Analysis of the Wright–Fisher Equation

In an attempt to model the diffusion of a genetic trait in a population, Wright and Fisher introduced the equation the heat equation

$$\partial_t u = x(1-x)\partial_x^2 u \quad \text{for } (t,x) \in (0,\infty) \times (0,1),$$

which has been known ever since as the *Wright–Fisher equation.* In more recent times, people working on the human genome project have been using the W-F equation to model the evolution of a mutation in one of the nucleotides in a DNA strand. Thus, there is renewed interest in the W-F equation.

From an analytic standpoint, the W-F equation raises some interesting technical issues. Specifically, at the spacial boundary, the coefficient of ∂_x^2 degenerates in a way which makes it difficult to apply known regularity results. Using a clever application of orthogonal polynomials, Kimura was able to write down the eigenvalues and eigenfunctions for the operator $x(1-x)\partial_x^2$ and thereby arrived at an expression for the fundamental solution as a series. Kimura's result is particularly useful for studying the long time behavior of solutions, but it is essentially useless when looking at short time behavior. In this paper, I will report on recent work which I have been doing with the mathematical geneticist Nick Patterson, who has successfully mobilized the talents of several mathematicians in an effort to advance is research on the human genome.