Statistical Mechanics on Sparse Random Graphs

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Outline

1. What is this talk about, and why should one care
2. Interesting phenomena
3. A few results
   - Ferromagnetic Ising model
   - Ising spin glass
   - Trees vs graphs: from reconstruction to pure states
4. Conclusion

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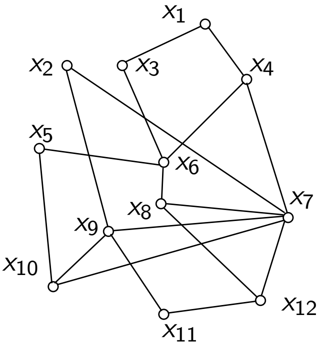
What is this talk about, and why should one care

‘Standard model’



$G = (V, E)$, $V = [n]$, $\underline{x} = (x_1, \ldots, x_n)$, $x_i \in \mathcal{X}$ (finite set).

$$\mu(\underline{x}) = \frac{1}{Z} \prod_{(ij) \in E} \psi_{ij}(x_i, x_j)\,.$$

'Standard model' (assumptions)

1. G has bounded degree (on average).

2. G has girth larger than 2ℓ with $\ell = \ell(n) \to \infty$ (apart from $o(n)$ vertices).

3. $0 \le \psi_{ij}(x_i, x_j) \le \psi_{\max} < \infty$. For each i exists x_i^{p} s.t. $0 < \psi_{\min} \le \psi_{ij}(x_i^{\mathrm{p}}, x_j)$.

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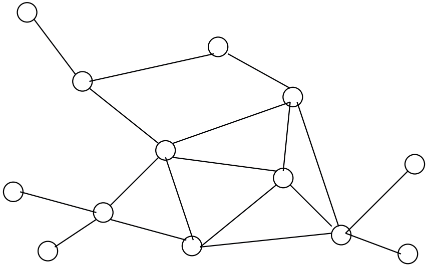
Example 1: q-coloring



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$\underline{x} = (x_1, x_2, \ldots, x_n)$, $x_i \in \{1, \ldots, q\}$ variables

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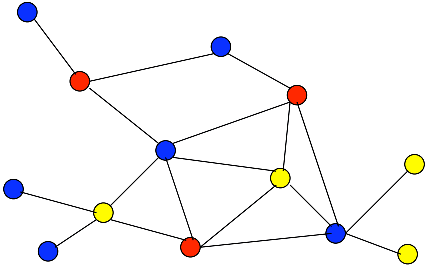

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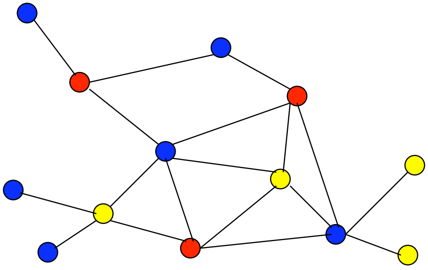
Uniform measure over proper colorings



$$\mu(\underline{x}) = \frac{1}{Z} \prod_{(i,j)\in E} \psi(x_i, x_j)\,, \qquad\qquad \psi(x,y) = \mathbb{I}(x \neq y)\,.$$

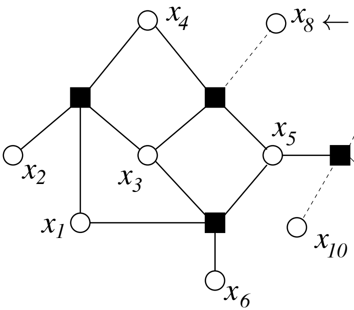
Example 2: k-satisfiability

n variables: $\underline{x} = (x_1, x_2, \ldots, x_n)$, $x_i \in \{0, 1\}$

m k-clauses

$$(x_1 \vee \overline{x_5} \vee x_7) \wedge (x_5 \vee x_8 \vee \overline{x_9}) \wedge \cdots \wedge (\overline{x_{66}} \vee \overline{x_{21}} \vee \overline{x_{32}})$$

Uniform measure over solutions



$\leftarrow$ variables $x_i \in \{0,1\}$

$\leftarrow$ clauses, e.g. $(x_5 \vee x_7 \vee \overline{x_9} \vee \overline{x_{10}})$

$$F = \cdots \wedge \underbrace{(x_{i_1(a)} \vee \overline{x}_{i_2(a)} \vee \cdots \vee x_{i_k(a)})}_{a\text{-th clause}} \wedge \cdots$$

$$\mu(\underline{x}) = \frac{1}{Z} \prod_{a=1}^{m} \psi_a(x_{i_1(a)}, \ldots, x_{i_k(a)})$$

Many other examples

- Communications (LDPC; XORSAT).
- Artificial intelligence (Bayesian networks; Graphical models).
- Statistics (Compressed sensing).
- ...

Interesting phenomena

1. 'Exact' predictions

Example: Free energy density

$$Z_n \equiv \sum_{\underline{x}} \prod_{(ij)\in E} \psi_{ij}(x_i, x_j)$$

$$\phi \equiv \lim_{n\to\infty} \frac{1}{n} \log Z_n \, .$$

[Cavity/Replica methods]

1. 'Exact' predictions

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$$\phi \equiv \lim_{n\to\infty} \frac{1}{n} \log Z_n .$$

[Cavity/Replica methods]

2. Mean field equations

'Set of $O(n)$ non-linear equations that determine local marginals in the large system limit'

Bethe-Peierls equations (replica symmetric cavity method):

$\mu_{i\to j}(\,\cdot\,) \equiv$ Marginal of x_i when replacing $\psi_{ij}(x_i, x_j)$ by 1

$$\mu_{i\to j}(x_i) \approx \frac{1}{z_{i\to j}} \prod_{l\in\partial i\setminus j} \sum_{x_l} \psi_{il}(x_i, x_l)\mu_{l\to i}(x_l)$$

General philosophy: approximate local marginals of $\mu(\,\cdot\,)$ in terms of measures on trees.

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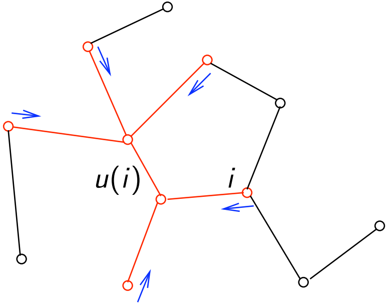
2. Bethe-Peierls approximation



$F = (U, E_U) \subseteq G$, $\mathrm{diam}(F) \leq 2\ell$, such that $\partial i \in U$ or $\partial i \cap U = \{u(i)\}$

$$\mu_U(\underline{x}_U) \approx \nu_U(\underline{x}_U) = \frac{1}{Z_U} \prod_{(i,j)\in E_U} \psi_{ij}(x_i, x_j) \prod_{i \in \partial U} \nu_{i \to u(i)}(x_i) \,.$$

Questions

$\{\nu_{i\to j}(\cdot)\}$ → 'set of messages' (aka cavity fields)

1. Is $\mu(\cdot)$ well-approximated by some $\{\nu_{i\to j}(\cdot)\}$?

2. How to find a good set of messages $\{\nu_{i\to j}(\cdot)\}$?

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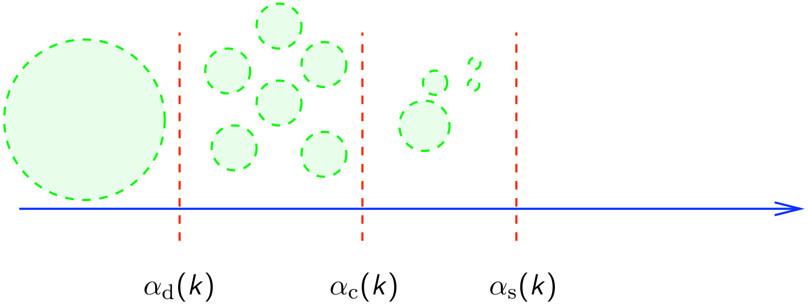
3. 'Dynamical' phase transition

'The free energy density is analytic but the measure μ splits into lumps'

3. 'Dynamical' phase transition

Example: k-satisfiability, the space of solutions

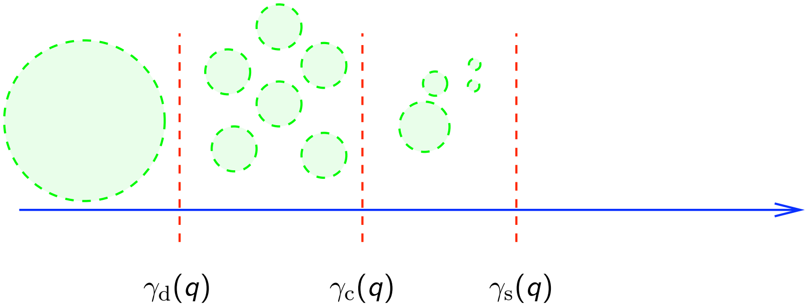


$\alpha = m/n$ fixed, $n \to \infty$.

[Biroli,Monasson,Weigt 00, Mézard,Parisi,Zecchina 02, Krzákala et al 07]

3. 'Dynamical' phase transition

Example: q-COL, the space of solutions



[Edges taken independently with probability γ/n each, γ fixed, $n \to \infty$]
[same references + Achlioptas,Ricci 06]

4. 'Non-self averaging'

$\underline{x}^{(1)}, \underline{x}^{(2)}$ independent configurations, same disorder (replicas)

$d(\underline{x}^{(1)}, \underline{x}^{(2)})$ Hamming distance

$\mu(d(\underline{x}^{(1)}, \underline{x}^{(2)}) > n\delta) \rightarrow$ non-degenerate random variable

[$\sim$ SK model]

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A few results

Ferromagnetic Ising model

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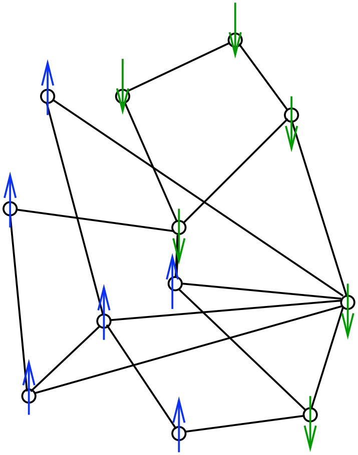

Ferromagnetic Ising model

$G_n = (V_n \equiv [n], E_n)$

$x_i \in \{+1, -1\}$

$\beta \geq 0$

$$\mu(\underline{x}) = \frac{1}{Z} \exp\left\{ \beta \sum_{(ij) \in E_n} x_i x_j + B \sum_{i=1}^{n} x_i \right\}$$

[in sparse random graphs: Johnston, Plechác 98/ Leone et al 04]

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Free energy density

Theorem (D., Montanari, 10)

If $\{G_n\}$ is uniformly sparse and converges locally to MGW $\mathsf{T}(P, \rho, \infty)$, then

$$\phi = \phi_*(P, \beta, B) \,.$$

[moment condition relaxed in Dommers, Giardina, van der Hofstad 10; extended to all limiting trees and to ferromagnetic Potts models with regular limiting tree in D., Montanari, Sly, Sun, 11]

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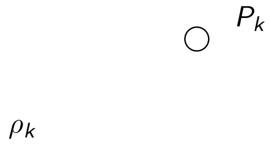
$\mathsf{T}(P, \rho, t)$



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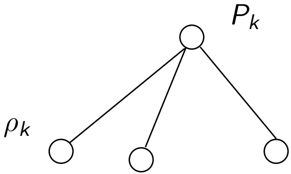

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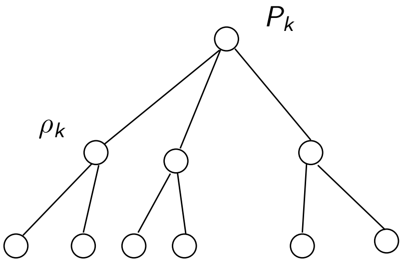

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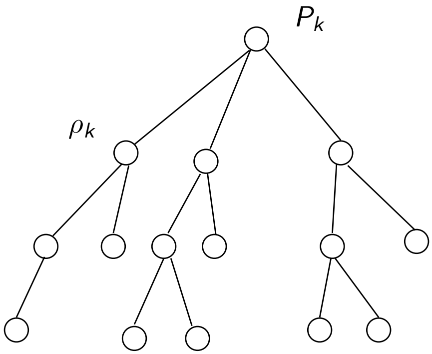
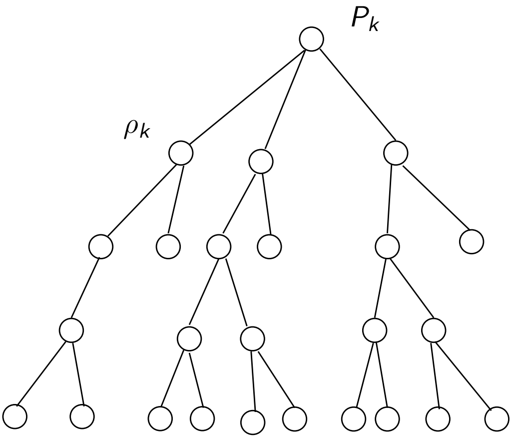


T(P, ρ, t)



'Converges locally'

$P \equiv \{P_k\}_{k \geq 0}$	Degree distribution, law of L (of mean $\overline{P} > 0$)
$\rho \equiv \{\rho_k\}_{k \geq 0}$	Size-biased P, law of K (degree of uniform edge)
$\mathsf{T}(P, \rho, t)$	t-generations GW tree (root degree P, else ρ)
$\mathsf{B}_i(t)$	Ball of radius t in G_n centered at node i

Definition

$\{G_n\}$ *converges locally* to $\mathsf{T}(P, \rho, \infty)$ if for uniformly random $I \in [n]$ and fixed t, law of $\mathsf{B}_I(t)$ converges as $n \to \infty$ to $\mathsf{T}(P, \rho, t)$.

[in framework of Benjamini & Schramm 01, Aldous & Lyons 07]

$\phi_*(P, \beta, B)$

For $B \geq 0$, let $\theta \equiv \tanh \beta$, $h^{(0)} > 0$, and for iid $h_i^{(t)}$,

$$h^{(t+1)} \stackrel{\mathrm{d}}{=} \tanh\left\{B + \sum_{i=1}^{K-1} \operatorname{atanh}(\theta\, h_i^{(t)})\right\},$$

Then $h^{(t)} \stackrel{\mathrm{d}}{\to} h^*$ and for iid h_i^* independent of L,

$$\phi_*(P, \beta, B) = \log\cosh B + \frac{\overline{P}}{2} \log\cosh \beta - \frac{\overline{P}}{2} \mathbb{E} \log(1 + \theta h_1^* h_2^*) +$$
$$+ \mathbb{E} \log\left\{(1 + \tanh B) \prod_{i=1}^{L} (1 + \theta h_i^*) + (1 - \tanh B) \prod_{i=1}^{L} (1 - \theta h_i^*)\right\}.$$

[Variational (LD) formulation for ϕ_*, see D., Montanari, Sun, 11]

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[Variational (LD) formulation for ϕ_*, see D., Montanari, Sun, 11]

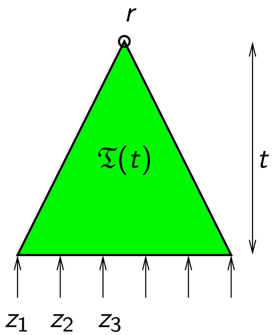
Proof strategy

0. Take $B > 0$.
1. Reduce to expectations of local quantities

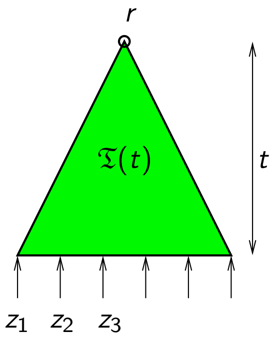
$$\frac{\mathrm{d}}{\mathrm{d}\beta} \log Z_n(\beta, B) = \sum_{(ij)\in E_n} \langle x_i x_j \rangle_n$$

   ($\langle \cdot \rangle_n$ denote expectation under Ising on G_n).
2. Prove convergence of local expectations to tree values.

2. Convergence to tree values



$\mathfrak{T}$	infinite tree with max degree $k_{\max}$
$\mathfrak{T}(t)$	first t generations
$\mu^{t,z}(\cdot)$	Ising model on $\mathfrak{T}(t)$ boundary condition z
$\mu_r^{t,z}$	root spin expectation



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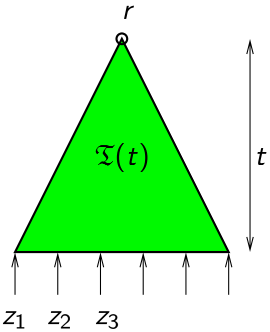

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Decorrelation

Uniform (Gibbs measure uniqueness)

$$|\mu_r^{t,z(1)} - \mu_r^{t,z(2)}| \leq | \mu_r^{t,+} - \mu_r^{t,-} | \to 0\,.$$

Easier

True only at high temperature ($\beta = O(1/k_{\max})$)

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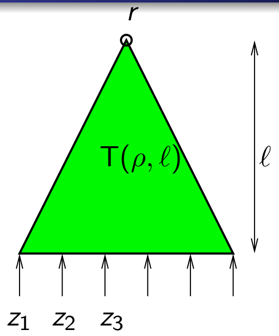
Why non-uniform control? Phase transition...

For $\beta > \beta_c \equiv \mathrm{atanh}(1/\overline{\rho})$

$$\lim_{B\to 0+} \lim_{n\to\infty} \mathbb{E}\langle x_I \rangle_n = - \lim_{B\to 0-} \lim_{n\to\infty} \mathbb{E}\langle x_I \rangle_n > 0$$

. . . and its tree counterpart



$$z = (+1, +1, \dots, +1) \quad \Rightarrow \quad \lim_{\ell \to \infty} \langle x_r \rangle_\ell > 0$$

$$z = (-1, -1, \dots, -1) \quad \Rightarrow \quad \lim_{\ell \to \infty} \langle x_r \rangle_\ell < 0$$

Decorrelation

Non uniform

$$|\mu_r^{t,+} - \mu_r^{t,\mathrm{free}}| \to 0\,.$$

Any temperature

Trick

$$0 \leq \mu_r^{t,+} - \mu_r^{t,\mathrm{free}} \leq \epsilon \{ \mu_r^{t,\mathrm{free}} - \mu_r^{t-1,\mathrm{free}} \} \to 0$$

($\mu_r^{t,\mathrm{free}}$ monotone by Griffiths)

[Ising specific, but strategy extended to Potts, Independent Sets]

Ising spin glass

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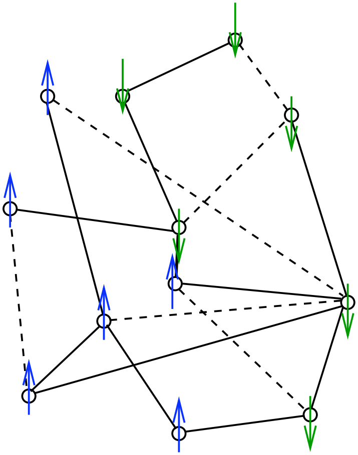

Ising spin glass

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$J_{ij} \in \{+1, -1\}$ uniformly random

[Viana, Bray 1985]

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[Viana, Bray 1985]

Free energy density: brief survey

Theorem

If G_n uniformly random with average degree γ and $\beta < \beta_*(B, \gamma)$, then

$$\phi = \widehat{\phi}_*(\gamma, \beta, B) \,.$$

Guerra, Toninelli 2003

$$B = 0, \ \beta_* = \mathrm{atanh}(1/\sqrt{\gamma})$$

Talagrand 2001, 2003

$$B \neq 0, \ \beta_* = O(1/\gamma)$$

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Free energy density

Theorem (D., Gerschenfeld, Montanari 11)

If G_n uniformly sparse and converges locally to $\mathsf{T}(P, \rho, \infty)$ and $\beta < \beta_*(B, P)$, then

$$\phi = \widehat{\phi}_*(P, \beta, B) \, .$$

$K \sim k_{\rm typ} \gg 1 \Rightarrow \beta_*(B, P) \simeq \frac{f(B)}{\sqrt{k_{\rm typ}}}$ and $0 < f(B) \uparrow \infty$ with B.

Free energy density

Theorem (D., Gerschenfeld, Montanari 11)

If G_n uniformly sparse and converges locally to $\mathsf{T}(P, \rho, \infty)$ and $\beta < \beta_*(B, P)$, then

$$\phi = \widehat{\phi}_*(P, \beta, B)\,.$$

$K \sim k_{\rm typ} \gg 1 \Rightarrow \beta_*(B, P) \simeq \frac{f(B)}{\sqrt{k_{\rm typ}}}$ and $0 < f(B) \uparrow \infty$ with B.

$\widehat{\phi}_*(P, \beta, B)$

For iid $\theta_i \in \{+\tanh\beta, -\tanh\beta\}$ uniformly at random, independent of K, L and iid $h_i^{(t)}$, let

$$h^{(t+1)} \stackrel{\mathrm{d}}{=} \tanh\Big\{B + \sum_{i=1}^{K-1} \operatorname{atanh}(\theta_i\, h_i^{(t)})\Big\},$$

Then $h^{(t)} \stackrel{\mathrm{d}}{\to} h^*$ and for iid h_i^* independent of L,

$$\widehat{\phi}_*(P, \beta, B) = \log\cosh B + \frac{\overline{P}}{2}\log\cosh\beta - \frac{\overline{P}}{2}\,\mathbb{E}\log(1+\theta_0 h_1^* h_2^*) +$$
$$+\mathbb{E}\,\log\left\{(1+\tanh B)\prod_{i=1}^{L}(1+\theta_i h_i^*) + (1-\tanh B)\prod_{i=1}^{L}(1-\theta_i h_i^*)\right\}.$$

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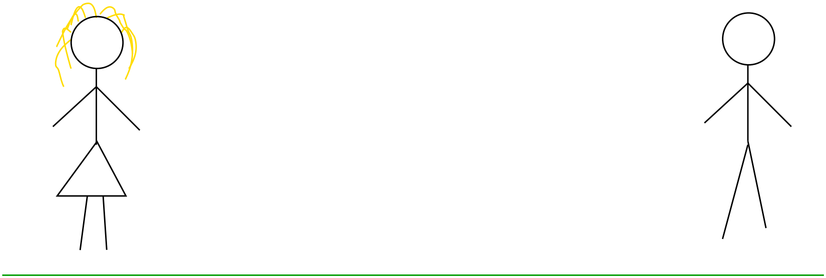
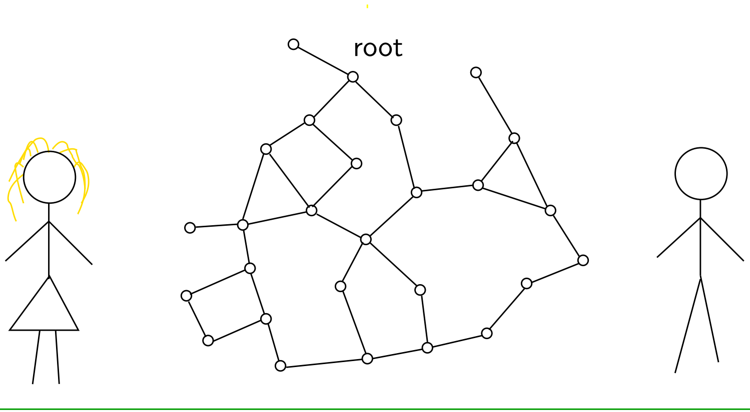
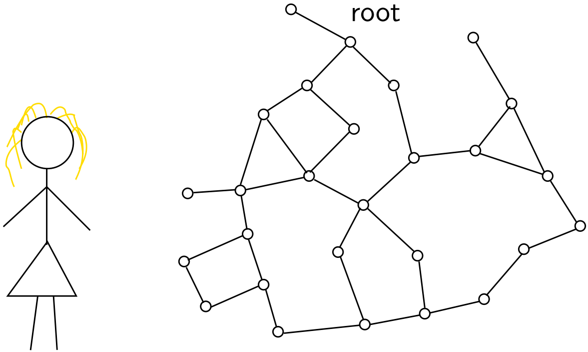
Trees vs graphs: from reconstruction to pure states



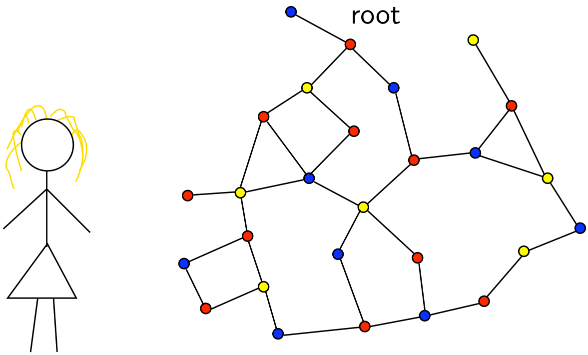
Alice, Bob and G



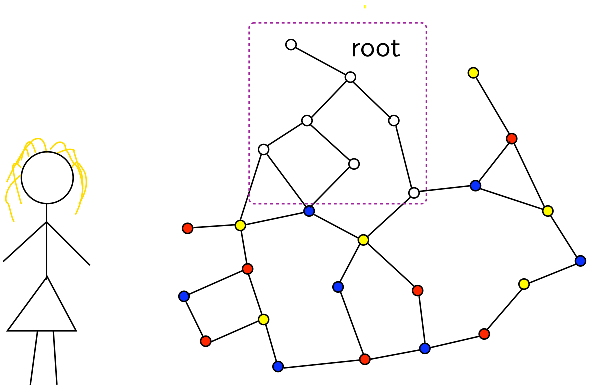
Exit Bob



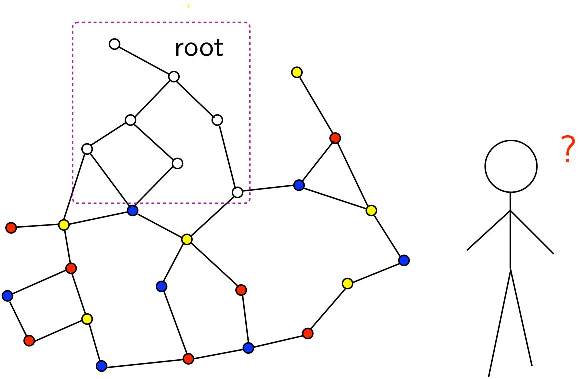
Alice samples a proper coloring (uniformly). . .



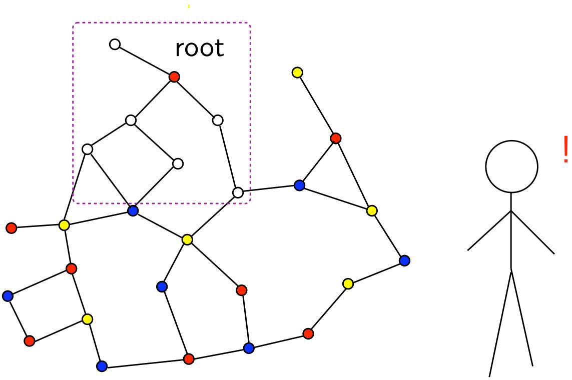
. . . and hides a ball B(root, t)



Bob. . .



. . . guesses right!



The problem

Does Bob have a chance?

Formally

$X = \{X_i : i \in V\}$ uniformly random proper coloring.

$\mu_U(\cdot\,|G)$ distribution of $X_U \equiv \{X_i : i \in U \subseteq V\}$

$\overline{\mathrm{B}}(r,t) = \{i \in V : d(i,r) \geq t\}$

Definition

The reconstruction problem is solvable for the sequence of random rooted graphs $G_n = (V_n = [n], E_n)$ if for some $\varepsilon > 0$,

$$||\mu_{r,\overline{\mathrm{B}}(r,t)}(\,\cdot\,,\,\cdot\,|G_n) - \mu_r(\,\cdot\,|G_n)\mu_{\overline{\mathrm{B}}(r,t)}(\,\cdot\,|G_n)||_{\mathrm{TV}} \geq \varepsilon\,,$$

with positive probability (bounded away from 0 as $n \to \infty$).

When G =Tree

- → Bleher, Ruiz, Zagrebenov (1995): Ising model on b-ary trees
- → Evans, Kenyon, Peres, Schulman (2000): Ising on general trees
- → Mossel, Peres (2003): Non binary variables
- → Brightwell, Winkler (2004), Martin (2004): Independent sets.
- → Chayes et al. (2006): Asymmetric Ising.

Pure states decomposition in q-COL

Theorem

$$\gamma_{\rm d}(q) \;:=\; \textit{Non-extremality (a.k.a. multiple pure states or dRSB)}$$
$$\overset{(1)}{=} \;\textit{Tree reconstruction threshold}$$
$$\overset{(2)}{=} \;\textit{Graph reconstruction threshold}$$

[Conjectured by Mézard, Montanari 06 that also prove (1) for regular tree-like graphs; (2) proved for Erdös-Rényi graphs by Montanari, Restrepo and Tetali 2011]

Conclusion

Combinatorics/Probability problems on random sparse graphs.

Unifying approach: approximation by trees.

Naturally leads to many interesting problems.

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Many challenges

1. General models and trees (e.g. ferromagnetic Potts for general limiting tree).

2. Ising spin glass - push $\beta_*(0, P)$ to the RSB point.

3. Rigorous understanding of the 'one-step replica symmetry' phase (as in q-coloring beyond $\gamma_{\rm d}(q)$) and beyond.

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If you want to know more about this. . .

- M. Mézard and A. Montanari, *Information, Physics, Computation*, Oxford Univ. Press. 2009.
- A. Dembo and A. Montanari, *Gibbs measures and phase transitions on sparse random graphs*, Brazilian J. of Probab. and Stat. 2010.
- A. Dembo, A. Gerschenfeld and A. Montanari, *Spin glasses on locally tree-like graphs*, in preparation
- A. Dembo, A. Montanari and N. Sun, *Factor models on locally tree-like graphs*, posted on ArXiv.