Random matrices with entries from a moving average process

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Joint work with Rajat S. Hazra and Deepayan Sarkar

January 9, 2013

The problem

- $(Z_{i,j} : i, j \in \mathbb{Z})$ is a stationary, mean zero, variance one Gaussian process.
- Stationarity means that for $k, l \in \mathbb{Z}$,

$$\left(Z_{i+k,j+l} : i, j \in \mathbb{Z}\right) \stackrel{d}{=} \left(Z_{i,j} : i, j \in \mathbb{Z}\right).$$

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- For $i, j \geq 1$, set

$$X_{i,j} := Z_{i \wedge j, i \vee j} \, .$$

The problem (contd.)

- Let
$$A_n := ((X_{i,j}))_{n\times n},\ n \geq 1\,.$$
- Let $\lambda_1 \leq \ldots \leq \lambda_n$ denote the eigenvalues of A_n.
- Denote
$$\mu_n := \frac{1}{n}\sum_{i=1}^{n}\delta_{\{\lambda_i/\sqrt{n}\}}\,.$$

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$$\mu_n := \frac{1}{n}\sum_{i=1}^{n} \delta_{\{\lambda_i/\sqrt{n}\}}\,.$$
- The problem: to identify the weak limit of μ_n as $n \to \infty$.

Assumptions

- **Assumption 1:** $R(\cdot,\cdot)$ is symmetric, that is,

$$R(u,v) = R(v,u) \text{ for all } u,v \in \mathbb{Z}\,.$$

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- **Assumption 2:** $\sum_{u,v\in\mathbb{Z}} |R(u,v)| < \infty$.

Some combinatorics

- $NC_2(2m)$ is the set of non-crossing pair partitions of $\{1, 2, \ldots, 2m\}$.

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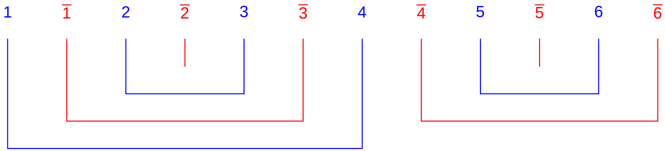

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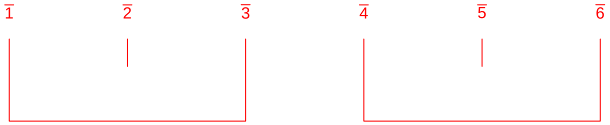

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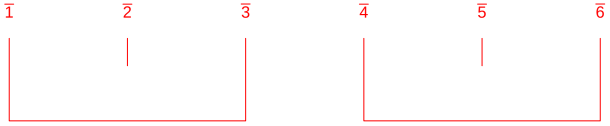

Therefore, $K(\sigma) = \{(\overline{1}, \overline{3}), (\overline{2}), (\overline{4}, \overline{6}), (\overline{5})\}$.

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- Define

$$S(\sigma) := \left\{ (k_1, \ldots, k_{2m}) \in \mathbb{Z}^{2m} : \sum_{j=1}^{l_i} k_{v_j^i} = 0, \, i = 1, \ldots, k+1 \right\} .$$

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- For $m \geq 1$, denote

$$\beta_{2m} := \sum_{\sigma \in NC_2(2m)} \sum_{(k_1,\ldots,k_{2m}) \in S(\sigma)} \prod_{(u,v) \in \sigma} R(k_u, k_v)\,.$$

The main result

Theorem (C., Hazra and Sarkar)

There exists a unique symmetric distribution μ with bounded support, whose $(2m)$-th moment is β_{2m} for $m = 1, 2, \ldots$. Furthermore, for all $a, b \in \mathbb{R}$,

$$\mu_n([a, b]) \xrightarrow{P} \mu([a, b])$$

as $n \to \infty$.

Wick's formula

Fact

If $(Y_1, \ldots, Y_{2k})$ *has a multivariate normal distribution with mean zero, then*

$$E\left(\prod_{i=1}^{2k} Y_i\right) = \sum_{\pi} \prod_{(u,v)\in\pi} E\left(Y_u Y_v\right),$$

where the sum runs over all pair partitions π *of* $\{1, 2, \ldots, 2k\}$.

- Fix $N \geq 1$.
- For $i, j, k, l \geq 1$, declare $(i, j) \sim (k, l)$ if

$$|i \wedge j - k \wedge l| \vee |i \vee j - k \vee l| \leq N \, .$$

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- Let $(i_1, \ldots, i_{2m}) \in \{1, \ldots, n\}^{2m}$ be such that for any pair partition π of $\{1, \ldots, 2m\}$, there exists $(j, k) \in \pi$ with

$$(i_{j-1}, i_j) \not\sim (i_{k-1}, i_k)\,,$$

where $i_0 := i_{2m}$.

- By Wick's formula,

$$E\left[X_{i_0 i_1} X_{i_1 i_2} \ldots X_{i_{2m-1} i_{2m}}\right]$$

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- **Difficulty:** There are n^{2m} such tuples, but we are scaling by n^{m+1}.
- "Assumption 2: $\sum_{u,v\in\mathbb{Z}} |R(u,v)| < \infty$" comes to our rescue.

Free product convloution

- Let $(\mathcal{A}, \phi)$ be a **non-commutative probability space**.
- Let μ be a probability measure on $\mathbb{R}$ with all moments finite. An element $a \in \mathcal{A}$ has **distribution** μ if

$$\phi(a^n) = \int_{\mathbb{R}} x^n \mu(dx),\ n \geq 1\,.$$

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- $a, b \in \mathcal{A}$ are **freely independent** if for all $k \geq 1$ and polynomials $p_1, \ldots, p_k, q_1, \ldots, q_k$,

$$\phi\left(p_1(a) q_1(b) \ldots p_k(a) q_k(b)\right) = 0 \, ,$$

whenever

$$\phi(p_i(a)) = \phi(q_i(b)) = 0, \; i = 1, \ldots, k \, .$$

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- If a and b have distributions μ and ν respectively, and a and b are freely independent, then $\mu \boxtimes \nu$ is the distribution of ab.

Fact

Let μ be a distribution supported on a compact subset of $[0,\infty)$, whose k-th moment is m_k, $k = 1, 2, \ldots$. Let μ_s be the Wigner semicircular law (WSL), given by

$$\mu_s(dx) = \frac{\sqrt{4-x^2}}{2\pi}\mathbf{1}(|x| \le 2)dx\,.$$

Then,

$$\int_{\mathbb{R}} x^{2k} \mu \boxtimes \mu_s(dx) = \sum_{\sigma \in NC_2(2k)} \prod_{i=1}^{k+1} m_{l_i(\sigma)}\,,$$

where $l_1(\sigma), \ldots, l_{k+1}(\sigma)$ denote the block sizes of $K(\sigma)$.

Special case 1.

Theorem (C., Hazra and Sarkar)

Assume that

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and let μ_r *denote the law of* $r(U)$ *where* U *follows Uniform(0,1). Then the LSD* μ *is given by*

$$\mu = \mu_r \boxtimes \mu_s\,,$$

where μ_s *is the WSL.*

Example

Let $(G_{i,j} : i, j \geq 1)$ be i.i.d. standard normal. Fix $N \geq 1$ and define

$$X_{i,j} := \sum_{k=0}^{N} \sum_{l=0}^{N} G_{i+K,j+l},\ i, j \geq 1\,.$$

Then, the hypothesis of the previous theorem is satisfied.

Special case 2

Theorem (C., Hazra and Sarkar)

Assume that

$$R(k,0) = 0 \text{ for all } k \neq 0\,.$$

Then, μ is the WSL.

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- and for all $j \neq k$,

$$\sum_{i=1}^{\infty} C_{ij} C_{ik} = 0 .$$

Example (contd.)

- $(G_{ij} : i, j \geq 1)$ i.i.d. standard Gaussian.
- For $i, j \geq 1$,

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- Then, the LSD of the matrix $A_n := ((X_{i,j}))_{1 \leq i,j \leq n}$ is WSL.

Invariance

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Invariance

- Chatterjee's invariance principle allows us to claim that for a **finite order** moving average (MA) process, standard Gaussian can be replaced by any distribution with mean zero and variance one.
- Would be nice if this can be generalized to infinite order MA processes.

The model

- Suppose that $\{X_{i,j} : i,j \geq 1\}$ is a family of i.i.d. random variables such that

$$P(|X_{1,1}| > \cdot) \in RV(-\alpha) \text{ for some } \alpha > 0\,,$$

that is,

$$\lim_{t\to\infty} \frac{P(|X_{1,1}| > tx)}{P(|X_{1,1}| > t} = x^{-\alpha},\ x > 0\,.$$

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- Define

$$Y_{k,l} := \sum_{i=0}^{N} \sum_{j=0}^{N} c_{ij} X_{i+k,j+l},\ 1 \leq k \leq l\,.$$

The model (contd.)

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- For a matrix B, let
$$\sigma_{\max}(B) := \sqrt{\text{largest eigenvalue of } B^T B}\,.$$

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- Problem: To find the asymptotics of $\sigma_{\max}(A_n)$ as $n \to \infty$.

The result

Define

$$b(t) := \inf\left\{x : P(|X_{11}| > x) \le t^{-1}\right\},\ t > 0\,,$$

$$C := \begin{bmatrix} 0 & \dots & 0 & c_{NN} & \dots & c_{N0} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & c_{0N} & \dots & c_{00} \\ c_{NN} & \dots & c_{0N} & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ c_{N0} & \dots & c_{00} & 0 & \dots & 0 \end{bmatrix}_{(2N+1)\times(2N+1)} .$$

The result (contd.)

Theorem (C., Hazra and Sarkar)

If $0 < \alpha < 1$, *then*

$$\frac{\sigma_{\max}(A_n)}{b(n^2/2)} \Longrightarrow \sigma_{\max}(C)Z\,,$$

as $n \to \infty$, *where* Z, *a Fréchet* (α) *random variable, has c.d.f.*

$$P(Z \le x) = \exp\left(-x^{-\alpha}\right),\ x > 0\,.$$

- It can be shown that

$$\frac{\sigma_{\max}(A_n)}{\max_{1\leq i\leq j\leq n}|X_{i,j}|}\xrightarrow{P}\sigma_{\max}(C)\,.$$

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- It is known that if $Z_1, Z_2, \ldots$ are i.i.d. copies of X_{11}, then

$$\frac{\max_{1\le j\le n}|Z_j|}{b(n)} \Longrightarrow \text{Fréchet}(\alpha)\,.$$

Theorem (Soshnikov(2004))

Let $\{X_{ij} : 1 \leq i \leq j\}$ *be i.i.d. such that* $P(|X_{11}| > \cdot) \in RV(-\alpha)$ *for some* $0 < \alpha < 2$*. If* W_n *is the* $n \times n$ *Wigner matrix constructed from* X_{ij}*'s, then*

$$\frac{\sigma_{\max}(W_n)}{\max_{1\leq i\leq j\leq n}|X_{ij}|} \xrightarrow{P} 1$$

as $n \to \infty$*.*

Idea of Soshnikov's proof

- If
$$(i^*, j^*) := \arg \max_{1 \le i \le j \le n} |X_{ij}| \,,$$
then $X_{i^* j^*}^{-1} W_n$ is approximately equal to the matrix whose (i^*, j^*)-th and (j^*, i^*)-th entries are one, rest are zero.

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- Soshnikov showed that

$$X_{i^*j^*}^{-1} \max_{1 \le i \le n} \left| \sum_{j=1}^{n} X_{ij} \right| \xrightarrow{P} 1 \,.$$

- Consider a simple example

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then $X_{i^*,j^*}^{-1} A_n$ is approximately

$$\left[\begin{array}{c|ccc} & i^* & j^* & j^*+1 \\ \hline i^* & 0 & 1 & 1 \\ j^* & 1 & 0 & 0 \\ j^*+1 & 1 & 0 & 0 \end{array}\right].$$

- For the matrix
$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$
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- However, on squaring this matrix, the two norms equal.
- In general, we looked at the r-th power, and let $r \to \infty$.

Future research

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LSD	solved	future work
Edge	future work	solved

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THANK YOU