Persistence Probabilities

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[1]Jointly with Jian Ding, Fuchang Gao, Sumit Mukherjee, Bjorn Poonen, Qi-Man Shao and Ofer Zeitouni.

Persistence probabilities

$(X_t, t \geq 0)$ is $\mathbb{R}$-valued $(X_0 = 0)$. First passage time

$$\tau_z = \inf\{t > 0 : X_t > z\}\,.$$

- Typically $z = 0$.
- Persistence (power law) exponent b if $\mathbb{P}(\tau_z \geq T) = T^{-b+o(1)}$.
- Large deviations problem (hitting probab. of Markov processes).
- In many examples, little gained from general theories.
- Focus on few examples of much interest.

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- Up to factor 2 same as $\mathbb{P}(Q_n(\cdot)$ has no roots in $J)$.

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- $p_{\mathbb{R}}(n) \sim \mathbb{P}(Q_n$ has no real roots$) = \mathbb{P}(N_n = 0)$.
- $\mathbb{E}(N_n) \sim c_\alpha \log n$ if ξ_i attracted to $\alpha \in (0, 2]$ stable [Ibragimov-Maslova/Logan-Shepp '68-'71; following Kac '43].

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- Much work on complex roots (e.g. [Edelman-Kostlan '95; Ibragimov-Zeitouni '97], more).

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$\mathcal{C}^\infty(\mathbb{R})$-valued, stationary centered Gaussian process $t \mapsto Y_t^{(\kappa)}$ of auto-covariance $(\mathrm{sech}((t-s)/2))^{\kappa+1}$, $\kappa > -1$ has

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Valid if ξ_i of zero mean and finite moments
(by KMT reduce to $\xi_i \sim N(0,1)$).

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  - Auto-correlation of $Q_n(e^{-e^{-s}})$ explicit $\&$ close to that of $Y_s^{(0)}$.

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  - When $T_n \epsilon_n \to \infty$ persistence power exponent can be discontinuous (e.g. $\sqrt{1-\epsilon_n} Y_s + \sqrt{\epsilon_n} Z$).
  - Resolved in [Dembo-Mukherjee '12] by new theorem about persistence exponent continuity for *summable*, non-negative Gaussian auto-correlations.

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  - α-stable ξ_i no result/method;
    For $\alpha = 1$, numerically persistence exponent near $0.86 > 4b_0$.

Heat equation initiated by white noise

- Classical solution of d-dimensional heat equation

$$\frac{\partial \phi_d(\mathbf{x}, t)}{\partial t} = \Delta \phi_d(\mathbf{x}, t)$$

with $\phi_d(\mathbf{x}, 0) = \psi(\mathbf{x})$ centered, Gaussian field of covariance $\delta_d(\mathbf{x} - \mathbf{y})$ (white noise).

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- No intuitive explanation of this connection!

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$S_i^{(0)} = \xi_i$ i.i.d. of zero mean; $S_0^{(\ell)} = 0$. For $\ell, k \geq 1$,

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  - $r_\ell \downarrow b_0$ for $\ell \uparrow \infty$ [Li-Shao '05].

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  - Handle $\ell = 2$ and infinite variance (see [Aurzada-Simon '12]).

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Sharp asymptotics beyond r_2

$n \mapsto n^{r_\ell} q_\ell(n)$ uniformly bounded above and below for:

- $\ell = 1$ any ξ_i of finite second moment [Feller '71].
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  - [Dembo-Ding-Gao '12]: approximate factorization and show $\arg\max_k S_k^{(2)}$ not concentrated near $\{0, n\}$.
- Rigid local maximizers for $\ell \geq 3$ ruin factorization approach!

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For $\xi_i \sim N(0,1)$ i.i.d.

$$X_k = 0, \forall k \leq 0, \quad \& \quad X_k = \sum_{i=1}^{L} a_i X_{k-i} + \xi_k, \qquad \forall k \geq 1$$

a Gaussian auto-regressive sequence.

Gaussian AR sequences

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Stability of $k \mapsto X_k$ according to $\rho := \max\{|z| : z \in \Lambda\}$,

$$\Lambda := \{z \in \mathbb{C} : z^L - \sum_{i=1}^{L} a_i z^{L-i} = 0\}.$$

Persistence for AR sequences

- [Dembo-Ding '13] study asymptotics of

$$q_{\underline{a}}(n) := \mathbb{P}(X_k < 0, \forall k \in [1, n])\,.$$

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    if $\rho = \beta = 1$ or $\rho > 1$ and $\beta \in (0,1)$.
- $\theta \mapsto r_{\underline{a}}$ discontinuous at $\theta \in \mathbb{Q}$ when $\Lambda = \{1, e^{2\pi i\theta}, e^{-2\pi i\theta}\}$!

Thank you!