability of an ODE-related to the singularity structure of its solutions in the complex plane
- Linear ODE's are integrable-singularities are fixed-suitable redefinition of solution is possible

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- Nonlinear ODE-singularities are movable- the problem is movable singularity
- The most successful integrability detector for nonlinear ODE's is Painlevé property -successfully extended to PDE's also

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- In 1884, Fuchs showed that if the first order equation

$$\frac{dw}{dz} = F(z, w)$$

where F is rational in 'w' and analytic in z,does not contain any movable critical points then, it should be of the form

- $$\frac{dw}{dz} = F(z, w) = a(z) + b(z)w + c(z)w^2$$

for some analytic functions a,b and c. This is called a generalised Riccati equation.

- Hence Riccati equation is the only first order ODE possessing Painlevé property.

- Painlevé and Gambier solved the problem of classifying all the second order ordinary differential equations of the form

$$y'' = F(y', y, z) \tag{1}$$

where F is rational in y, polynomial in y', and locally analytic in z

- They identified only 50 second order non-linear equation possessing Painlevé property. Among the above 50-only six are new- introducing new special functions -Painlevé transcendents

DISCRETE INTEGRABLE SYSTEMS

- Why discrete?-due to the characteristic time -many physical models are discrete
- Appearance of discrete systems is as early as1931 -identified only after the knowledge of continuous limits in 1991
- Discrete dynamical systems-more fundamental -continuous systems through continuous limits
- The explosive growth of interest in discrete integrable systems forced to look for Integrability detectors for discrete systems
- continum integrability is well established -Discrete integrability is new much to explore
- owing to the roaring success of Painlevé property-What is the analogous of Painlevé property for discrete equations- a number of methods are introduced in recent years

SINGULARITY CONFINEMENT

- singularity confinement -first discrete integrability test- Grammaticos et al is a simple elegant test

SINGULARITY CONFINEMENT CRITERION

If the dynamics leads to a singularity then after a finite number of iterations the singularity should disappear without essential loss of initial information

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- $$X_{n+1} + X_{n-1} = \frac{a}{X_n} + \frac{1}{X_n^2} \tag{2}$$
- $$X_{-1} = u, \qquad X_0 = \epsilon$$
- $$\epsilon \to 0, X_1 \Rightarrow \infty, X_2 \Rightarrow 0$$
- and
  $$X_3 \to u$$
- The singularity pattern is $\{0, \infty, 0, u\}$.

Salient features of singularity Confinement

- It is simple and elegant- practically simple to check
- Singularity confinement is used to derive discrete Painlevé equations.
- Singularity patterns are expected to be the same for autonomous and non autonomous systems. So this test is used to generate integrable non autonomous from integrable autonomous systems.

SINGULARITY CONFINEMENT AND CHAOTIC SYSTEMS

- Jarmo Hietarinta and Claude Viallet countered that singularity confinement test need not be sufficient for discrete integrability.
- consider the map ,

$$X_{n+1} + X_{n-1} = X_n + \frac{a}{X_n^2} \tag{3}$$

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- The integrability of discrete systems depends on the asymptotic structure of its solutions at ∞

Example

$u_{n+1} - u_n = 1/(n-c)^N$

$u_n = u_0 + \sum_0^{n-1} 1/(i-c)^N$

$y(z+1) - y(z) = 1/(z-c)^N$

$y(z) = (-1)^{N-1}/(N-1)!(d/dz)^N \log[\Gamma(z-c) + \pi(z)]$ where π is a periodic function but $dy/dz = 1/(z-c)^N$, logarithmic singularity forN=1

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From Continuous To Discrete

- Arbitrary periodic functions - analogous to arbitrary constants in continuous systems
- singularity structure for integrable and non integrable systems reveal the secret-integrability of discrete systems is encoded with the singularity structure of the solution at ∞
- solutions of well known discrete integrable systems are of finite order
- **counter example(Viallet)**
  $y(z+1) + y(z-1) = y(z) + \alpha/y^2(z)$ **solution exists but of infinite order**
- **solutions of finite order is a necessary condition for discrete integrability and so the study at ∞ is important**
- **Important observation- Arnold and Veselov -growth properties -essential for integrability to discrete equations**

- solutions of integrable as well as non integrable discrete systems -meromorphic functions in the complex plane Yanagihara
- including $y(z+1) - y(z-1) = \mu y(z)(1 - y(z))$ has non-trivial meromorphic solutions of finite order if $max(p, q) \leqslant 1$
- Ablowitz et al $y(z+1) - y(z-1) = R_{p,q}(z, y)$ and $y(z+1)y(z-1) = R_{p,q}(z, y)$ admit finite order meromorphic solutions if $max(p, q) \leqslant 2$
- including $y(z+1) - y(z-1) = ((\alpha z + \beta)y(z) + \gamma)/(1 - y^2(z))$
- singularity structure for integrable and non integrable systems reveal the secret-integrability of discrete systems is encoded with the singularity structure of the solution at ∞
- Again examples -Finite order solution is necessary but not sufficient for integrability

Why Nevanlinna?

- The continuous limit for
  $y(z+1) - y(z) = hF(y(z)), y(z) = u(x), x = hz$
  $du/dx = F(u(x))$, when $F(u) = u^3$ the general solution is branched -when x is finite,as h $\to 0$,$z \to \infty$
- Behaviour of solutions at a finite point in the differential equation is reflected at infinity for the corresponding solution of the discrete system
  For discrete system the study of the behaviour of its solution at infinity is essential-the only way for this is the **Nevanlinna Theory**
- **From differential to difference two major observations are made-For integrability of discrete equations solutions should be of finite order and it is only necessary but not sufficient**
- **Discrete equations can be represented as delay equations in the complex plane**

- Ablowitz and Halburd introduced a novel,complex analysis based - discrete integrability detector
- difference equations - delay equations-used complex analysis tools for the study
- AHH hypothesis-Infinite order solution is is an indication for nonintegrability indicating finite order solution is a necessary condition for integrability-but not sufficient
- what could be complemented with?
- Absence of digamma functions in the series expansion of the solution -according to Ablowitz et al but
- Check for the absense of digamma function in the series expansion of solution is practically difficult - how to make sure the absence of other worse singularities
- hence we(Grammaticos et al) blended Nevanlinna theory and singularity confinement-a simple and powerful discrete integrability detector
- Aim of this lecture is to explore this discrete integrability detector

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Fundamental Theorem of Algebra

- Every nth degree polynomial has exactly 'n' roots,

$$a_0 + a_1 z + ... + a_n z^n = 0, \;\; a_n \neq 0;$$
$$a_0 + a_1 z + ... + a_n z^n = \alpha;$$

Value Distribution Theory

Every Polynomial assumes every complex value exactly the same number of times including ∞ .

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- This common number - Degree of the polynomial - order at which $M(r, P)$ grows,

$$M(r, P) = \sup_{|z|=r} |P(z)|$$

Essence

"The Value Distribution of a polynomial is the degree of the polynomial - is also the degree of growth of Max function - is the rate at which the number of zeros of the polynomial in a disc, $|z| < r$ grows as $r \longrightarrow \infty$"

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Evolution of Functions

- Polynomials
- Entire Functions
- Meromorphic Functions

What is the Value Distribution Theory for Entire Functions?

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- Entire functions
- Taylor's Theorem
- Entire functions - Sort of Polynomials "of degree ∞".
- Entire functions have exceptions e.g $e^z \neq 0$

Little Picard's Theorem - Analogous of Fundamental Theorem of Algebra

"Every entire function assumes every complex number the same number of times including ∞, infinitely many times with atmost one exception."

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Big Picard's Theorem - Analogous of Liouville's Theorem

"An Entire Function with two exceptions reduces to a constant."

Liouville's Theorem

"Every bounded Entire function is a constant."

Growth of Entire functions

Non-constant Entire Functions have unbounded growth.

How to measure the rate of growth?

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Growth of Entire Functions - Type and Order

- How to distinguish the growth of $e^z, e^{2z}, e^{3z}, e^{z^2}, e^{e^z}$
- All are Entire Functions
- All are totally different from the rest
- One has to distinguish these functions
- For this two important concepts: Exponential Type, Order are introduced

Exponential Type of Entire Function

$$T(f) = \limsup_{r \longrightarrow \infty} \frac{\log M(r,f)}{\log r}$$

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Order of an Entire Function

$$\sigma(f) = \limsup_{r \longrightarrow \infty} \frac{\log \log M(r, f)}{\log r}$$

Distinguishing growth in terms of Type and Order

	P(z)	e^z	e^{2z}	e^{3z}	e^{z^2}	e^{e^z}
Type	0	1	2	3	∞	∞
Order	0	1	1	1	2	∞

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Order of a Meromorphic function

- Before we proceed, the following points should be taken care
- Entire functions have exceptions e.g $e^z \neq 0$
- entire or meromorphic functions have only finitely many zeros inside a disc
- for meromorphic functions $M(r, f) \to \infty$ even in a finite disc
- So it is better to consider the rate at which the number of zeros in a disc of radius r grows as $r \to \infty$

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- The *proximity function* is

$$m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})|d\theta,$$

where

$$\log^+ x := \max(\log x, 0)$$

- The *enumerative function* is

$$N(r,f) := \int_0^r \frac{n(t,f)-n(0,f)}{t}dt + n(0,f)\log r$$
$$= \sum_{k=1}^{n} \log\frac{r}{|b_k|} + n(0,f)\log r,\ \ b_k \text{ are poles}$$

where $n(r,f)$ is the number of poles of f (counting multiplicities) in $|z| \le r$

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- The *Nevanlinna characteristic function,*

$$T(r,f) = m(r,f) + N(r,f)$$

measures "the affinity" of f for infinity.

Poisson-Jensen's Theorem - Analogous of Fundamental Theorem of Algebra for Meromorphic Functions

Replace f by $(f - a)$ Poisson -Jensen's theorem assures that the sum $T(r, f, a) = N(r, f, a) + m(r, f, a)$ is independent of a and hence the sum is simply denoted by $T(r, f)$.

Characteristic function

The characteristic function of a meromorphic function **f will be defined as,** $T(r,f) = \ m(r,f) + N(r,f)$

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The order $\sigma(f)$ of a meromorphic function **f is defined by,**

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PROPERTIES OF NEVANLINNA CHARACTERISTIC

- The two basic relations which reproduce the statement on the affinity of f for ∞, 0 or a are:
- $T(r; 1/f) \asymp T(r; f)$
- $T(r; f - a) \asymp T(r; f)$
- Valiron: $T\left(r; \frac{P(f)}{Q(f)}\right) \asymp \sup(p, q) T(r; f)$
  where P and Q are polynomials in f with constant coefficients, of degrees p and q respectively, provided the rational expression P/Q is irreducible

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- $T(r; fg) \preceq T(r; f) + T(r; g)$
- $T(r; f + g) \preceq T(r; f) + T(r; g)$
- $T(r; fg + gh + hf) \preceq T(r; f) + T(r; g) + T(r; h)$
- $T\Big(r; \sum_{J \subseteq I} \alpha_J (\prod_{j \in J} f_j)\Big) \preceq \sum_{i \in I} T(r; f_i)$ for constant α_J's.
- $T(r; f(z \pm 1)) \preceq (1 + \epsilon) T(r + 1; f(z))$ (Halburd et al)

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The three-tiered approach to discrete integrability

- The criterion of non-infinite order of the solution of a given difference equation can be complemented with singularity confinement so as to become a discrete integrability detector
- First step-Use Nevanlinna characteristic techniques-to estimate the rate of growth-restricting to autonomous case -reduce the complication of growth due to coefficients-requirement of slow growth-severe restriction on the difference equations
- second step- sigularity confinement-all autonomous equations that do not satisfy singularity confinement are rejected at this second step.
- The third step consists in the deautonomisation of the system-using once again the singularity confinement criterion. We obtain thus a mapping which satisfies the Nevanlinna criterion for low-growth of the solutions +confined singularities

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- The discrete equations we shall examine here are three-point mappings of the general form:

$$A(x_n, x_{n-1}, x_{n+1}) = B(x_n)$$

  A is polynomial and B is rational, with coefficients which *do not* depend on the independent variable n -consider A linear separately in $x_{n\pm1}$.
- Following AHH - delay equation in complex domain - evaluate the Nevanlinna characteristic
- $u(1+\epsilon)T(r+1;x) + vT(r;x) \succeq wT(r;x)$
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- $T(r+1;x) \succeq \frac{w-v}{u(1+\epsilon)} T(r;x)$
- Now if $w > u + v$, for r large and ϵ small $\lambda \equiv \frac{w-v}{u(1+\epsilon)}$ becomes strictly greater than unity. $T(r+1;x) \geq \lambda T(r;x) - c$ for some c independent of r . The case c negative is trivial: $T(r+k;x) \geq \lambda^k T(r;x)$.

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- d-P_{IV}

$$(x_{n+1} + x_n)(x_n + x_{n-1}) = \frac{P(x_n)}{Q(x_n)}$$

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CONCLUSION

- **Search for discrete analogue of Painlevé property**
- This is a practically simple discrete integrabilty detector in the light of Nevanlinna theory
- we beleive the requirement of slow growth is not disturbed in deautonomization
- This approach is extended to all QRT mappings which include discrete Painlevé equations
- For a large number of discrete Painlevé equations even Lax Pairs are known
- Algebraic entropy is a strong intgrability detector

- For most of the discrete Painlevé equations geometrical discriptions are given -Weyl Groups in particular the solutions of discrete Painlevé equations are constructed from the solutions of non-autonomous Hirota-Miwa equations So we beleive a much strong group theoretical based discrete integrability detector can come
- our discrete integrability again confirms that dicrete Painlevé equations are the only integrable sytems from QRT family
- concerning special limits to dP_{IV} and qP_{VI}- yet to be explored
- Problem is still open to lattice equation
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