Chaos of a Markov operator and the fourth moment condition

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fourth-moment theorem

D. Nualart, G. Peccati (2005)

condition for a Wiener chaos

to be close to Gaussian

multiple Wiener integrals

multiple Wiener integrals

simplified (finite-dimensional) model

Wiener (Gaussian) chaos of order k

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Wiener (Gaussian) chaos of order k

$$F = F(x) = \sum_{i_1,\dots,i_k=1}^{N} a_{i_1,\dots,i_k}\, x_{i_1} \cdots x_{i_k}, \qquad x = (x_1, \dots, x_N) \in \mathbb{R}^N$$

multi-linear form

$a_{i_1,\dots,i_k} \in \mathbb{R}$ symmetric, vanishing on diagonals

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multi-linear form

$a_{i_1,\ldots,i_k} \in \mathbb{R}$ symmetric, vanishing on diagonals

$$\int_{\mathbb{R}^N} F^2 d\gamma_N = 1 \qquad \Big(\int_{\mathbb{R}^N} F\, d\gamma_N = 0\Big)$$

$d\gamma_N(x) = e^{-|x|^2/2}\, \dfrac{dx}{(2\pi)^n}$ standard Gaussian measure on $\mathbb{R}^N$

multiple Wiener integrals

simplified (finite-dimensional) model

Wiener (Gaussian) chaos of order k

$$F = F(X) = \sum_{i_1,\ldots,i_k=1}^{N} a_{i_1,\ldots,i_k} X_{i_1} \cdots X_{i_k}$$

$X_1, \ldots, X_N$ independent standard normal

$a_{i_1,\ldots,i_k} \in \mathbb{R}$ symmetric, vanishing on diagonals

$$\mathbb{E}\big(F(X)^2\big) = 1 \qquad \big(\mathbb{E}\big(F(X)\big) = 0\big)$$

$$F = F_n, \quad n \in \mathbb{N} \qquad k\text{-chaos} \quad (k \quad \text{fixed})$$

$$N = N_n \to \infty$$

$$\int_{\mathbb{R}^{N^n}} F_n^2 \, d\gamma_{N_n} = 1 \qquad (\text{or} \quad \to 1)$$

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Theorem by **D. Nualart, G. Peccati (2005)**

distribution of F_n converges to γ_1 (standard normal on $\mathbb{R}$)

if and only if

$$\int_{\mathbb{R}^{N_n}} F_n^4 \, d\gamma_{N_n} \to 3$$

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striking reduction of the moment method

original proof

stochastic calculus

original proof

stochastic calculus

Wiener chaos (multiple Wiener integrals)

$$F = \int_{[0,1]^k} f(t_1, \ldots, t_k) dB_{t_1} \cdots dB_{t_k}$$

$f \in \mathrm{L}^2([0,1]^k; \mathbb{R})$ symmetric

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$$F = I_k(f)$$

main tool : multiplication formula

$$I_k(f)\, I_\ell(g) \;=\; \sum_{r=0}^{k\wedge\ell} r!\binom{k}{r}\binom{\ell}{r}\, I_{k+\ell-2}(f\,\widetilde{\otimes}_r g)$$

main tool : multiplication formula

$$H_k \, H_\ell = \sum_{r=0}^{k \wedge \ell} r! \binom{k}{r} \binom{\ell}{r} H_{k+\ell-2r}$$

H_k Hermite polynomials

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contraction

$$f \otimes_r g = \int_{[0,1]^r} f(t_1,\ldots,t_{k-r},s_1,\ldots,s_r)\times$$

$$g(t_{k-r+1},\ldots,t_{k+\ell-2r},s_1,\ldots,s_r)ds_1\cdots ds_r$$

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$\int_{\mathbb{R}^{N_n}} F_n^4\, d\gamma_{N_n} \,\to\, 3$ implies

$$\big\| f_n \,\widetilde{\otimes}_p\, f_n \big\|_2 \,\to\, 0, \qquad p = 1,\dots,k-1$$

combinatorial arguments

D. Nualart, G. Peccati (2005)

$I_k(f_n) = W_{T_n}$ time change $T_n \to 1$

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moments $\mathbb{E}\big(I_k(f_n)^q\big) \to \int_{\mathbb{R}} x^q d\gamma_1$

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further equivalence

$\mathrm{Var}_{\gamma_{N_n}}\big(|\nabla F_n|^2\big) \to 0$

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I. Nourdin, J. Rosinski (2012)

covariance criterion

first objectives

Gaussian k-chaos

$$F = F(x) = \sum_{i_1,\ldots,i_k=1}^{N} a_{i_1,\ldots,i_k} \, x_{i_1} \cdots x_{i_k}$$

first objectives

Gaussian k-chaos

$$F = F(x) = \sum_{i_1,\ldots,i_k=1}^{N} a_{i_1,\ldots,i_k} x_{i_1} \cdots x_{i_k}$$

understand what is used on F

why a fourth moment condition $\int_{\mathbb{R}^N} F^4 d\gamma_N \sim 3$

connection with $\mathrm{Var}_{\gamma_N}\big(|\nabla F|^2\big)$

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first feature : **eigenfunction,** eigenvalue k

$$-\mathrm{L}F = k\,F$$

$\mathrm{L} = \Delta - x \cdot \nabla$ Ornstein-Uhlenbeck operator on $\mathbb{R}^N$

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invariant (reversible) measure $\gamma_N = \gamma$

integration by parts

$$\int_{\mathbb{R}^N} f(-\mathrm{L}g)d\gamma = \int_{\mathbb{R}^N} \nabla f \cdot \nabla g \, d\gamma$$

connection with $\mathrm{Var}_{\gamma_N}(|\nabla F|^2)$

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$$k \int_{\mathbb{R}^N} F^4 d\gamma$$

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$$k\int_{\mathbb{R}^N} F^4 d\gamma = \int_{\mathbb{R}^N} F^3(-\mathrm{L}F)d\gamma$$

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$$k\int_{\mathbb{R}^N} F^4 d\gamma = \int_{\mathbb{R}^N} F^3(-\mathrm{L}F)d\gamma = 3\int_{\mathbb{R}^N} F^2|\nabla F|^2 d\gamma$$

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similarly $$k \int_{\mathbb{R}^N} F^2 d\gamma = \int_{\mathbb{R}^N} F\,(-\mathrm{L}F) d\gamma = \int_{\mathbb{R}^N} |\nabla F|^2 d\gamma$$

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normalization $\int_{\mathbb{R}^N} F^2 d\gamma = 1$

$$\int_{\mathbb{R}^N} |\nabla F|^2 d\gamma = k$$

connection with $\mathrm{Var}_{\gamma_N}(|\nabla F|^2)$

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normalization $\quad \int_{\mathbb{R}^N} F^2 d\gamma = 1$

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$$\mathrm{Var}_\gamma(|\nabla F|^2) = \int_{\mathbb{R}^N} \big(|\nabla F|^2 - k\big)^2 d\gamma$$

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technical task

from $\displaystyle\int_{\mathbb{R}^N} F^2 \big(|\nabla F|^2 - k \big) d\gamma$ to $\displaystyle\int_{\mathbb{R}^N} \big(|\nabla F|^2 - k \big)^2 d\gamma$

main step

if $\int_{\mathbb{R}^N} F^4 d\gamma \sim 3$

then $\mathrm{Var}_\gamma\big(|\nabla F|^2\big) = \int_{\mathbb{R}^N} \big(|\nabla F|^2 - k\big)^2 d\gamma \sim 0$

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if $|\nabla F|^2 \sim k$ $\quad \big(\mathrm{Var}_\gamma(|\nabla F|^2) \sim 0\big)$

then the distribution of F

is approximatively Gaussian

second (main) step

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first step

if $$|\nabla F|^2 \sim k \qquad \big(\mathrm{Var}_\gamma\big(|\nabla F|^2\big) \sim 0\big)$$

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first step

Ornstein-Uhlenbeck operator on $\mathbb{R}^N$

$$\mathrm{L} = \Delta - x \cdot \nabla$$

first step

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$F : \mathbb{R}^N \to \mathbb{R}$ Gaussian chaos

F eigenfunction of L

$$-\mathrm{L}F = \lambda F \qquad (\lambda > 0)$$

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$\varphi : \mathbb{R} \to \mathbb{R}$ smooth

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chain rule formula for L (Laplacian)

first step

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chain rule formula for L (Laplacian)

$$\mathrm{L}(\varphi \circ F)$$

first step

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$$\mathrm{L}(\varphi \circ F) = \varphi'(F)\,\mathrm{L}F + \varphi''(F)\,|\nabla F|^2$$

first step

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$$\mathrm{L}(\varphi \circ F) = \varphi'(F)\,\mathrm{L}F + \varphi''(F)\,|\nabla F|^2 = -\lambda F \varphi'(F) + \varphi''(F)\,|\nabla F|^2$$

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if $|\nabla F|^2 = \lambda$ then

$$\mathrm{L}(\varphi \circ F) = -\lambda F \varphi'(F) + \lambda \varphi''(F)$$

$$\mathrm{L}(\varphi \circ F) = \lambda\,(\mathcal{L}\varphi)(F)$$

$\mathcal{L}\psi = \psi'' - x\psi'$ on $\mathbb{R}$ (one-dimensional O-U operator)

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$\gamma_{\#F}$ distribution of $F : \mathbb{R}^N \to \mathbb{R}$ under γ

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$\gamma_{\#F}$ distribution of $F : \mathbb{R}^N \to \mathbb{R}$ under γ

$$0 = \int_{\mathbb{R}^N} \mathrm{L}(\varphi \circ F) d\gamma = \lambda \int_{\mathbb{R}} \mathcal{L}\varphi\, d\gamma_{\#F}$$

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$\gamma_{\#F}$ invariant measure of $\mathcal{L}$

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$\gamma_{\# F}$ invariant measure of $\mathcal{L}$

$$\gamma_{\# F} = \gamma_1$$

Stein's method

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quantify the preceding

I. Nourdin, G. Peccati (2009)

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$$-\mathrm{L}\,F = \lambda\,F$$

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quantify the preceding

I. Nourdin, G. Peccati (2009)

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$\gamma_{\# F}$ distribution of $F : \mathbb{R}^N \to \mathbb{R}$ under γ

$$\left| \int_{\mathbb{R}} \varphi \, d\gamma_{\# F} - \int_{\mathbb{R}} \varphi \, d\gamma_1 \right| \leq \frac{C_\varphi}{\lambda} \, \mathrm{Var}_\gamma \big(|\nabla F|^2 \big)^{1/2}$$

sufficiently many smooth $\varphi : \mathbb{R} \to \mathbb{R}$

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$\gamma_{\#F}$ distribution of $F : \mathbb{R}^N \to \mathbb{R}$ under γ

$$\left| \int_{\mathbb{R}} \varphi \, d\gamma_{\#F} - \int_{\mathbb{R}} \varphi \, d\gamma_1 \right| \leq \frac{C_\varphi}{\lambda} \, \mathrm{Var}_\gamma\big(|\nabla F|^2\big)^{1/2}$$

sufficiently many smooth $\varphi : \mathbb{R} \to \mathbb{R}$

if $\mathrm{Var}_\gamma\big(|\nabla F|^2\big) \sim 0$

then distribution $\gamma_{\#F}$ close to Gaussian γ_1

second (main) step

when does

$$\int_{\mathbb{R}^N} F^4 d\gamma \sim 3$$

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more information is needed

Γ **calculus**

convenient framework

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Markov operator L on state space E

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μ invariant symmetric probability measure

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Γ (bilinear) operator

$$\Gamma(f,g) = \frac{1}{2}\left[\mathrm{L}(fg) - f\,\mathrm{L}g - g\,\mathrm{L}f\right]$$

$f, g : E \to \mathbb{R}$ in some nice algebra $\mathcal{A}$

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integration by parts

$$\int_E f(-\mathrm{L}g)d\mu = \int_E \Gamma(f,g)d\mu$$

example

Ornstein-Uhlenbeck operator on $E = \mathbb{R}^N$

$$\mathrm{L} = \Delta - x \cdot \nabla$$

invariant measure $\mu = \gamma$

γ standard Gaussian measure on $\mathbb{R}^N$

$$\Gamma(f, g) = \nabla f \cdot \nabla g$$

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iterated gradients

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D. Bakry, M. Émery (1985)

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additional datum

Gaussian chaos

$$F = F(x) = \sum_{i_1,\ldots,i_k=1}^{N} a_{i_1,\ldots,i_k}\, x_{i_1} \cdots x_{i_k}$$

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intrinsic description in terms of the Γ_m

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$$S = \{0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots\}$$

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bilinear forms on $\mathcal{A} \times \mathcal{A}$

$$Q_k(\Gamma)(f,f) = Q_k(\Gamma)(f)$$

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Definition *An eigenfunction* F *of* $-\mathrm{L}$ *with eigenvalue* λ_k

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is said to be a chaos of degree $k \geq 1$ *relative to* $S = (\lambda_n)_{n \in \mathbb{N}}$ *if*

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further polynomials

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Theorem *Let* F *be a* *k-chaos eigenfunction with eigenvalue* λ_k. *Set* $\Gamma = \Gamma(F)$. *Then*

$$\pi_{k-1} \int_E \Gamma^2 d\mu = \pi_k \int_E F^2 \, \Gamma \, d\mu + (-1)^k \int_E \Gamma \, T_{k+1}\big(\tfrac{\mathrm{L}}{2}\big) \, \Gamma \, d\mu$$

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Corollary *Recall the spectrum* $S = (\lambda_n)_{n \in \mathbb{N}}$ *of* $-\mathrm{L}$. *If*

$$(-1)^k \, T_{k+1}\big(-\tfrac{\lambda_n}{2}\big) \leq 0 \quad \textit{for every } n \in \mathbb{N}$$

then

$$\int_E \Gamma^2 d\mu \leq \lambda_k \int_E F^2 \, \Gamma \, d\mu$$

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F normalized in $\mathrm{L}^2(\mu)$

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$$\int_E \Gamma \, d\mu = \int_E F(-\mathrm{L}F) d\mu = \lambda_k \int_E F^2 d\mu = \lambda_k$$

$$\mathrm{Var}_\mu(\Gamma) \leq \lambda_k \bigg(\int_E F^2 \, \Gamma \, d\mu - \lambda_k \bigg)$$

$$k \int_{\mathbb{R}^N} F^4 d\gamma = \int_{\mathbb{R}^N} F^3(-\mathrm{L}F) d\gamma = 3 \int_{\mathbb{R}^N} F^2 |\nabla F|^2 d\gamma$$

$$\int_{\mathbb{R}^N} |\nabla F|^2 d\gamma = k$$

$$k\left(\frac{1}{3} \int_{\mathbb{R}^N} F^4 d\gamma - 1\right) = \int_{\mathbb{R}^N} F^2 \big(|\nabla F|^2 - k\big) d\gamma$$

$$\int_{\mathbb{R}^N} F^4 d\gamma \sim 3 \quad \Longrightarrow \quad |\nabla F|^2 \sim k$$

$$\mathrm{Var}_\gamma\big(|\nabla F|^2\big) = \int_{\mathbb{R}^N} \big(|\nabla F|^2 - k\big)^2 d\gamma$$

technical task

from $\displaystyle\int_{\mathbb{R}^N} F^2 \big(|\nabla F|^2 - k\big) d\gamma$ to $\displaystyle\int_{\mathbb{R}^N} \big(|\nabla F|^2 - k\big)^2 d\gamma$

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if $\int_E F^4 d\mu = 3$ then $\Gamma = \Gamma(F)$ constant

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Stein's method (first step)

distribution of F is Gaussian

$$\mathrm{Var}_\mu(\Gamma) \,\leq\, \lambda_k \bigg(\int_E F^2\,\Gamma\, d\mu - \lambda_k \bigg)$$

recall : integration by parts $\int_E F^2 d\mu = 1$

$$\lambda_k \bigg(\frac{1}{3} \int_E F^4 d\mu - 1 \bigg) \,=\, \int_E F^2\,\Gamma\, d\mu - \lambda_k$$

$$\mathrm{Var}_\mu(\Gamma) \,\leq\, \lambda_k^2 \bigg(\frac{1}{3} \int_E F^4 d\mu - 1 \bigg)$$

if $\int_E F^4 d\mu = 3$ then $\Gamma = \Gamma(F)$ constant

Stein's method (first step)

distribution of F is Gaussian

Nualart-Peccati theorem

main statement

$$R_{k+1}(X) = \frac{1}{X^2}\big[Q_{k+1}(X) - q_1 X\big] = \sum_{i=2}^{k+1} q_i X^{i-2}$$

$$T_{k+1}(X) = R_{k+1}(X+\lambda_k) - R_{k+1}(\lambda_k)$$

$$\pi_k = \lambda_1 \cdots \lambda_k, \quad k \geq 1 \qquad (\pi_0 = 1)$$

Theorem *Let F be a k-chaos eigenfunction with eigenvalue λ_k. Set $\Gamma = \Gamma(F)$. Then*

$$\pi_{k-1} \int_E \Gamma^2 d\mu = \pi_k \int_E F^2 \, \Gamma \, d\mu + (-1)^k \int_E \Gamma \, T_{k+1}\big(\tfrac{\mathrm{L}}{2}\big) \, \Gamma \, d\mu$$

key argument of the **proof**

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$$\Gamma_m = \big(\tfrac{1}{2}\,\mathrm{L} + \lambda\,\mathrm{Id}\big)^{m-1}\Gamma$$

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Corollary *Recall the spectrum* $S = (\lambda_n)_{n\in\mathbb{N}}$ *of* $-\mathrm{L}$. *If*

$$(-1)^k\, T_{k+1}\big(-\tfrac{\lambda_n}{2}\big) \,\leq\, 0 \quad \textit{for every } n \in \mathbb{N}$$

then

$$\int_E \Gamma^2 d\mu \,\leq\, \lambda_k \int_E F^2\,\Gamma\, d\mu$$

$$\mathrm{Var}_\mu(\Gamma) \,\leq\, \lambda_k \bigg(\int_E F^2\,\Gamma\, d\mu - \lambda_k \bigg)$$

spectral condition

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Wiener (Gaussian) chaos

Nualart-Peccati theorem

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elementary exercise

extensions

infinite dimensional Wiener chaos

extensions

infinite dimensional Wiener chaos

abstract Markov chaos

extensions

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abstract Markov chaos

continuous and discrete

extensions

infinite dimensional Wiener chaos

abstract Markov chaos

continuous and discrete (cube $S = (\lambda_n)_{n \in \mathbb{N}} = \mathbb{N}$)

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convergence to other distributions (gamma)

I. Nourdin, G. Peccati (2009)

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I. Nourdin, G. Peccati (2009)

Wigner chaos (free probability) ?

T. Kemp, I. Nourdin, G. Peccati, R. Speicher (2012)

convergence to gamma distributions

convergence to gamma distributions

Theorem F *k-chaos with eigenvalue* λ_k *such that* $\int_E F^2 d\mu = p > 0$. *Set* $\Gamma = \Gamma(F)$. *Under the spectral condition*

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convergence to gamma distributions

Theorem *F k-chaos with eigenvalue λ_k such that $\int_E F^2 d\mu = p > 0$. Set $\Gamma = \Gamma(F)$. Under the spectral condition*

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it holds

$$\mathrm{Var}_\mu(\Gamma - \lambda_k F) \leq \lambda_k \int_E F^2 \, \Gamma \, d\mu + A_k \int_E F \, \Gamma \, d\mu - pB_k - p^2 \lambda_k^2$$

where

$$A_k = \frac{2(-1)^k \lambda_k}{\pi_{k-1}} R_{k+1}\Big(\frac{\lambda_k}{2}\Big) \qquad \text{and} \qquad B_k = \frac{(-1)^k \lambda_k^2}{\pi_{k-1}} R_{k+1}\Big(\frac{\lambda_k}{2}\Big)$$

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k even

$$\frac{3}{k^2}\,\mathrm{Var}_\mu(\Gamma - k\,F) \leq \int_E F^4 d\mu - 6\int_E F^3 d\mu + 6p - 3p^2$$

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$(F_n)_{n\in\mathbb{N}}$ sequence of k-chaos $\int_E F_n^2\,d\mu = p$

if $\displaystyle\int_E F_n^4 d\mu - 6\int_E F_n^3 d\mu + 6p - 3p^2 \to 0$

then $(F_n + p)_{n\in\mathbb{N}}$ converges in distribution

to gamma distribution p

I. Nourdin, G. Peccati (2009)