An asymptotic result on Skorokhod problem in an orthant

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Joint work with
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Abstract

We consider the problem of asymptotic irrelevance of initial condition for the Skorokhod problem in an orthant. A characterization of this property is given. Also a useful sufficient condition is presented. Some implications for stochastic processes are also pointed out. This is a joint work with Offer Kella

SP in an orthant

$G = \mathbb{R}^d_+ = \{x \in \mathbb{R}^d : x_i \geq 0 \forall i\}$ $d-$dimensional orthant

$a \geq b$ denotes $a_i \geq b_i \forall i$
$a > b$ denotes $a_i > b_i \forall i$

$R = I - P$ $d \times d$ *reflection matrix*

$P = ((P_{ij})), P_{ii} = 0, P_{ij} \geq 0, j \neq i,$
spectral radius of $P < 1$
$R^{-1} = (I - P)^{-1} = I + P + P^2 + \cdots$ has nonnegative entries

Given $a \in G$, $X(\cdot)$ an $\mathbb{R}^d-$ valued r.c.l.l. function on $[0, \infty)$, with $X(0) = 0$

SP$(a + X(\cdot), R)$: contd.

Find $\mathbb{R}^d-$ valued r.c.l.l. functions $Y^{(a)}, Z^{(a)}$ such that

- $Z^{(a)}(t) = a + X(t) + RY^{(a)}(t)$ for all t; (*Skorokhod equation*)
- $Z^{(a)}(t) \in G$ for all $t \geq 0$ (*constraint*)
- $Y^{(a)}(0) = 0$, $Y_i^{(a)}(\cdot)$ nondecreasing for all i;
- $Y_i^{(a)}(t) - Y_i^{(a)}(s) = \int_{(s,t]} 1_{\{0\}}(Z_i^{(a)}(u))dY_i^{(a)}(u), 0 \leq s \leq t$;
  So $Y_i^{(a)}(\cdot)$ can increase only when $Z_i^{(a)}(\cdot) = 0$ (*minimality*)

$Z^{(a)}(\cdot)$ *regulated/ reflected part*
$Y^{(a)}(\cdot)$ *pushing part*

SP$(a + X(\cdot), R)$: contd.

$d = 1 \Rightarrow G = [0, \infty), R = I = 1$; normal reflection
Explicit solution given by

- $y^{(a)}(t) = \sup_{0 \leq s \leq t} \max\{0, -(a + x(s))\}$
- $z^{(a)}t = a + x(t) + y^{(a)}(t), \;\; t \geq 0$

For $d \geq 2$, one-dimensional result above + spectral radius condition $\Rightarrow$ appropriate map on $(D([0, \infty) : G)) \times (D([0, \infty) : G))$ having unique fixed point $\Rightarrow$ unique solution to SP$(a + X(\cdot), R)$

Also solution map $a + X(\cdot) \mapsto (Y^{(a)}(\cdot), Z^{(a)}(\cdot))$ Lipschitz continuous

Harrison and Reiman (1981), many others ...

Question (Q)

When is

$$\lim_{t\to\infty} Z^{(a)}(t) - Z^{(0)}(t) = 0$$

for all $a \in G$?

Notation

For $\mathbb{R}^d-$ valued r.c.l.l. function f with $f(0) \in G$ denote

$\Psi(f)$: regulated part (z part) of solution to SP(f, R)
$\Phi(f)$: pushing part (y part) of solution to SP(f, R)

So $Y^{(a)}(\cdot) = \Phi(a + X(\cdot)), Z^{(a)}(\cdot) = \Psi(a + X(\cdot))$

Significance of (Q)

Suppose X arises as sample paths of a stochastic process, again denoted $X(\cdot)$ such that

$X(0) = 0$ a.s.,

$X(\cdot)$ has stationary ergodic increments (not necessarily independent increments) with finite mean,

$R^{-1}(E(X(1) - X(0))) < 0$ (componentwise)

In particular, $\{X(t+s) - X(s) : t \geq 0\} =^d \{X(t) - X(0) : t \geq 0\}$ in distribution as processes for all s, and

$\frac{1}{t}X(t) \to E(X(1)) - X(0)) \in \mathbb{R}^d$ a.s.

Significance of (Q) : contd.

Then it is known (Kella and Whitt (1996)) there is $G-$ valued random variable ξ such that

- $\Psi(X)(t) \to^d \xi$
- $\Psi(\xi + X(\cdot))$ is a stationary process

So $\Psi(\xi + X(t)) =^d \xi$ for all t,
but ξ may not be independent of the process $X(\cdot)$

This means: If the assumption $X(0) = 0$ is dropped, then there is $G-$ valued random variable $\tilde{\xi}$ such that $\Psi(\tilde{\xi} + X(\cdot))$ is stationary, and $\Phi(\tilde{\xi} + X(\cdot))$ has stationary increments; that is, $\{\Psi(X(t)) : t \geq 0\}$ is tight and has a stationary version

Significance of (Q) : contd.

Missing aspects:

Q_1 Does there exist a limiting distribution for any initial $X(0)$?

Q_2 Does the limiting distribution depend on $X(0)$?

Q_3 Is the stationary version unique?

Note: $X(\cdot) =$ Brownian motion with mean vector μ and $R^{-1}\mu < 0$ imply "Yes" to $(Q), (Q_1) - (Q_3)$ (thanks to positive recurrence of RBM $\Psi(X)$ (Harrison and Williams(1987))

Not clear even when $X(\cdot)$ is a Levy process

"Yes" to $(Q) \Rightarrow$ "Yes" to $(Q_1), (Q_2), (Q_3)$

See Kella and Whitt (1996), Konstantopoulos, Last and Lin (2004)

Some conjectures

Some suggested sufficient conditions (weaker to stronger):

- $\lim_{t\to\infty} R^{-1}X(t) = -\infty$
- $\limsup_{t\to\infty} \frac{1}{t}R^{-1}X(t) < 0$
- $\lim_{t\to\infty} \frac{1}{t}X(t) = x$ exists, $R^{-1}x < 0$

All componentwise; (here $X(\cdot)$ r.c.l.l. function)

Results: Theorem 1

Theorem

Fix $1 \le i \le d$. *If*

$$\liminf_{t\to\infty} X_i(t) = -\infty, \tag{1}$$

or

$$\liminf_{t\to\infty} (R^{-1}X)_i(t) = -\infty, \tag{2}$$

then for every $a \in G$,

$$\lim_{t\to\infty} Z_i^{(a)}(t) - Z_i^{(0)}(t) = 0$$

Results: Theorem 2

Theorem

Fix $1 \le i \le d$. *The following are equivalent:*

- $\lim_{t\to\infty} Z_i^{(a)}(t) - Z_i^{(0)}(t) = 0$, *for all* $a \in G$
- $\lim_{t\to\infty} Y_i^{(a)}(t) = +\infty$, *for some* $a \in G$
- $\lim_{t\to\infty} Y_i^{(a)}(t) = +\infty$, *for all* $a \in G$

Some implications, examples

Corollary

Let $X(\cdot)$ be a $d-$diml. Levy process such that
$E|X(1) - X(0)| < \infty$, $R^{-1}E(X(1) - X(0)) < 0$.
Then the corresponding reflected Levy process $Z(\cdot) = \Psi(X)(\cdot)$, a Markov process, has a unique stationary probability distribution, and converges to this stationary distribution for any initial condition

(1),(2) weaker than any of the earlier suggested sufficient conditions. So (Q) and hence $(Q_1) - (Q_3)$ have satisfactory answers

Implications, examples, ...

Set up deterministic; no probabilistic assumptions

Markovian structure or existence of stationary distribution not required

So "uniqueness" question separated from "existence" question

Example

$X(\cdot)$ $d-$diml. Brownian motion, $R = I$.
So $Z^{(a)}(\cdot)$ reflected standard BM with normal reflection, starting at $a \in G$
We know $\liminf_{t\to\infty} X_i(t) = -\infty$ a.s. for all i
Hence $Z^{(a)}(t) - Z^{(0)}(t) \to 0$ a.s. as $t \to \infty$ for all $a \in G$
For $d \geq 3$ $Z(\cdot)$ is transient; for $d = 1, 2$ $Z(\cdot)$ is null recurrent
So no stationary probability distribution

Implications, examples, ...

Example

$X_1(\cdot), \cdots, X_d(\cdot)$ independent *renewal risk processes (Sparre-Andersen processes)*

$$X_i(t) \;=\; c_i t - \sum_{\ell=1}^{N_i(t)} U_\ell^{(i)}(t), \; t \geq 0, 1 \leq i \leq d$$

$\{N_j(\cdot)\}, \{U_\ell^{(i)} : \ell \geq 1\}, 1 \leq i, j \leq d$ independent families of r.v.'s, $N_i(\cdot)$ renewal counting process with i.i.d. interarrivals $A_\ell^{(i)}, \ell \geq 1$, for $1 \leq i \leq d$. Assume

$$c_i \;=\; \frac{E(U_1^{(i)})}{E(A_1^{(i)})}, \; 1 \leq i \leq d$$

Example: contd.

Example

So $E(X_i(t)) = 0, \forall t, i$; it can be shown that

$$\limsup_{t\to\infty} X_i(t) = +\infty, \text{ a.s.}$$
$$\liminf_{t\to\infty} X_i(t) = -\infty, \text{ a.s.}, \; 1 \le i \le d$$

Take $X(\cdot) = (X_1(\cdot), \cdots, X_d(\cdot)), R = I$, so normal reflection
Corresponding reflected process $Z^{(a)}(\cdot)$ is not Markov, in general; also no limiting probability distribution
By Theorem 1, $Z^{(a)}(t) - Z^{(0)}(t) \to 0$ with probability 1

Example

Sufficient condition (1),(2) in Theorem 1 not necessary for $d \geq 2$

$$\begin{aligned} X_2(t) = -X_1(t) &= t|\sin t|, \; t \geq 0, \\ R &= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \end{aligned}$$

Then

$$\begin{aligned} Y_1^{(0)}(t) &\to +\infty \\ Y_2^{(0)}(2\pi n) &\leq Y_1^{(0)}(2\pi n) \to +\infty \end{aligned}$$

So by Theorem 2

$$\lim_{t\to\infty} Z^{(a)}(t) - Z^{(0)}(t) = 0, \; a \in G$$

Implications, examples, ...

Example

Theorem 2 is *not* equivalent to

$$\lim_{t\to\infty} Z^{(a)}(t) - Z^{(0)}(t) \quad = \quad 0, \quad \textit{for} \quad \textit{some} \quad a \in G$$

One dimensional counterexample: $X(t) = -\min\{t, 1\}$
Then for $a \geq 0$, $Z^{(a)}(t) = \max\{0, a - \{t, 1\}\}$
So, if $a \leq 1$, then $Z^{(a)}(t) = 0$ for all $t \geq a$
If $a > 1$, then $Z^{(a)}(t) = a - 1$ for all $t \geq 1$
Note that $Z^{(0)}(\cdot) \equiv 0$
Thus Theorem 2 holds for $0 \leq a \leq 1$, but does not hold for $a > 1$

Next example indicates domains other than orthants may not be very simple to handle

Example

$D \subset \mathbb{R}^2$ bounded domain, $x, \hat{x} \in D$
$U(\cdot)$, $\hat{U}(\cdot)$ reflected Brownian motion with normal reflection starting at $x, \hat{x}$ resply.
When does

$$|U(t) - \hat{U}(t)| \quad \to \quad 0? \tag{3}$$

If D is "Lip" domain, or if the boundary of D is a polygon or finite union of disjoint polygons, then (3) holds. Uniform distribution on D is the unique stationary distribution, spectral theory of self-adjoint operators, connection with complex function theory, ... among facts/ tools used; Burdzy and Chen (2002). See also Cranston and Le Jan (1990), Burdzy, Chen and Jones (2006)

Thank you

Talk based on

• O. Kella and S. Ramasubramanian: Asymptotic irrelevance of initial conditions for Skorokhod reflection mapping on the nonnegative orthant. *Mathematics of Operations Research* **37** (2012) 301 – 312.

Extension of sufficient conditions in Theorem 1 to some cases with nonconstant reflection and drift are also given