Potts and independent set models on $d$-regular graphs

Amir Dembo    Andrea Montanari    Allan Sly    Nike Sun

Stanford University    UC Berkeley

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1. The Potts and independent set models

2. Locally tree-like graphs and the Bethe prediction

3. Previous work and results

4. Verifying the Bethe prediction: proof ideas
The Potts and independent set models

Locally tree-like graphs and the Bethe prediction

Previous work and results

Verifying the Bethe prediction: proof ideas
Graphical models

A. Dembo, A. Montanari, A. Sly, N. Sun

Factor models on $d$-regular graphs
Graphical models

\[ G = (V, E) \text{ finite undirected graph} \]
Graphical models

$G = (V, E)$ finite undirected graph

**Spin configuration** $\sigma \in \mathcal{X}^V$

($\mathcal{X}$ finite alphabet)
\( G = (V, E) \) finite undirected graph

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(\( \mathcal{X} \) finite alphabet)

**Graphical model:**
Graphical models

\[ G = (V, E) \] finite undirected graph

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(\( \mathcal{X} \) finite alphabet)

**Graphical model:**
Model of random spin configuration
Graphical models

\[ G = (V, E) \] finite undirected graph

**Spin configuration** \( \sigma \in \mathcal{X}^V \)

(\( \mathcal{X} \) finite alphabet)

**Graphical model:**
Model of random spin configuration defined by **local** interactions
Factor models

\[ \mathbf{G} = (V, E) \]

\[ \text{judge interaction } \psi(\sigma_i, \sigma_j) \]

\[ \bar{\psi}(\sigma_i) \]

\[ \text{Rexternal field} \]

Taking product over all edges gives factor model

\[ \nu_G(\sigma) = \frac{1}{Z} \prod_{(ij) \in E} \psi(\sigma_i, \sigma_j) \prod_{i \in V} \bar{\psi}(\sigma_i) \]

\[ Z = \text{normalizing constant or partition function} \]
Factor models

Factor model on $G = (V, E)$:

$\nu_G(\sigma) = \frac{1}{Z} \prod_{(ij) \in E} \psi(\sigma_i, \sigma_j) \prod_{i \in V} \bar{\psi}(\sigma_i)$

$Z$ is the normalizing constant or partition function.
Factor models on $G = (V, E)$:

$$\psi(\sigma_i, \sigma_j)$$

Edge interaction $\psi(\sigma_i, \sigma_j)$,
Factor models

Factor model on $G = (V, E)$:

Edge interaction $\psi(\sigma_i, \sigma_j)$, vertex factor $\overline{\psi}(\sigma_i)$ (external field)
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Edge interaction $\psi(\sigma_i, \sigma_j)$, vertex factor $\bar{\psi}(\sigma_i)$ (external field)
Taking product over all edges, vertices gives factor model
The Potts model on a graph $G = (V, E)$ is defined by the partition function $Z_G(\beta, B)$, where $\beta$ is the inverse temperature and $B$ is the external field. The partition function is given by:

$$Z_G(\beta, B) = \prod_{(ij) \in E} e^{\beta \mathbb{1}\{\sigma_i = \sigma_j\}} \prod_{i \in V} e^{B \mathbb{1}\{\sigma_i = 1\}}$$

Here, $\mathbb{1}\{\cdot\}$ is the indicator function, taking the value 1 if the condition inside the brackets is true, and 0 otherwise. The parameter $\beta = \frac{1}{kT}$, where $k$ is the Boltzmann constant and $T$ is the temperature.

The external field $B$ acts on the distinguished spin 1, and $V$ is the set of spins in the graph.

Factor models on $d$-regular graphs can be analyzed within this framework.
\[ X = [q] \equiv \{1, \ldots, q\} \]
$\mathcal{X} = [q] \equiv \{1, \ldots, q\}$

**q-Potts model** on $G = (V, E)$:
\[ \mathcal{X} = \{q\} \equiv \{1, \ldots, q\} \]

**q-Potts model** on \( G = (V, E) \):

\[
\nu_{G}^{\beta, B}(\sigma) = \frac{1}{Z_G(\beta, B)} \prod_{(ij) \in E} e^{\beta \mathbf{1}\{\sigma_i = \sigma_j\}} \prod_{i \in V} e^{B \mathbf{1}\{\sigma_i = 1\}}
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**q-Potts model** on \( G = (V, E) \):

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\[ \beta^{-1} = \text{temperature} \]
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Potts model

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- \( \beta^{-1} = \) temperature
- \( B = \) external field, in direction of distinguished spin 1
- \( Z_{G}(\beta, B) = \) partition function
Potts model

Figure: David Wilson
\[ \nu^\beta_B G (\sigma) = \frac{1}{Z_G (\beta, B)} \prod_{(i,j) \in E} e^{\beta 1 \{\sigma_i = \sigma_j\}} \prod_{i \in V} e^{B 1 \{\sigma_i = 1\}} \]

is ferromagnetic

\[ \beta > 0 \]

is antiferromagnetic

- Ising model

- \( \beta = -\infty \)
Potts model

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- \( \beta > 0 \) is ferromagnetic; \( \beta < 0 \) is anti-ferromagnetic (AF)
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- $q = 2$: Ising model
Potts model

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- \( \beta > 0 \) is ferromagnetic; \( \beta < 0 \) is anti-ferromagnetic (AF)
- \( q = 2 \): Ising model
- \( \beta = -\infty \): random proper \( q \)-colorings
Independent set (hard-core) model

\[ \mathcal{X} = \{0, 1\} \]

The independent set \( \mathcal{R} \) of the hard-core model on \( G = (V, E) \) is defined as:

\[
\mathcal{G}(\lambda) = \frac{1}{Z_{\mathcal{G}}(\lambda)} \prod_{(i, j) \in E} \left( \sigma_i \sigma_j - 1 \right) \prod_{i \in V} \lambda \sigma_i
\]

Where \( \lambda \) is the fugacity or activity, and \( Z_{\mathcal{G}}(\lambda) \) is the partition function with \( Z_{\mathcal{G}}(1) \) being the number of independent sets.
Independent set (hard-core) model

\[ \mathcal{X} = \{0 = \text{unoccupied}, 1 = \text{occupied}\} \]
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The **independent set (IS)** or **hard-core model** on \( G = (V, E) \):

\[ Z_G(\lambda) = \text{partition function} \]

\[ Z_G(1) = \text{number of independent sets} \]
$\mathcal{X} = \{0 = \text{unoccupied}, 1 = \text{occupied}\}$

The independent set (IS) or hard-core model on $G = (V, E)$:

$$\nu^\lambda_G(\sigma) = \frac{1}{Z_G(\lambda)} \prod_{(ij) \in E} 1\{\sigma_i \sigma_j \neq 1\} \prod_{i \in V} \lambda^{\sigma_i}$$
\[ \mathcal{X} = \{0 = \text{unoccupied}, 1 = \text{occupied}\} \]

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- \( 1\{\sigma_i \sigma_j \neq 1\} \): hard constraints; repulsive interactions
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- \( Z_G(\lambda) \): partition function, with \( Z_G(1) = \) number of independent sets
Independent set (hard-core) model

Figure: David Wilson
Free energy density

Consider a sequence of random graphs $G_n$ with $n$ vertices in the thermodynamic limit $n \rightarrow \infty$.

Asymptotics of partition function $Z_n \equiv Z_{G_n}$?

Free energy $\phi_n \equiv n^{-1} E_n \left[ \log Z_n \right]$

Free energy density $\phi \equiv \lim_{n \rightarrow \infty} \phi_n$

Random growth rate of $Z_n$

Does $\phi$ exist? Can its value be computed?

The purpose of this work is to give an answer in the setting of locally tree-like graphs.

Factor models on $d$-regular graphs
Consider a sequence of (random) graphs $G_n$ ($n$ vertices) in the thermodynamic limit $n \to \infty$. 

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1. The Potts and independent set models

2. Locally tree-like graphs and the Bethe prediction

3. Previous work and results

4. Verifying the Bethe prediction: proof ideas
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2. Locally tree-like graphs and the Bethe prediction

3. Previous work and results

4. Verifying the Bethe prediction: proof ideas
Locally tree-like graphs
In what sense is the random 3-regular graph locally tree-like?
Locally tree-like graphs

In what sense is the random 3-regular graph locally like $T_3$?

random 3-regular graph

first few levels of $T_3$
$I_n \in V_n$ unif. random

$B_t(I_n)$ radius $t$ ball about $I_n$

Isomorph to $T_d^t$
(first $t$ levels of $T_d$)?
Locally tree-like graphs

$I_n \in V_n$ unif. random

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Isomorphic to $T^t_d$ (first $t$ levels of $T_d$)?
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Locally tree-like graphs

\[ I_n \in V_n \text{ unif. random} \]

\[ B_t(I_n) \text{ radius } t \text{ ball about } I_n \]

Isomorphic to \( T_d^t \)

(first \( t \) levels of \( T_d \))?
Locally tree-like graphs: definition

\[ G_n = (V_n, E_n) \] random graph sequence
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\[ I_n \in V_n \] uniformly random vertex
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Definition.
Locally tree-like graphs: definition

\[ G_n = (V_n, E_n) \] random graph sequence
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**Definition.**

\[ G_n \text{ converges locally to } T_d \text{ if for all } t \geq 0, \]
Locally tree-like graphs: definition

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**Definition.**
\[ G_n \text{ converges locally} \text{ to } T_d \text{ if for all } t \geq 0, \]
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Locally tree-like graphs: definition

\( G_n = (V_n, E_n) \) random graph sequence
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\( G_n \) converges locally to \( T_d \) if for all \( t \geq 0 \),
\( B_t(I_n) \) converges in probability to \( T_d^t \)

Notation: \( G_n \to_{loc} T_d \)
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Notation: \[ G_n \rightarrow_{loc} T_d \]

[Can also make definition with general (random) limiting tree]
Examples.
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The random $d$-regular graph converges locally to $T_d$.
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More generally, so does the random $k$-partite $d$-regular graph.
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The Erdős-Rényi graph $G(n, \gamma/n)$ converges locally to the Pois($\gamma$) Galton–Watson tree.
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Examples.

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$T_d^t$ does not converge locally to $T_d$, but rather to the random $d$-canopy tree.
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The random $d$-regular graph converges locally to $T_d$
More generally, so does the random $k$-partite $d$-regular graph
The Erdős-Rényi graph $G(n, \gamma/n)$ converges locally to
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$T_d^t$ does not converge locally to $T_d$,
but rather to the random $d$-canopy tree

Local weak limits are unimodular measures
on the space of rooted graphs.
For factor models on graph sequence $G_n \rightarrow_{loc} T$, the Bethe prediction is defined only in terms of limiting tree - not the finite graphs $G_n$. The Bethe prediction gives an explicit prediction for the free energy density $\phi \equiv \lim_{n \rightarrow \infty} \frac{1}{n} E_n \log Z_n$. 
For factor models on graph sequence $G_n \rightarrow_{loc} T$,

non-rigorous methods of statistical physics give an explicit prediction for free energy density $\phi \equiv \lim_n n^{-1} \mathbb{E}_n[\log Z_n]$:
The Bethe prediction

For factor models on graph sequence $G_n \to_{loc} T$, non-rigorous methods of statistical physics give an explicit prediction for free energy density $\phi \equiv \lim_n n^{-1} E_n [\log Z_n]$: the Bethe prediction (or replica symmetric solution)
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Bethe prediction is defined only in terms of limiting tree — not the finite graphs $G_n$. 

The Bethe prediction: definition

\[ \phi \equiv \lim_{n \to \infty} \frac{1}{n} \log Z_n \]
exists and equals the rethe free energy
\[ \Phi \equiv \Phi(h) \]
for \( h \in \Delta_R X \) simple a distinguished fixed point of the Bethe or belief propagation recursion

\[ h(\sigma) \sim \bar{\psi}(\sigma) \left( \sum \sigma' \psi(\sigma, \sigma') h(\sigma') \right) \]
The Bethe prediction: definition

Bethe prediction for factor models on $G_n \rightarrow_{loc} T_d$: \[ \phi \equiv \lim_{n \to \infty} \frac{1}{n} \log Z_n \] exists and equals the rethe free energy $\Phi \equiv \Phi(h)$ for $h \in \Delta$.
The Bethe prediction: definition

Bethe prediction for factor models on $G_n \rightarrow_{loc} T_d$:
$$\phi \equiv \lim_n n^{-1} \mathbb{E}_n[\log Z_n] \text{ exists}$$
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The Bethe prediction: definition

Bethe prediction for factor models on $G_n \to_{loc} T_d$:

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for $h \in \Delta$ ($\mathcal{X}$-simplex) a distinguished fixed point of the Bethe or belief propagation (BP) recursion:
The Bethe prediction: definition

Bethe prediction for factor models on $G_n \to_{loc} T_d$:

$$\phi \equiv \lim_{n} n^{-1} \mathbb{E}_n [\log Z_n]$$
exists and equals the Bethe free energy

$$\Phi \equiv \Phi(h)$$

for $h \in \Delta$ ($\mathcal{K}$-simplex) a distinguished fixed point of the Bethe or belief propagation (BP) recursion:

$$h(\sigma) \approx \psi(\sigma) \left( \sum_{\sigma'} \psi(\sigma, \sigma') h(\sigma') \right)^{d-1}$$
The Bethe prediction: functional form

\[ \Phi \equiv \Phi_{\text{vx}} - \Phi_{\text{e}} \]

where

\[ \Phi_{\text{vx}} \equiv \log \left( \sum \sigma \bar{\psi}(\sigma) \right) \]

\[ \Phi_{\text{e}} \equiv d^2 \log \left( \sum \sigma, \sigma' \psi(\sigma, \sigma') h(\sigma') \right) \]
The Bethe prediction: functional form

Functional form:
Functional form: $\Phi \equiv \Phi^v - \Phi^e$ where
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Functional form: $\Phi \equiv \Phi^{\text{vx}} - \Phi^e$ where

$$\Phi^{\text{vx}} \equiv \log \left\{ \sum_\sigma \overline{\psi}(\sigma) \left( \sum_{\sigma'} \psi(\sigma, \sigma') h(\sigma') \right)^d \right\}$$
The Bethe prediction: functional form

Functional form: $\Phi \equiv \Phi^{vx} - \Phi^e$ where

$$\Phi^{vx} \equiv \log \left\{ \sum_{\sigma} \bar{\psi}(\sigma) \left( \sum_{\sigma'} \psi(\sigma, \sigma') h(\sigma') \right)^d \right\}$$

$$\Phi^e \equiv \frac{d}{2} \log \left\{ \sum_{\sigma, \sigma'} \psi(\sigma, \sigma') h(\sigma) h(\sigma') \right\}$$
Interpretation of the BP fixed point:
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Suppose factor model $\nu_n$ on $G_n$ has local weak limit $\nu$. —
Interpretation of the BP fixed point:
Suppose factor model $\nu_n$ on $G_n$ has local weak limit $\nu$ — trans.-inv. Gibbs measure for factor model on $T_d$
The Bethe prediction: interpretation of BP recursion

**Interpretation of the BP fixed point:**
Suppose factor model $\nu_n$ on $G_n$ has local weak limit $\nu$ — trans.-inv. Gibbs measure for factor model on $T_d$

![Diagram of factor model]

Ignore long cycles
Interpretation of the BP fixed point:
Suppose factor model $\nu_n$ on $G_n$ has local weak limit $\nu$ — trans.-inv. Gibbs measure for factor model on $T_d$

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Boundary data $\approx$ i.i.d.
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Marginal of $\nu$ on $U \approx$
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Consistent family of marginals
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Suppose factor model $\nu_n$ on $G_n$ has local weak limit $\nu$ — trans.-inv. Gibbs measure for factor model on $T_d$

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Marginal of $\nu$ on $U \approx$
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Consistent family of marginals precisely when $h$ is a BP fixed point
Interpretation of the BP fixed point:
Suppose factor model $\nu_n$ on $G_n$ has local weak limit $\nu$ — trans.-inv. Gibbs measure for factor model on $T_d$

Ignore long cycles
Boundary data $\approx$ i.i.d.
Marginal of $\nu$ on $U \approx \nu(\sigma_{U} | \sigma_{\partial U}) \times \prod_{\nu \in \partial U} h(\sigma)$

Consistent family of marginals precisely when $h$ is a BP fixed point

BP fixed point $h$
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BP fixed point $h$

$\nu \equiv \nu_h$ candidate local weak limit of $\nu_n$
BP fixed point $h \leftrightarrow \nu \equiv \nu_h$ candidate local weak limit of $\nu_n$
BP fixed point $h \longleftrightarrow \nu \equiv \nu_h$ candidate local weak limit of $\nu_n$

Heuristic for $\Phi \equiv \Phi^{vx} - \Phi^e$: 

\[ \Phi^{vx}(h) = \log W_{\text{partition of marginal of } \nu_h \text{ on star graph } T} \]

\[ \Phi^e(h) = \frac{1}{2} \log W_{\text{partition on } d \text{ disjoint edges}} \]

\[ n \text{ stars } V \text{ contribution } \approx n \cdot \Phi^{vx} \]

\[ n \cdot \Phi^e \text{ to correct for overcounting} \]

\[ V \text{ subtract } n \cdot \Phi^e \]

Only a heuristic $G_n$ are typically not trees $J$
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Only a heuristic: $G_n$ are typically not trees!
The Bethe prediction: multiple fixed points

\[ \Phi(h) \equiv \Phi(\nu h) \]

is the heuristic formula for \(\phi\) assuming \(\nu \to \text{loc}\) for a fixed point \(h\) is non-unique. Therefore, the prediction becomes the supremum of \(\Phi(h)\) over fixed points.
\[ \Phi(h) \equiv \Phi(\nu_h) \text{ is (heuristic) formula for } \phi \text{ assuming } \nu_n \to_{loc} \nu_h \]
The Bethe prediction: multiple fixed points

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If BP fixed point \( h \) is non-unique, assume \( \nu_n \to_{loc} \text{mixture}(\nu_h) \)
The Bethe prediction: multiple fixed points

\[ \Phi(h) \equiv \Phi(\nu_h) \text{ is (heuristic) formula for } \phi \text{ assuming } \nu_n \to_{\text{loc}} \nu_h \]

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Bethe prediction becomes supremum of \( \Phi(h) \) over fixed points \( h \)
The Bethe prediction: some remarks

[Text content]

A. Dembo, A. Montanari, A. Sly, N. Sun

Factor models on $d$-regular graphs
Bethe prediction computation **only** involves infinite tree $T_d$, **not** specific graph sequence $G_n$. 
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The Bethe prediction: some remarks

Bethe prediction computation \textbf{only} involves infinite tree $T_d$, \textbf{not} specific graph sequence $G_n$.

Can make prediction for general (random) limiting trees.

Bethe prediction is “\textit{replica symmetric}” in the sense that there is a fixed Gibbs measure $\nu$ in definition of $\Phi(\nu)$ — equivalently, take \textit{same} $h$ at each boundary vertex.
Bethe prediction specialized to ferromagnetic Potts:
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Translation-invariant Gibbs measures
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Translation-invariant Gibbs measures
\( \nu^f \) (free) and \( \nu^1 \) (maximally 1-biased)
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Bethe prediction specialized to IS and AF Ising:
Bethe prediction for AF two-spin systems

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For $G_n$ non-bipartite, same prediction believed to hold in uniqueness regimes only
1. The Potts and independent set models

2. Locally tree-like graphs and the Bethe prediction

3. Previous work and results

4. Verifying the Bethe prediction: proof ideas
Outline

1. The Potts and independent set models
2. Locally tree-like graphs and the Bethe prediction
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4. Verifying the Bethe prediction: proof ideas
Previous work: ferromagnetic Ising

Ferromagnetic Ising:
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[Dembo–Montanari AAP ’10] verified Bethe prediction for all $\beta \geq 0$, $B \in \mathbb{R}$, for graphs converging locally to Galton-Watson trees.
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Moment condition on root vertex degree later removed

[Dommers–Giardinà–van der Hofstad JSP ’10]
Ferromagnetic Ising:

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Proofs use an interpolation scheme, comparing $\partial_\beta \phi_n$ with $\partial_\beta \Phi$. 
Results: Ferro. Potts on general limiting tree

Theorem

The prediction $\phi = \Phi$ holds on locally tree-like graphs with general limiting tree for Potts at any $B \in \mathbb{R}$ with $B \geq 0$ with $\beta$ sufficiently low.
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Results: Potts on $T_d$

Theorem Ronvanov Potts model on $G_n \to \text{loc} T_d$

$\lim \inf \frac{\phi_n}{n} \geq \Phi$ for all $\beta, B \geq 0$

Theorem Ronvanov Slyv SunabS

Factor models on $d$-regular graphs
Can obtain sharper results when $G_n \rightarrow_{loc} T_d$: 
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Previous work: AF two-spin free energy density

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IS, AF Ising:
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\textbf{IS, AF Ising:}

Bethe prediction $\phi = \Phi$ holds for random regular graphs below uniqueness threshold

[Bandyopadhyay–Gamarnik SODA '06]
**Previous work: AF two-spin free energy density**

**IS, AF Ising:**

Bethe prediction $\phi = \Phi$ holds for random regular graphs below uniqueness threshold

[Bandyopadhyay–Gamarnik SODA ’06]

**Existence** of $\phi$ for random regular graphs and Erdős-Rényi graphs

[Bayati–Gamarnik–Tetali STOC ’10]
Results: AF two-spin free energy density
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For the Ising and IS models on $G_n \rightarrow_{loc} T_d$ with $G_n$ bipartite, $\phi = \Phi$ for all parameter values.
Two-spin systems — algorithmic results:
Complexity of two-spin systems

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Ferromagnetic:
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**Ferromagnetic:**

FPRAS for ferro. Ising at all temperatures, arbitrary magnetic field

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AF two-spin systems have uniqueness thresholds on $T_d$: 
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AF two-spin systems have uniqueness thresholds on $T_d$:
$\lambda_c(d)$ for IS, $\beta_{af}(B, d) < 0$ for AF Ising
Complexity of two-spin systems

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- **FPTAS** for IS partition function \( Z_G(\lambda) \) on bdd. deg. graphs,
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- FPTAS for IS partition function $Z_G(\lambda)$ on bdd. deg. graphs,
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- FPTAS for AF Ising partition function $Z_G(\beta, B)$ on bdd. deg.
  graphs, $\beta_{c}^{af}(B, d) < \beta < 0$ [Sinclair–Srivastava–Thurley ’11]
Complexity of AF two-spin systems

Hardness results for IS:
Complexity of AF two-spin systems

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\[ Z_G(\lambda) \text{ hard to approximate on } d\text{-regular graphs when } \lambda > c/d \]  
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Complexity of AF two-spin systems

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Complexity of AF two-spin systems

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**Phase transition at } \lambda_c(d):**
Complexity of AF two-spin systems

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Phase transition at \( \lambda_c(d) \):

- [Mossel–Weitz–Wormald PTRF ’09] Local \( \text{MCMC} \) mixes slowly on random bipartite \( d\)-reg. graphs, \( \lambda_c(d) < \lambda < \lambda_c(d) + \epsilon(d) \)
Complexity of AF two-spin systems

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Complexity of AF two-spin systems

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Complexity of AF two-spin systems

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Complexity of AF two-spin systems

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  — first rigorous indication that computational transition for finite \(d\)-regular graphs \(\leftrightarrow\) statistical physics phase transition for the model on \(T_d\)
- Subsequently improved to \(\lambda > \lambda_c(d)\) for \(d \neq 4, 5\) [Galanis–Ge–Štefankovič–Vigoda–Yang '11]
Recent: complexity of AF two-spin systems

Theorem

RSlyV Sun 'abS

RaS vor

\[ d \geq 3, \quad \lambda > \lambda_c(d) \]

the partition function \( Z_G(\lambda) \)

is hard to approx

on the class of \( d \)

regular graphs.

RbS vor

\[ d \geq 3, \quad \beta \leq \beta_{af}(B, d) \]

the partition function \( Z_G(\beta, B) \)

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NonWtrivial twoWspin systems on \( d \)

regular graphs reduce to yS

RaS O

[Weitz STOC 'ov]

complete classification of hardWcore complexity except at \( \lambda_c(d) \)

RbS O

[Jerrum–Sinclair ALP '–o]

complete classification of ysing complexity except at \( \beta_{af}(B, d) \)

ynterpolation O methods from

[Montanari–Mossel–Sly PTRF 'pr]

circumvent di ffi

valanis–ˇ Stefankoviˇc–Vigoda R and RbS with

B = 0

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Factor models on \( d \)-regular graphs
Theorem (Sly, Sun ’12).
Recent: complexity of AF two-spin systems

**Theorem** (Sly, Sun ’12).

(a) For \( d \geq 3, \lambda > \lambda_c(d) \) the IS partition function \( Z_G(\lambda) \) is hard to approx. on the class of \( d \)-regular graphs.
Recent: complexity of AF two-spin systems

**Theorem** (Sly, Sun ’12).

(a) For $d \geq 3, \lambda > \lambda_c(d)$ the IS partition function $Z_G(\lambda)$ is hard to approx. on the class of $d$-regular graphs.

(b) For $d \geq 3, \beta < \beta_{af}(B,d)$, the Ising partition function $Z_G(\beta, B)$ is hard to approx. on the class of $d$-regular graphs.
Recent: complexity of AF two-spin systems

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(b) For $d \geq 3, \beta < \beta^\text{af}_c(B, d)$, the Ising partition function $Z_G(\beta, B)$ is hard to approx. on the class of $d$-regular graphs.

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Independently, Galanis–Štefankovič–Vigoda ’12 establish (a), and (b) with \( B = 0 \).
1. The Potts and independent set models

2. Locally tree-like graphs and the Bethe prediction

3. Previous work and results

4. Verifying the Bethe prediction: proof ideas
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4. Verifying the Bethe prediction: proof ideas
Proof ideas: interpolation scheme for factor models

Recall \( \phi_n = n^{-1} \log Z_j \Rightarrow \partial_B \phi_n = \text{avg}[\text{local observable wrt } \nu] \)

\[ \limsup_n \left[ \phi_n(B_1) - \phi_n(B_0) \right] \leq \Phi(B_1) - \Phi(B_0) \]

San show \( \partial_B \Phi(\nu) = \text{avg}[\text{same observable at root of } T \text{ wrt } \nu] \)

Wibbs measure unique \( \Rightarrow \) observable averages on \( G \) converge to averages on \( T \) by general theory \( \Rightarrow \)

Sometimes obtain beyond uniqueness from model-specific monotonicity properties
Proof ideas: interpolation scheme for factor models

(Dembo, Montanari, Sun '11)
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Generalized interpolation scheme for abstract factor models
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Basic idea: if \( \limsup_n \partial_B \phi_n \leq \partial_B \Phi \) \( \star \)
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Can sometimes obtain \( \star \) beyond uniqueness from
(model-specific) \((\text{anti-})\text{monotonicity}\) properties
Proof ideas: BP recursion on $T_d$

The recursion on $T_d$ is simply a map $\Delta \to \Delta$

where $\sigma \sim \bar{\psi}(\sigma) = \sum_{\sigma'} \psi(\sigma, \sigma') h(\sigma')$.

By explicitly analyzing this mapping, we can obtain more exact results for $T_d$ than are implied by the interpolation scheme for general trees.
Proof ideas: BP recursion on $T_d$

BP recursion on general limiting trees is complicated, but
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BP recursion on general limiting trees is complicated, but BP recursion on $T_d$ is simply a map $\Delta \rightarrow \Delta$: 

$$\bar{\psi}(\sigma) = \sum_{\sigma'} \psi(\sigma, \sigma') h(\sigma')^{d-1}$$
Proof ideas: BP recursion on $T_d$

BP recursion on general limiting trees is complicated, but BP recursion on $T_d$ is simply a map $\Delta \rightarrow \Delta$:

$$h(\sigma) \approx \bar{\psi}(\sigma) \left( \sum_{\sigma'} \psi(\sigma, \sigma') h(\sigma') \right)^{d-1}$$
Proof ideas: BP recursion on $T_d$

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For AF two-spin systems on bipartite graphs, complete Bethe prediction can be verified by interpolation with a good choice of the local observable.
Proof ideas: AF two-spin systems on bipartite graphs

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\[ \partial_B \phi_n = \mathbb{E}_n[\sigma_{I_n}] \] (with \( B \equiv \log \lambda \) for IS)
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With obvious observable \( i \mapsto \sigma_i \), can show \( \phi = \Phi \) for \( \lambda \leq \lambda_c \)
Proof ideas: AF two-spin systems on bipartite graphs

For AF two-spin systems on bipartite graphs, complete Bethe prediction can be verified by interpolation with a good choice of the local observable

$$\partial_B \phi_n = \mathbb{E}_n[\sigma_{I_n}] \quad \text{(with } B \equiv \log \lambda \text{ for IS)}$$

With obvious observable $i \mapsto \sigma_i$, can show $\phi = \Phi$ for $\lambda \leq \lambda_c$

But by taking observable $i \mapsto (\sigma_i + d^{-1} \sum_{j \in \partial i} \sigma_j)/2$

can show $\phi = \Phi$ for all $\lambda > 0$
IS BP recursion (in terms of $h(0)$)

\[ \lambda = 0.5 \]
IS BP recursion

IS BP recursion (in terms of $h(0)$)

$\lambda = 0.9$
IS BP recursion (in terms of $h(0)$)

$\lambda = 1.5875$
IS BP recursion (in terms of $h(0)$)

\[ \lambda = 1.6875 \]
IS BP recursion (in terms of $h(0)$)

Semi-translation-invariant solutions arise above $\lambda_c$
IS BP recursion (in terms of $h(0)$)

$\lambda = 1.6975$

Semi-translation-invariant solutions arise above $\lambda_c$
IS BP recursion

IS BP recursion (in terms of $h(0)$)

$\lambda = 1.7875$

Semi-translation-invariant solutions arise above $\lambda_c$
IS BP recursion (in terms of $h(0)$)

$\lambda = 2.$

Semi-translation-invariant solutions arise above $\lambda_c$
IS BP recursion (in terms of $h(0)$)

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Semi-translation-invariant solutions arise above $\lambda_c$
IS BP recursion (in terms of $h(0)$)

$\lambda = 4.$

Semi-translation-invariant solutions arise above $\lambda_c$
Use bipartite property to interpolate semiWtrans\[Winv\[ fixed point from λ = ∞
IS free energy density

\[ IS(\lambda) = S(\lambda) - S(\infty) \]

The graph illustrates the free energy density \( IS(\lambda) \) as a function of \( \lambda \). The graph shows the behavior of the free energy density under varying values of \( \lambda \), with \( \phi \) on the y-axis and \( \lambda \) on the x-axis. The bipartite property is used to interpolate the semi-W transform from a fixed point.
IS free energy density

\[ \phi(\lambda) \]

Use bipartite property to interpolate semi-Wtrans from fixed point.
IS free energy density

Use bipartite property to interpolate semi $W_{\lambda}$ from fixed point $\lambda = \infty$.
Use bipartite property to interpolate semi-trans.-inv. fixed point from $\lambda = \infty$
In Potts model, $\partial_B \phi_n = \mathbb{E}_n \mathbb{E}_{\nu_n} [\delta_{\sigma_{I_n}, 1}]$, 
Proof ideas: interpolation for Potts

In Potts model, \( \partial_B \phi_n = \mathbb{E}_n \mathbb{E}_{\nu_n} [\delta_{\sigma_{I_n}}, 1] \), so local observable is simply \( \nu \mapsto \delta_{\sigma_v, 1} \)
In Potts model, \( \partial_B \phi_n = \mathbb{E}_n \mathbb{E}_{\nu_n} [\delta_{\sigma_{I_n},1}] \),
so local observable is simply \( v \mapsto \delta_{\sigma_v,1} \)

Similarly \( \partial_\beta \phi_n = \mathbb{E}_n \mathbb{E}_{\nu_n} [\sum_{j \in \partial I_n} \delta_{\sigma_{I_n},\sigma_j}] \)
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In non-uniqueness regimes, can take advantage of
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In non-uniqueness regimes, can take advantage of random-cluster (FK) representation for Potts model
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In Potts model, \( \partial_B \phi_n = \mathbb{E}_n \mathbb{E}_\nu_n [\delta_{\sigma_{I_n},1}] \), so local observable is simply \( \nu \mapsto \delta_{\sigma,1} \)

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In non-uniqueness regimes, can take advantage of random-cluster (FK) representation for Potts model to get monotonicity properties, thereby restricting range of admissible Gibbs measures
Ising vs. Potts

Ising BP (in terms of $\log[h(+) / h(-)]$)

$\beta = 0.1$
Ising vs. Potts

**Ising BP** (in terms of $\log[h(+) / h(-)]$)

\[ \beta = 0.5 \]
Ising vs. Potts

Ising BP (in terms of $\log[h(+) / h(-)]$)

$\beta = \log(2)$
Ising vs. Potts

Ising BP (in terms of $\log[h(+) / h(-)]$)

\[ \beta = 1.3 \]
Ising vs. Potts

Ising BP (in terms of $\log[h(+) / h(-)]$)

BP solutions as function of $\beta$

$\beta = 1.3$
Ising vs. Potts

**Ising BP** (in terms of $\log[h(+)/h(-)]$)

![Graph showing Ising BP solutions as function of $\beta$.]

$\beta = 1.3$

**BP solutions as function of $\beta$**

![Graph showing BP solutions as function of $\beta$.]

Adding small field $B > 0$ resolves non-uniqueness
Potts BP (in terms of \( \log[h(1)/h(2)] \))

\[ \beta = 0.5 \]
Ising vs. Potts

Potts BP (in terms of $\log[h(1)/h(2)]$)

$\beta = 1.5$
Potts BP (in terms of $\log[h(1)/h(2)]$)

$\beta = 1.7162$
Ising vs. Potts

Potts BP (in terms of $\log[h(1)/h(2)]$)

$\beta = 2$
Ising vs. Potts

Potts BP (in terms of $\log[h(1)/h(2)]$)

$\beta = 2.6$
Ising vs. Potts

Potts BP (in terms of $\log[h(1)/h(2)]$)

$\beta = 4$
Ising vs. Potts

Potts BP (in terms of $\log[h(1)/h(2)]$)

$\beta = 4$

BP solutions as function of $\beta$

Free
Ising vs. Potts

Potts BP (in terms of $\log[h(1)/h(2)]$)

$\beta = 4$

BP solutions as function of $\beta$

maximally 1-biased

free
Ising vs. Potts

Potts BP (in terms of $\log[h(1)/h(2)]$)

$\beta = 4$

BP solutions as function of $\beta$

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Potts BP (in terms of $\log[h(1)/h(2)]$)

\[ \beta = 4 \]

Adding $B > 0$ not enough to resolve non-uniqueness
Potts: $\phi \geq \Phi$ by interpolation
Potts: $\phi \geq \Phi$ by interpolation
Potts: $\phi \geq \Phi$ by interpolation
Potts: $\phi \geq \Phi$ by interpolation
Potts: $\phi \geq \Phi$ by interpolation

Factor models on $d$-regular graphs
Potts: $\phi \geq \Phi$ by interpolation
Potts: $\phi \geq \Phi$ by interpolation

An Dembo An Montanari An Slyl Nn Sun

Factor models on $d$-regular graphs
Interpolation gives $\phi \geq \Phi$. 
Interpolation gives $\phi \geq \Phi$, with equality for $(\beta, B) \notin \mathcal{R}_\neq$ (shaded)
Interpolation gives $\phi \geq \Phi$, with equality for $(\beta, B) \notin \mathcal{R}_\neq$ (shaded).

Different approach needed to obtain equality inside $\mathcal{R}_\neq$. 
Potts: $\phi \leq \Phi$ by graph deconstruction
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Potts: $\phi \leq \Phi$ by graph deconstruction

Delete a vertex

\[ \text{Diagram showing vertex deletion process.} \]
Potts: $\phi \leq \Phi$ by graph deconstruction

Delete a vertex

Match up half edges
Potts: $\phi \leq \Phi$ by graph deconstruction

Delete a vertex
Match up half edges
Show decrease in $\log Z$
at each step is $\leq \Phi$ ★
Potts: $\phi \leq \Phi$ by graph deconstruction

Delete a vertex

Match up half edges

Show decrease in $\log Z$

at each step is $\leq \Phi \star$

Matching not done u.a.r.

but to guarantee $\star$

\[ \phi \leq \Phi \text{ by graph deconstruction} \]
Potts: $\phi \leq \Phi$ by graph deconstruction

Delete a vertex

Match up half edges

Show decrease in $\log Z$ at each step is $\leq \Phi$

Matching **not** done u.a.r. but to guarantee ★

Argue graphs remain uniformly locally tree-like
Potts: $\phi \leq \Phi$ by graph deconstruction

Delete a vertex

Match up half edges

Show decrease in $\log Z$ at each step is $\leq \Phi \star$

Matching not done u.a.r. but to guarantee $\star$

Argue graphs remain uniformly locally tree-like

This procedure reduces the upper bound to showing $\star$, which is a difficult (but tractable) calculus problem
Two questions

We make crucial use of the fact that the limiting tree is $T_d$. Can these methods be extended to more general graph ensembles? The prediction is believed to be false for $S_1$ at high fugacity on typical nonbipartite graphs converging to $T_d$. Can we describe what happens in this case?
Two questions

- We make crucial use of the fact that the limiting tree is $T_d$. Can these methods be extended to more general graph ensembles, e.g. Erdős-Rényi?
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- We make crucial use of the fact that the limiting tree is $T_d$. Can these methods be extended to more general graph ensembles, e.g. Erdős-Rényi?

- The Bethe prediction is believed to be false for IS at high fugacity on typical non-bipartite graphs converging to $T_d$. Can one describe what happens in this case?