Expected Total Number of Infections for Virus Spread on a Finite Network

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- G is nothing but the *network* created out of these.

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- After an unit time each infected agent tries to infect all its uninfected or healthy neighbors independently with probability β and then dies out or gets removed from the network.
- The process continues till all infected sites are removed.

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- In *epidemiology* such a model is called a *Susceptible Infected Removed (SIR)* model.
- This is model is related to *i.i.d Bernoulli bond percolation* model.
- In fact, if we start with only one initial infected sites, say v_0 then it is easy to see that the collection of all the infected sites is nothing but the vertices in the open connected component of v_0 in the standard i.i.d. bond percolation at parameter β.

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- Note Y^{G,V_0} is also the total number of removed/dead sites ever.

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- We will like to find answers to above without fixing any specific network structure.
- However we may make assumptions on the qualitative properties of the graph G.

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- This model was proposed by Draief, Ganesh and Massoulie [Ann. Appl. Probab. 2008] where they found an *upper bound* on $\mathbf{E}\left[Y^G\right]$.

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Theorem [Draief, Ganesh and Massoulie, 2008]

Let A be the adjacency matrix of the graph G and $\lambda(A)$ be the eigenvalue with the largest absolute value. Suppose $\beta\lambda(A) < 1$. Then

$$\mathbf{E}\left[Y^{G,V_0}\right] \leq \frac{\sqrt{n\,|V_0|}}{1-\beta\lambda(A)},$$

where n is the number of vertices. Moreover, if G is a regular graph with degree $d \geq 2$, then for $\beta < \frac{1}{d}$

$$\mathbf{E}\left[Y^G\right] \leq \frac{|V_0|}{1-\beta d}.$$

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- They prove this using very simple matrix based calculations.
- The paper included several examples where this and similar theorems were used to find the *upper bound*.

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- Moreover the proposed upper bound only works for "*small*" values of β.
- The bound involves $\lambda(A)$ which can be difficult to compute for a general graph.
- More importantly $\lambda(A)$ may depend on n the number of vertices. Thus not giving much idea about what happens for a "*large*".

Our Approach

- We will provide a simple *lower bound* to $\mathbf{E}\left[Y^{G,V_0}\right]$ which will work for every $\beta \in (0,1)$.

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- Our lower bound can be computed using easy algorithm.
- We will also prove that for a large class of graphs this lower bound is a good approximation to the exact quantity if the network size is "*large*".

Obtaining a Lower Bound

Theorem 1 [B. and Sajadi]

Let G be an arbitrary finite graph and $v_0 \in V$ be a fixed vertex of it. Let T be a spanning tree of the connected component of G containing the vertex v_0 and rooted at v_0. Let $Y^{T,\{v_0\}}$ be the total number of vertices ever infected when the epidemic runs only on T and starting with exactly one infection at v_0. Then

$$\mathbf{E}\left[Y^{T,\{v_0\}}\right] \leq \mathbf{E}\left[Y^{G,\{v_0\}}\right] \text{ for all } 0 < \beta < 1\,.$$

Moreover, if $\mathcal{T}$ is a *breadth-first search (BFS) spanning tree* of the connected component of v_0 rooted at v_0, then

$$\mathbf{E}\left[Y^{T,\{v_0\}}\right] \leq \mathbf{E}\left[Y^{\mathcal{T},\{v_0\}}\right] \leq \mathbf{E}\left[Y^{G,\{v_0\}}\right] \text{ for all } 0 < \beta < 1\,.$$

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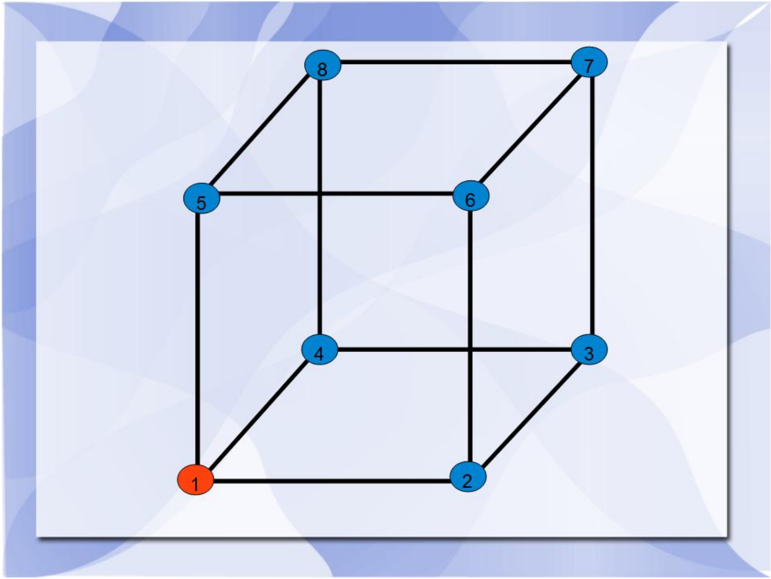
- In fact it can be obtained simply by counting the number of individuals in each generation. In other words using *branching process* (not necessarily a Galton-Watson process though).

Breadth-First-Search Algorithm (Example: Cube)



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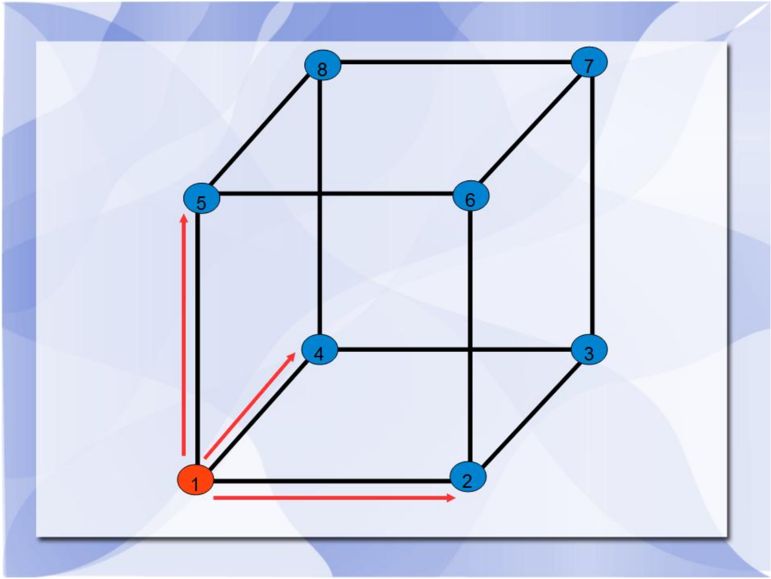

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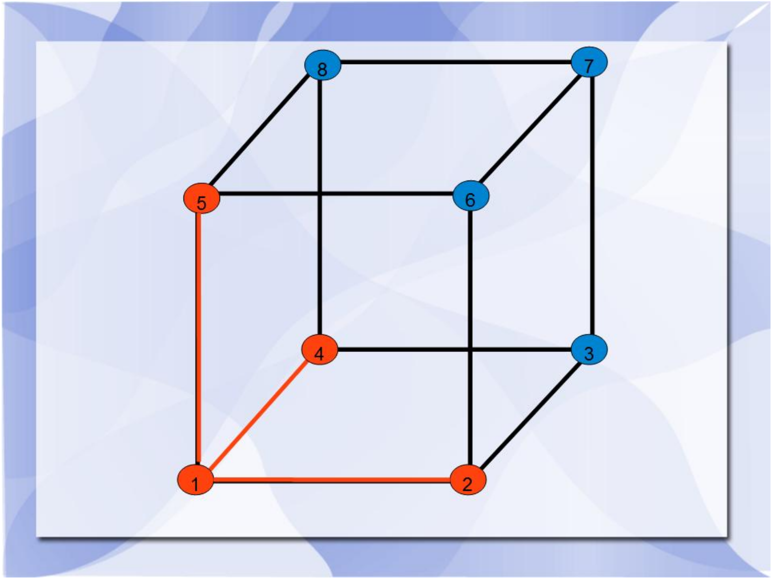

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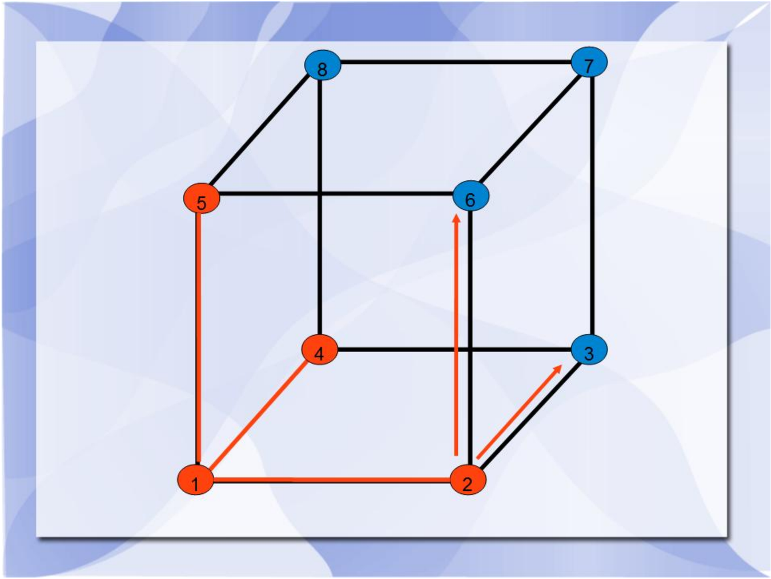

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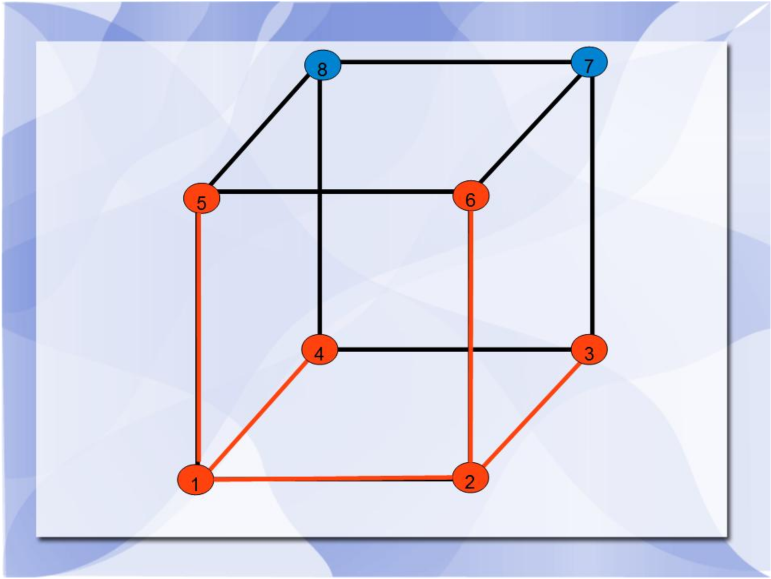

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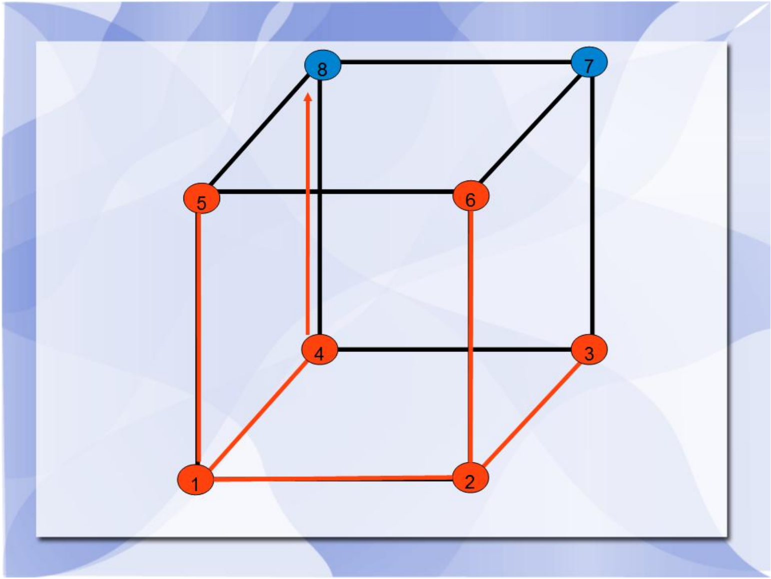

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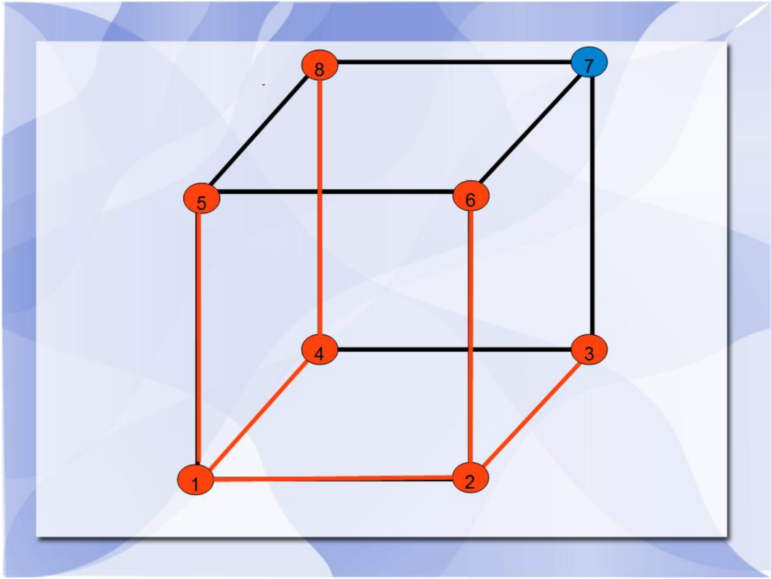

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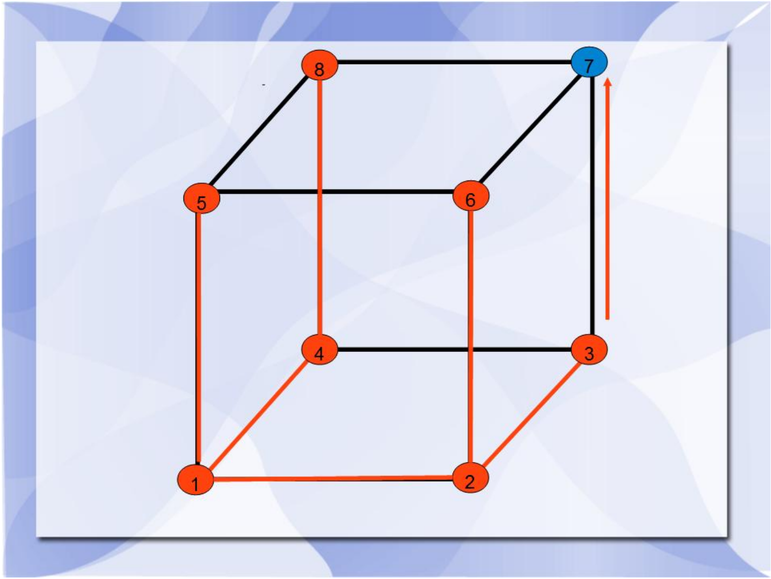

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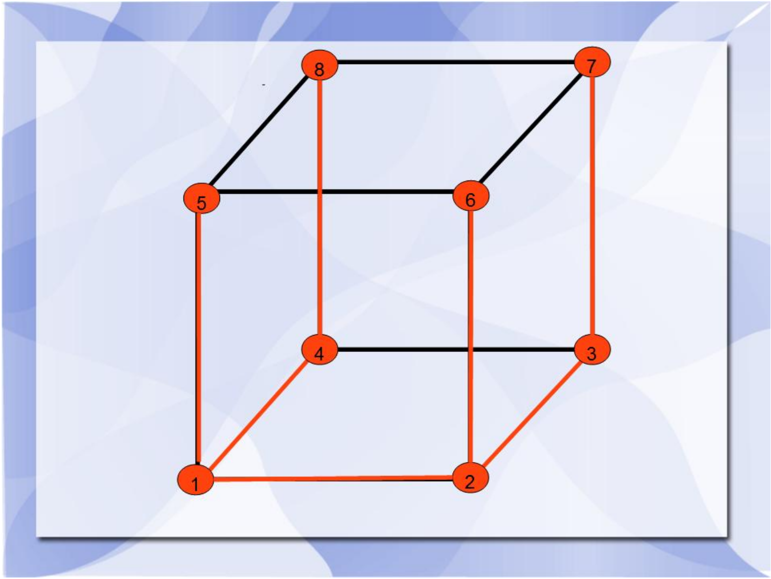
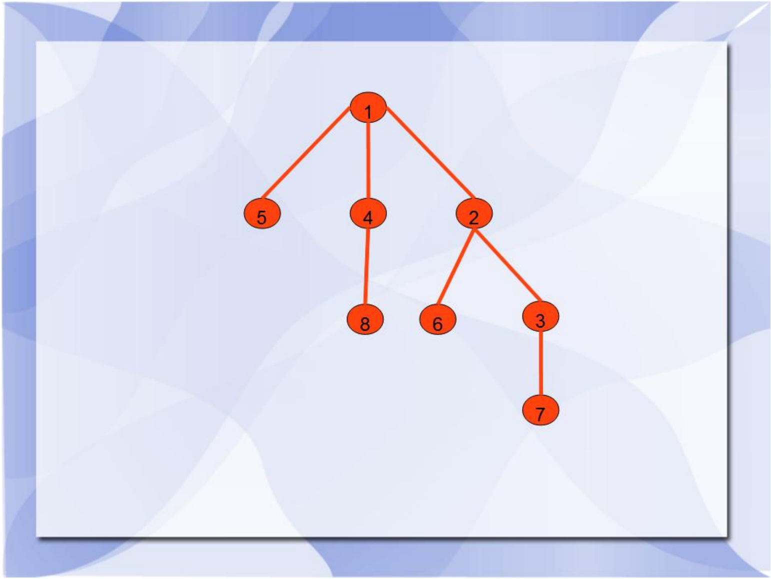


Breadth-First-Search (BFS) Spanning Tree of the Cube



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- This proves that $\mathbf{E}\left[Y^{T,\{v_0\}}\right] \leq \mathbf{E}\left[Y^{\mathcal{T},\{v_0\}}\right]$.

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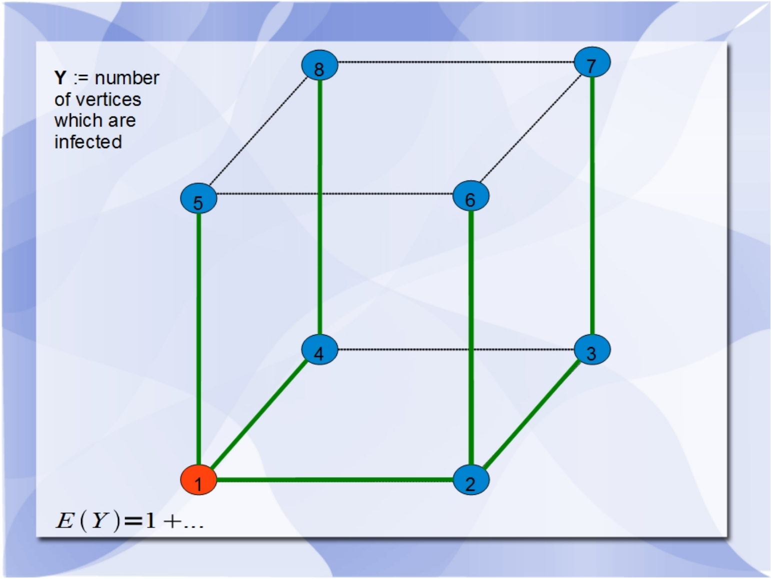

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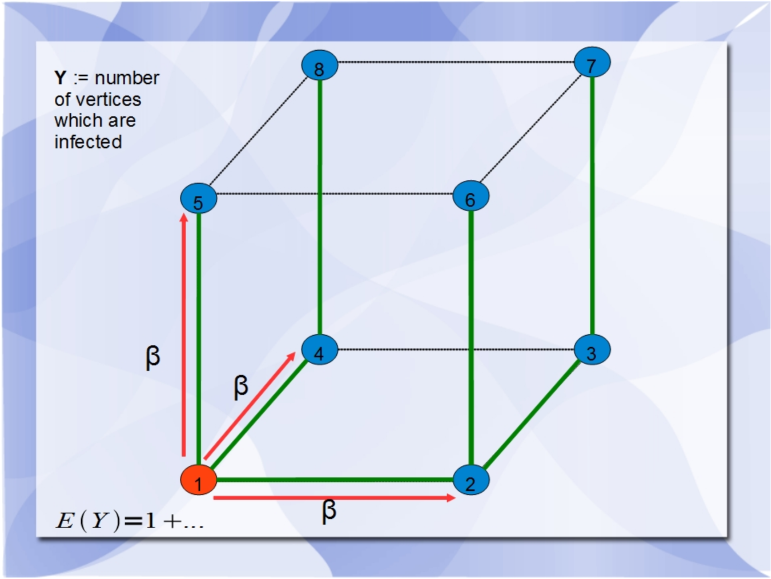

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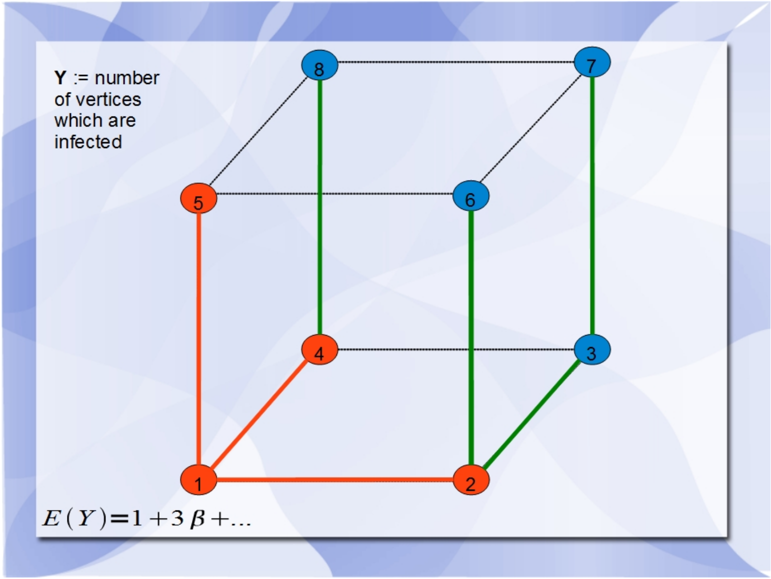

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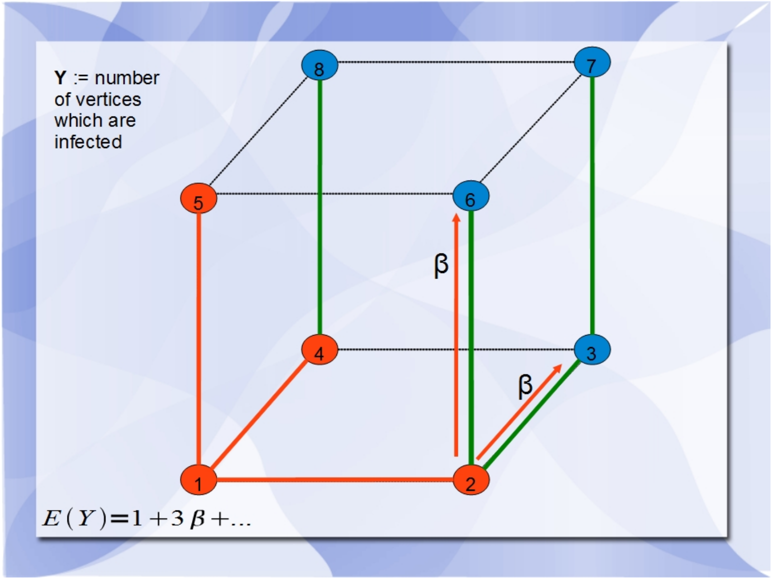

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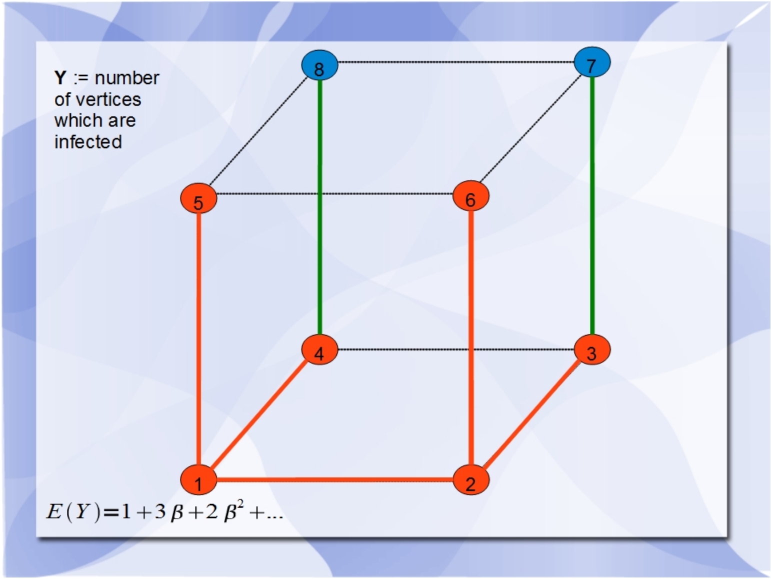

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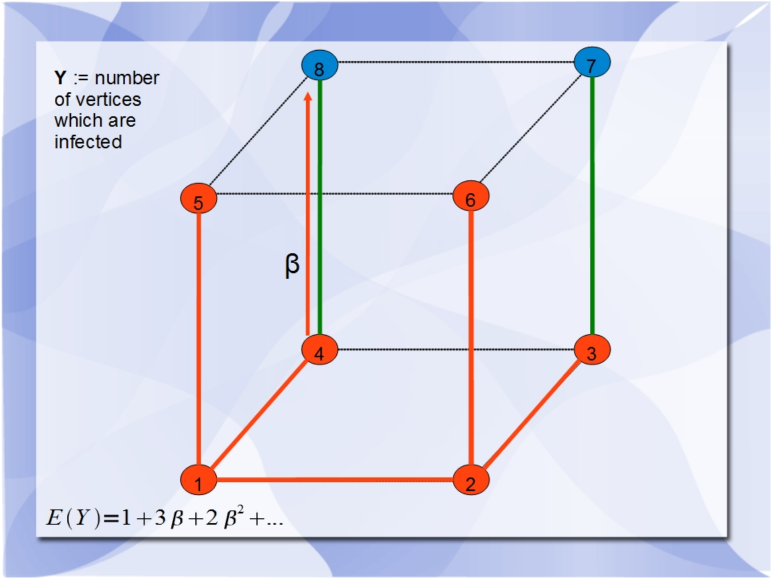

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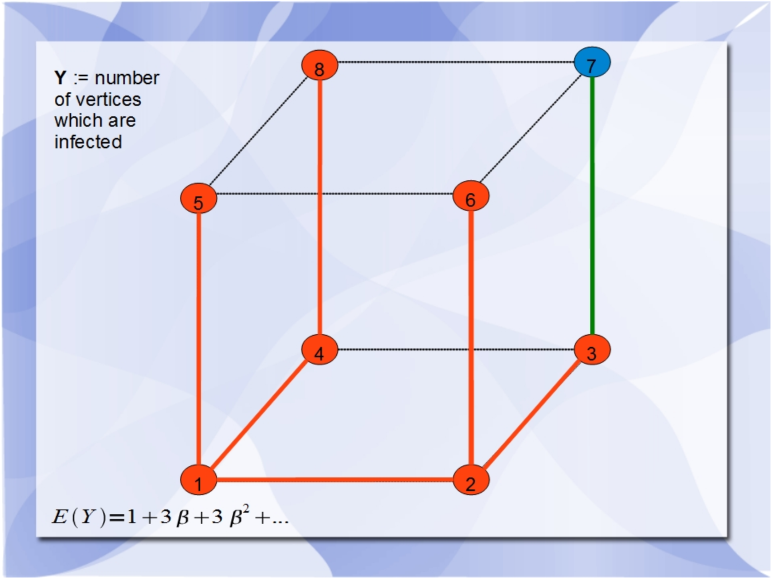

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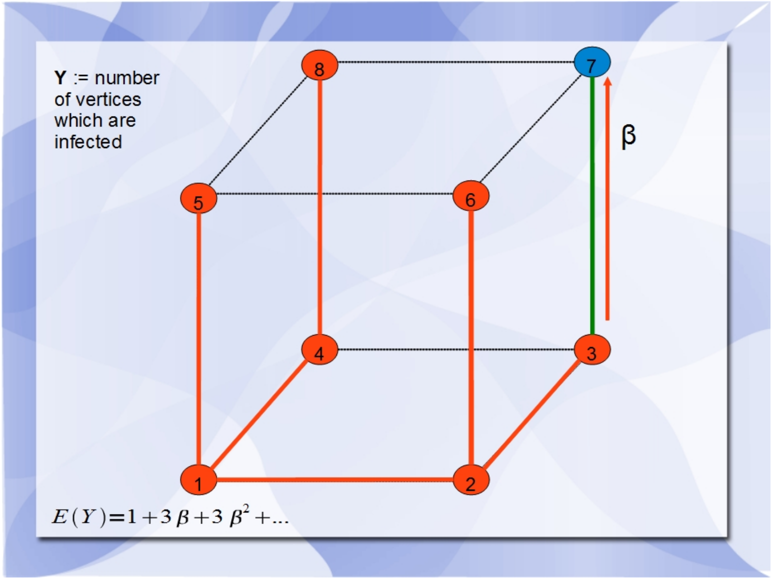

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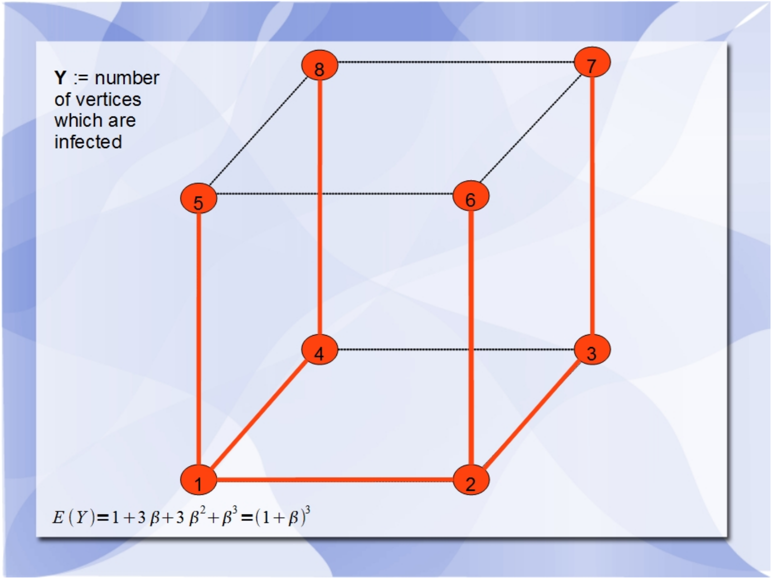

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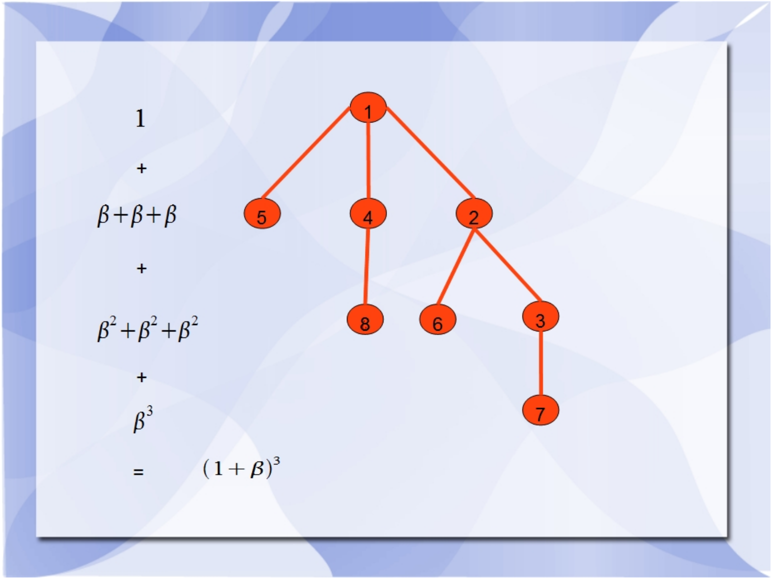

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Complete (K_n)	$1 + (n-1)\beta$	$\frac{1}{1-(n-1)\beta}$ for $\beta < \frac{1}{n-1}$

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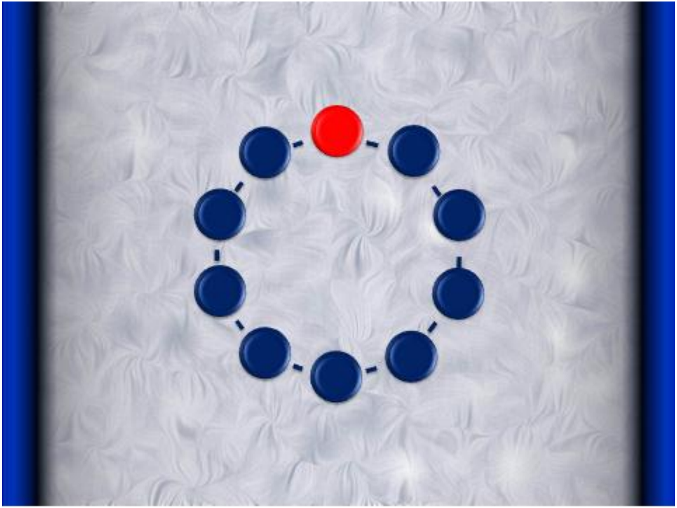
- What is our lower bound ?
- Which one is better ?

Finding the Lower Bound for C_n

Lower Bound $= 1 + \cdots$



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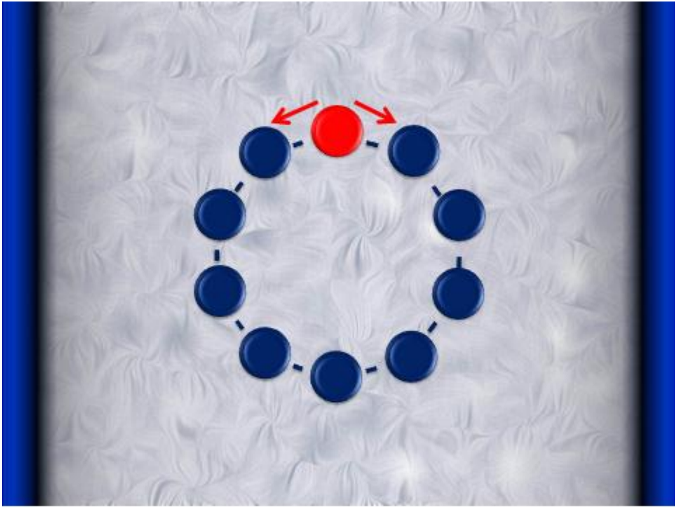

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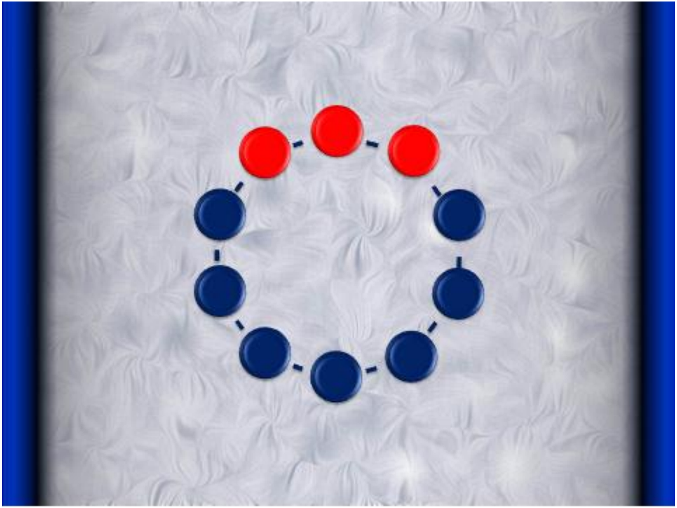

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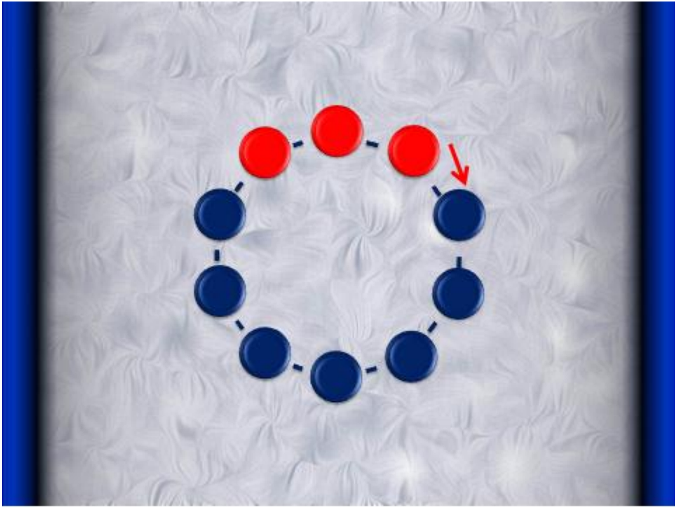

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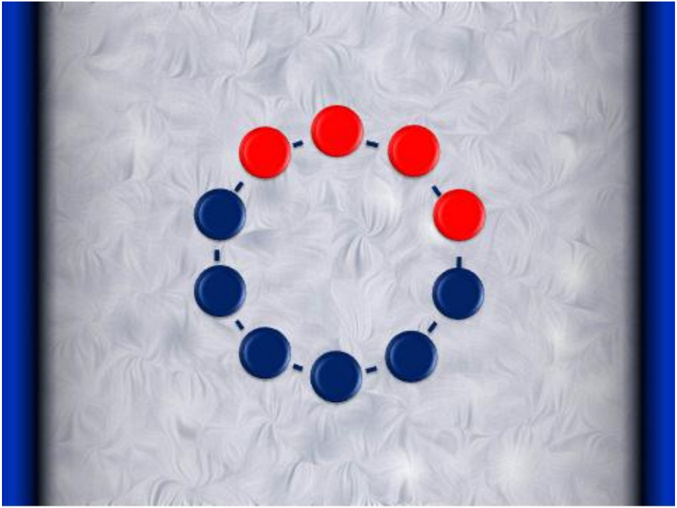

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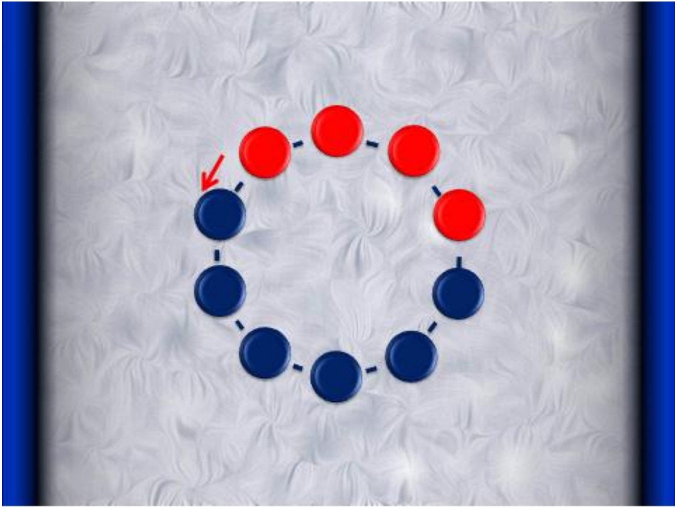

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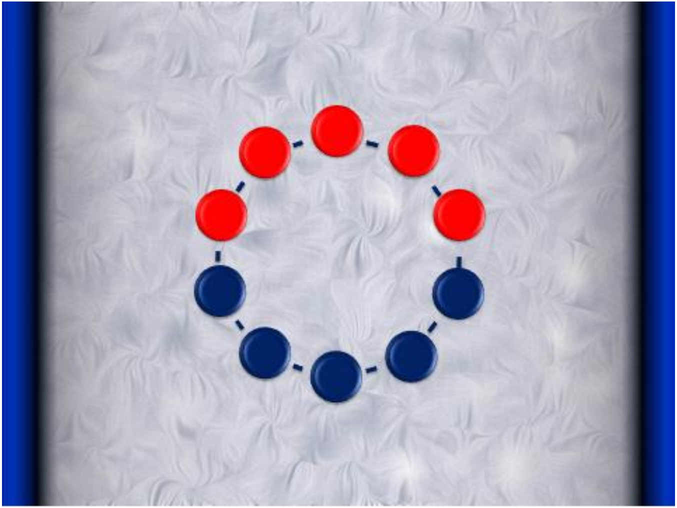

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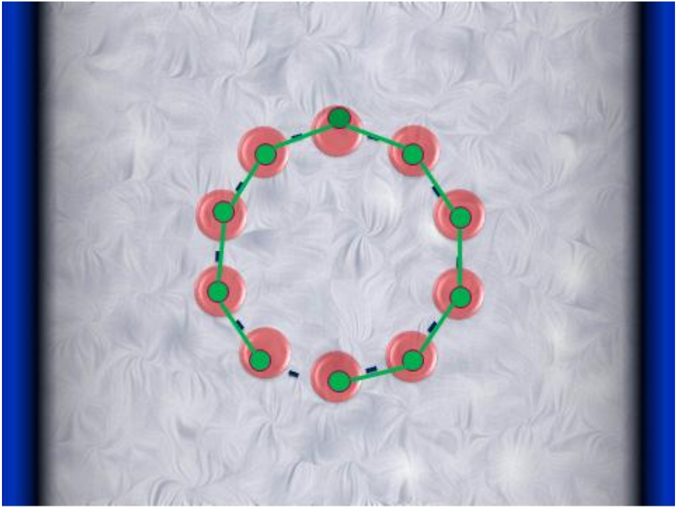
Lower Bound $= 1 + 2\beta + 2\beta^2 + \cdots + 2\beta^{\frac{n}{2} - 1} + \beta^{\frac{n}{2}}$



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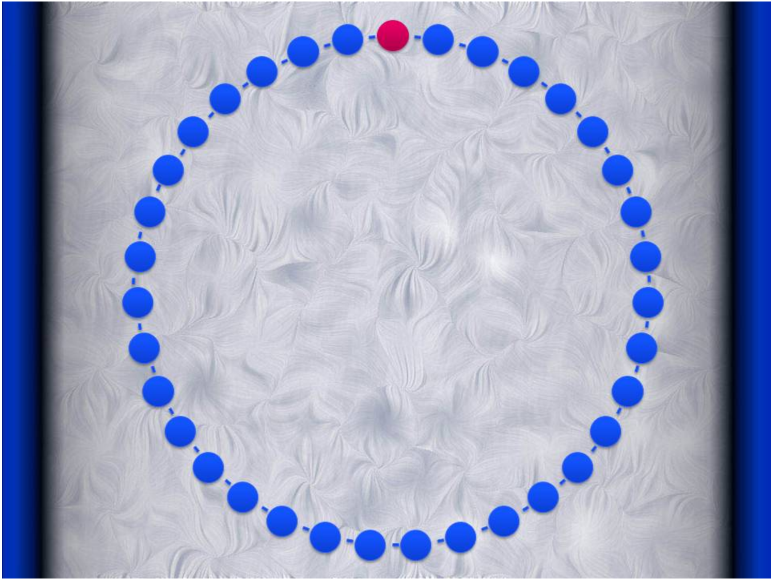
- Note the upper bound is $\frac{1}{1-2\beta}$ which only works for $\beta < \frac{1}{2}$. Moreover it also shows that even if $\beta < \frac{1}{2}$ there is still a positive *gap* between the upper and lower bound.

Finding the Exact Value of $\mathbf{E}\left[Y^{C_n}\right]$



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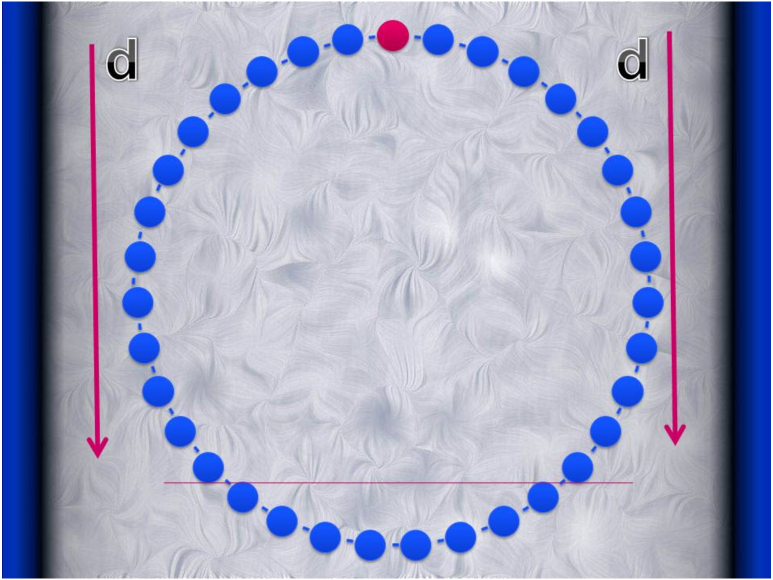

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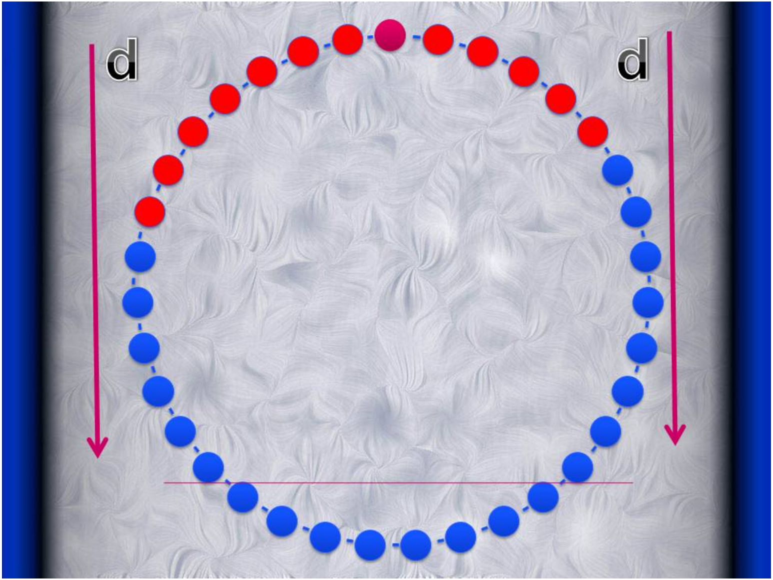

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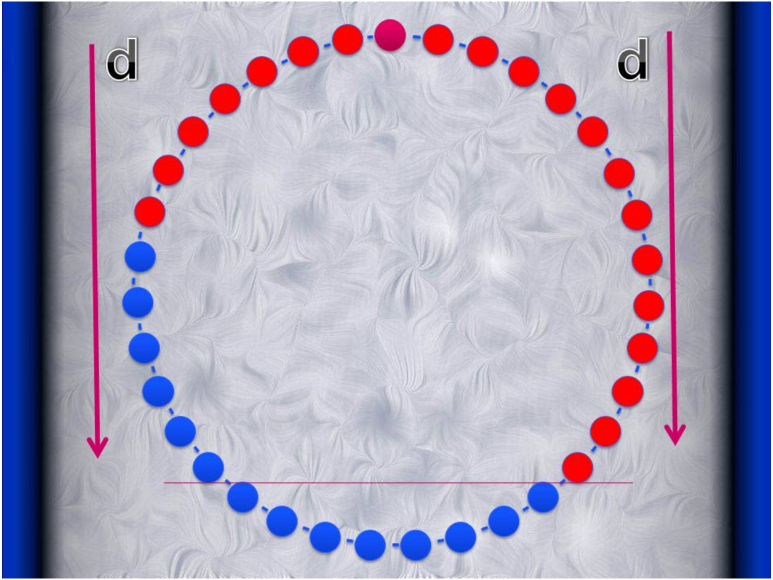

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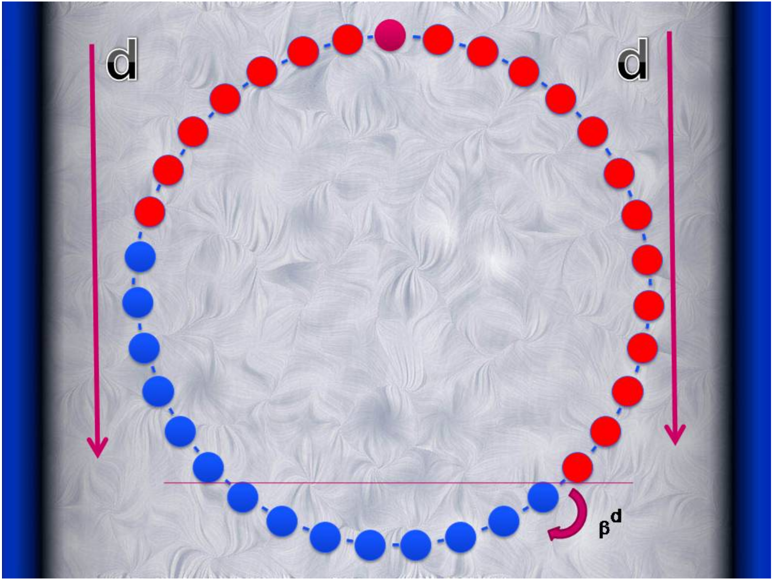

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- For any $\beta > 0$ it is now easy to see that for any $d \geq 1$

$$\text{Lower bound} \leq \mathbf{E}\left[Y^{C_n}\right] \leq \text{Lower bound} + 2\beta^d n\,.$$

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- Thus by taking $d = O\left(n\right) < n$ and letting n tends to infinity we can conclude

$$\left(\mathbf{E}\left[Y^{C_n}\right] - \text{Lower Bound}\right) \longrightarrow 0\,.$$

Asymptotic Correctness of the Lower Bound

Theorem 2 [B. and Sajadi]

Let $\{(G_n, v_0^n)\}_{n \geq 1}$ be a sequence of rooted graphs with roots $\{v_0^n\}_{n \geq 1}$ such that there exists a sequence $\alpha_n = O(\log n)$ with $N_{\alpha_n}(G_n, v_0^n)$ is a tree for all $n \geq 1$. Then, there exists $0 < \beta_0 \leq 1$, such that for all $0 < \beta < \beta_0$

$$\frac{\mathbf{E}\left[Y^{G_n, \{v_0^n\}}\right]}{\mathrm{LB}^{G_n, \{v_0\}}} \longrightarrow 1 \text{ as } n \to \infty .$$

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- But we can do better in a lot of cases!

Asymptotic Correctness of the Lower Bound

Theorem 3 [B. and Sajadi]

Let $\{(G_n, v_0^n)\}_{n \geq 1}$ be a sequence of rooted deterministic or random graphs with deterministic or randomly chosen roots $\{v_0^n\}_{n \geq 1}$. Suppose that for each G_n the maximum degrees of a vertex is bounded by Δ. Suppose there is a rooted deterministic or random tree $\mathcal{T}$ with root ϕ such that

$$(G_n, v_0^n) \xrightarrow{l.w.c.} (\mathcal{T}, \phi) \ \text{ as } \ n \to \infty\,.$$

Let $\mathrm{LB}^{G_n, \{v_0\}} := \mathbf{E}\left[Y^{\mathcal{T}_n, \{v_0^n\}}\right]$ where $\mathcal{T}_n$ is a BFS spanning tree rooted at v_0^n of the graph G_n. Then for $\beta < \frac{1}{\Delta}$

$$\left(\mathbf{E}\left[Y^{G_n, \{v_0^n\}}\right] - \mathrm{LB}^{G_n, \{v_0\}}\right) \longrightarrow 0 \ \text{ as } \ n \to \infty\,.$$

Moreover for $\beta < \frac{1}{\Delta}$ we also get

$$\lim_{n \to \infty} \mathrm{LB}^{G_n, \{v_0\}} = \lim_{n \to \infty} \mathbf{E}\left[Y^{G_n, \{v_0^n\}}\right] = \mathbf{E}\left[Y^{\mathcal{T}, \phi}\right]\,.$$

Asymptotic Correctness of the Lower Bound

Theorem 4 [B. and Sajadi]

Suppose G_n is a graph selected uniformly at random from the set of all r-regular graphs on n vertices where we assume nr is an even number. Let v_0^n be an uniformly selected vertex of G_n. Then for $\beta < \frac{1}{r}$

$$\lim_{n \to \infty} \mathbf{E}\left[Y^{G_n, \{v_0^n\}} \right] = \frac{1+\beta}{1-(r-1)\beta}.$$

Starting with More than One Initial Infected Vertices

- Suppose we start with initial infected sites $v_1, v_2, \ldots, v_k$ in the graph G.

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- Then we can prove theorems similar to Theorem 1, 2, 3 and 4.

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Theorem 5 [B. and Sajadi]

Suppose G_n is a graph selected uniformly at random from the set of all r-regular graphs on n vertices where we assume nr is an even number. Let $I_n := \left\{ v_{0,1}^n, v_{0,2}^n, \cdots, v_{0,k}^n \right\}$ be k uniformly and independently selected vertices of G_n. Then for $\beta < \frac{1}{r}$

$$\lim_{n \to \infty} \mathbf{E}\left[Y^{G_n, I_n} \right] = k \frac{1+\beta}{1-(r-1)\beta}.$$

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- If the graph **does not** *locally looks like a tree* then the lower bound is not necessarily a good approximation to the exact quantity.

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where LB_n is the lower bound obtained on the graph K_n.

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- Interesting enough in this case even the upper bound does not give good approximation!

Thank You