On the existence of paths between points in high level excursion sets of Gaussian random fields

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Let $\mathbf{X} = (X(\mathbf{t}), \, \mathbf{t} \in \mathbb{R}^d)$ be a real-valued sample continuous Gaussian random field.

Given a level u, the excursion set of $\mathbf{X}$ above the level u is the random set

$$A_u = \left\{ \mathbf{t} \in \mathbb{R}^d : \, X(\mathbf{t}) > u \right\}.$$

Much is know about the structure of the excursion set when the field is smooth, and u is large (Adler and Taylor (2007), Azaïs and Wschebor (2009)).

Consider a large level u.

Question: given that two points in $\mathbb{R}^d$ belong to the excursion set, what is the probability that they belong to the same path connected component of the excursion set?

Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$, $\mathbf{a} \neq \mathbf{b}$. A path in $\mathbb{R}^d$ connecting $\mathbf{a}$ and $\mathbf{b}$ is a continuous map $\xi : [0,1] \to \mathbb{R}^d$ with $\xi(0) = \mathbf{a}$, $\xi(1) = \mathbf{b}$.

Denote the collection of all such paths by $\mathcal{P}(\mathbf{a}, \mathbf{b})$. Estimate

$$P\left(\exists\, \xi \in \mathcal{P}(\mathbf{a}, \mathbf{b}) : X(\xi(v)) > u,\ 0 \le v \le 1 \,\middle|\, X(\mathbf{a}) > u, X(\mathbf{b}) > u\right).$$

The non-trivial part of the problem: estimate the probability

$$\Psi_{\mathbf{a},\mathbf{b}}(u) := P\left(\exists\ \xi \in \mathcal{P}(\mathbf{a},\mathbf{b}) :\ X(\xi(v)) > u,\ 0 \le v \le 1\right).$$

If the random field is stationary, we may assume that $\mathbf{b} = \mathbf{0}$, and use the notation $\Psi_{\mathbf{a}}$.

If the domain of a random field is restricted to $T \subset \mathbb{R}^d$, and $\mathbf{a}$, $\mathbf{b}$ are in T, we consider

$$\Psi_{\mathbf{a},\mathbf{b}}(u) = P\left(\exists \xi \in \mathcal{P}(\mathbf{a},\mathbf{b}) :\ \xi(v) \in T \text{ and } X(\xi(v)) > u,\ 0 \le v \le 1\right).$$

Large deviations setup

Let A be the open set

$$A \equiv A_{\mathbf{a},\mathbf{b}} := \Big\{\omega \in C_0(\mathbb{R}^d) : \exists\, \xi \in \mathcal{P}(\mathbf{a},\mathbf{b}),\ \omega(\xi(v)) > 1,\ 0 \le v \le 1\Big\},$$

$$C_0(\mathbb{R}^d) = \Big\{\omega = (\omega(\mathbf{t}),\, \mathbf{t} \in \mathbb{R}^d) \in C(\mathbb{R}^d) : \lim_{\|\mathbf{t}\|\to\infty} \omega(\mathbf{t})/\|\mathbf{t}\| = 0\Big\}.$$

We can write for $u > 0$

$$\Psi_{\mathbf{a},\mathbf{b}}(u) = P\left(u^{-1}\mathbf{X} \in A\right),$$

and use the large deviations results for Gaussian measures of Deutschel and Stroock (1989).

The reproducing kernel Hilbert space (RKHS) $\mathcal{H}$ of the random field $\mathbf{X}$, is a subspace of $C(\mathbb{R}^d)$.

Consider the space of finite linear combinations $\sum_{j=1}^{k} a_j X(\mathbf{t}_j)$ $a_j \in \mathbb{R}$, $\mathbf{t}_j \in \mathbb{R}^d$ for $j = 1, \dots, k$, $k = 1, 2, \dots$.

Its closure $\mathcal{L}$ in the mean square norm is identified with $\mathcal{H}$ via the injection $\mathcal{L} \to C(\mathbb{R}^d)$ given by

$$H \to w_H = \Big(E\big(X(\mathbf{t})H\big),\ \mathbf{t} \in \mathbb{R}^d \Big)$$

and the resulting norm

$$\|w_H\|_{\mathcal{H}}^2 = E(H^2)\,.$$

Let $\mathbf{X}$ be stationary with spectral measure $F_{\mathbf{X}}$.

The RKHS $\mathcal{H}$ can be identified with the subspace of $L^2(F_{\mathbf{X}})$ of functions with even real parts and odd imaginary parts, via the injection $L^2(F_{\mathbf{X}}) \to C_0(\mathbb{R}^d)$ given by

$$h \to S(h) = \left(\int_{\mathbb{R}^d} e^{i(\mathbf{t},\mathbf{x})}\, \bar{h}(\mathbf{x})\, F_{\mathbf{X}}(d\mathbf{x}),\ \mathbf{t} \in \mathbb{R}^d \right),$$

with the resulting norm

$$\|S(h)\|^2_{\mathcal{H}} = \|h\|^2_{L^2(F_{\mathbf{X}})} \;=\; \int_{\mathbb{R}^d} \|h(x)\|^2\, F_{\mathbf{X}}(d\mathbf{x}).$$

Theorem 1 Let $\mathbf{X} = (X(\mathbf{t}), \mathbf{t} \in \mathbb{R}^d)$ be a continuous stationary Gaussian random field, with covariance function satisfying

$$\limsup_{\|\mathbf{t}\|\to\infty} R_{\mathbf{X}}(\mathbf{t}) \le 0 \,.$$

Then

$$\lim_{u\to\infty} \frac{1}{u^2} \log \Psi_{\mathbf{a}}(u) = -\frac{1}{2}\mathcal{C}_{\mathbf{X}}(\mathbf{a}),$$

where

$$\mathcal{C}_{\mathbf{X}}(\mathbf{a}) := \inf \left\{ \int_{\mathbb{R}^d} \|h(\mathbf{x})\|^2 \, F_{\mathbf{X}}(d\mathbf{x}) : \text{ for some } \xi \in \mathcal{P}(\mathbf{0}, \mathbf{a}) \right.$$

$$\left. \int_{\mathbb{R}^d} e^{i(\xi(v),\mathbf{x})} \, \bar{h}(\mathbf{x}) \, F_{\mathbf{X}}(d\mathbf{x}) > 1, \, 0 \le v \le 1 \right\}.$$

The constraints in the optimization problem in Theorem 1 are not convex. However, for a fixed path, the constraints are convex, and one can use the convex Lagrange duality.

Theorem 2 For a continuous stationary Gaussian random field $\mathbf{X}$,

$$\mathcal{C}_{\mathbf{X}}(\mathbf{a}) = \left[\sup_{\xi \in \mathcal{P}(\mathbf{0},\mathbf{a})} \min_{\mu \in M_1^+([0,1])} \int_0^1 \int_0^1 R_{\mathbf{X}}\big(\xi(u) - \xi(v)\big)\, \mu(du)\, \mu(dv) \right]^{-1}.$$

Here $M_1^+([0,1])$ is the space of all probability measures on $[0,1]$.

An optimal path is a path of *maximal* R_X *capacity*.

Assume a path $\xi \in \mathcal{P}(\mathbf{0}, \mathbf{a})$ is fixed. Then the minimization problem

$$\min_{\mu \in M_1^+([0,1])} \int_0^1 \int_0^1 R_{\mathbf{X}}\big(\xi(u), \xi(v)\big)\, \mu(du)\, \mu(dv)$$

is the problem of finding a probability measures μ of *minimal energy*, or *capacitory measures*.

The set $\mathcal{W}_\xi \subseteq M_1^+([0,1])$ over which the minimum is achieved is a weakly compact convex subset of $M_1^+([0,1])$.

If the feasible set for the primary problem is non-empty then, for every $\varepsilon > 0$,

$$P\left(\sup_{0\le v\le 1}\left|\frac{1}{u}X(\xi(v)) - x_\xi(v)\right| \ge \varepsilon \middle| X(\xi(v)) > u,\, 0 \le v \le 1\right) \to 0$$

as $u \to \infty$. Here H_ξ is primary optimal, and

$$x_\xi(v) = E\big[X(\xi(v))H_\xi\big],\ 0 \le v \le 1\,.$$

Furthermore, there is a characterization of the optimal $H_\xi \in \mathcal{L}$ and the optimal $\mu \in \mathcal{W}_\xi$.

Theorem 3

(i) For every $\mu \in \mathcal{W}_\xi$ we have

$$H_\xi = \mathcal{C}_{\mathbf{X}}(\mathbf{a}, \mathbf{b}; \xi) \int_0^1 X\big(\xi(v)\big)\, \mu(dv)$$

with probability 1.

(ii) A probability measure $\mu \in M_1^+([0,1])$ is a measure of minimal energy if and only if

$$\min_{0 \le v \le 1} \int_0^1 R_{\mathbf{X}}\big(\xi(u), \xi(v)\big)\, \mu(du)$$

$$= \int_0^1 \int_0^1 R_{\mathbf{X}}\big(\xi(u_1), \xi(u_2)\big)\, \mu(du_1)\, \mu(du_2) > 0\,.$$

Remarks

- By Theorem 3, the function

$$v \mapsto \int_0^1 R_{\mathbf{X}}\big(\xi(u), \xi(v)\big)\, \mu(du),\ 0 \le v \le 1,$$

  is constant on the support of any measure $\mu \in \mathcal{W}_\xi$. This seems to indicate that the support of any measure of minimal energy may not be 'large'. However, this intuition holds only in some cases.
- If the random field is stationary, and the spectral measure is of the full support, then the image of any measure $\mu \in \mathcal{W}_\xi$ on the path ξ is unique.

One-dimensional case

In this case there is, essentially, a single path between two points.

Let $\mathbf{X} = \big(X(t),\ t \in \mathbb{R}\big)$ be a stationary continuous Gaussian process. We are interested in understanding how the probability

$$\Psi_a(u) = P\Big(X(t) > u,\, 0 \le t \le a\Big)$$

changes with $a > 0$, and what happens with the optimal probability measures μ_a and limiting shapes x_a.

For some processes (in particular, those with a finite second spectral moment), on short intervals we get an easy descrption.

Proposition 1 Suppose that for some $a > 0$

$$R_{\mathbf{X}}(t) + R_{\mathbf{X}}(a-t) \geq R_{\mathbf{X}}(0) + R_{\mathbf{X}}(a) > 0 \quad \text{for all } 0 \leq t \leq a.$$

Then a measure in $\mathcal{W}_a$ is given by

$$\mu = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1.$$

Furthermore,

$$C_{\mathbf{X}}(a) = \frac{2}{R_{\mathbf{X}}(0) + R_{\mathbf{X}}(a)},$$

$$x_a(t) = \frac{R_{\mathbf{X}}(t) + R_{\mathbf{X}}(a-t)}{R_{\mathbf{X}}(0) + R_{\mathbf{X}}(a)}, \qquad 0 \leq t \leq a\,.$$

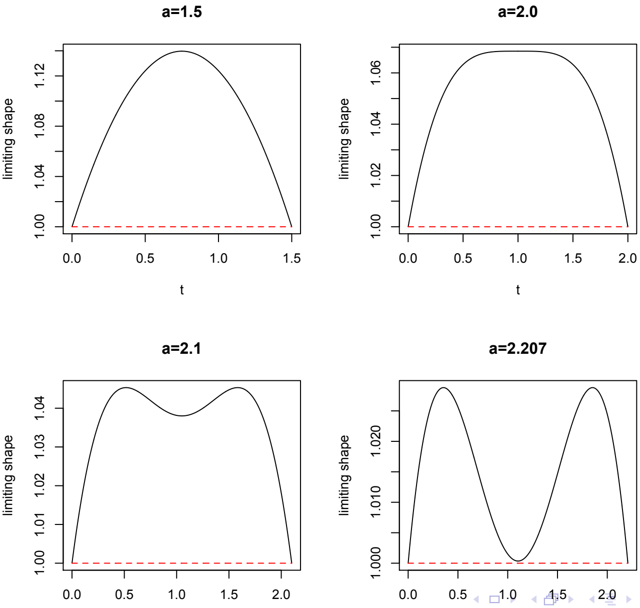
In any case, the measure $\mu = (\delta_0 + \delta_1)/2$ does NOT remain optimal for longer intervals.

Example 1 Consider the centered stationary Gaussian process with the Gaussian covariance function

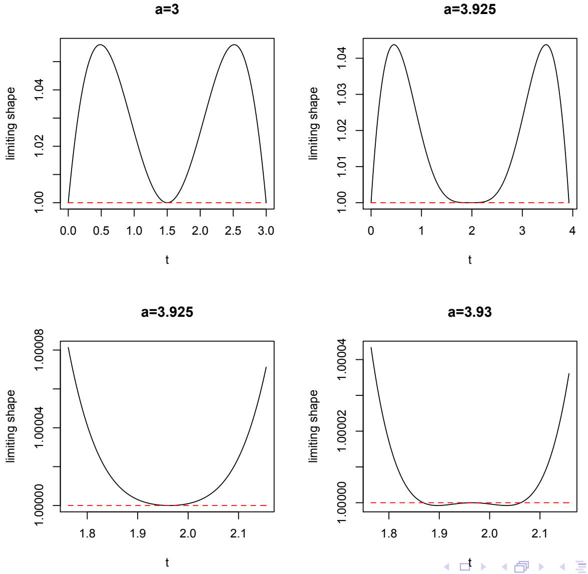
$$R(t) = e^{-t^2/2}, \ t \in \mathbb{R}.$$

Since the spectral measure has a Gaussian spectral density which is of full support in $\mathbb{R}$, for every $a > 0$ there is a unique (symmetric) measure of minimal energy. Furthermore, the second spectral moment is finite, so that, for $a > 0$ sufficiently small this process satisfies the conditions of Proposition 1.

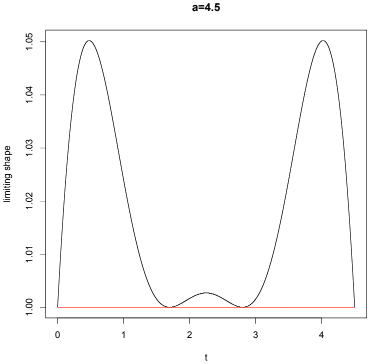
The measure $\mu = (\delta_0 + \delta_1)/2$ remains optimal for $a \leq a_1 \approx 2.2079$.



In the next regime the optimal measure acquires a point in the middle of the interval. This continues for $a_1 < a \leq a_2 \approx 3.9283$.



In the next regime the middle point of the optimal measure splits in two and starts moving away from the middle. This continues for $a_2 < a \le a_3 \approx 5.4508$.



Example 2 Consider an Ornstein-Uhlenbeck process, i.e. a centered stationary Gaussian process with the covariance function

$$R(t) = e^{-|t|},\ t \in \mathbb{R}\,.$$

For this process the spectral measure has a Cauchy spectral density, which has a full support in $\mathbb{R}$. Therefore, for every $a > 0$ there is a unique (symmetric) measure of minimal energy.

In this case even the first spectral moment is infinite. Proposition 1 does not apply here.

The optimal probability measure is

$$\mu = \frac{1}{a+2}\delta_0 + \frac{1}{a+2}\delta_1 + \frac{a}{a+2}\lambda\,,$$

where λ is Lebesgue measure on $(0,1)$.

There are no phase transitions. We have

$$C_{\mathbf{X}}(a) = (a+2)/2 \quad \text{for all } a > 0$$

and the limiting shape x_a is identically equal to 1 on $[0,a]$.

Long intervals

As the length a of the interval increases, a difference between short and long memory processes arises. In the short memory case, the uniform measure is asymptotically optimal.

Theorem 4 Assume that $R_{\mathbf{X}}$ is positive, and satisfies

$$\int_0^\infty R(t)\,dt < \infty\,.$$

Then, with λ denoting the uniform probability measure on $[0,1]$,

$$\lim_{a\to\infty} \frac{1}{a} C_{\mathbf{X}}(a) = \left(\lim_{a\to\infty} a \int_0^1 \int_0^1 R_{\mathbf{X}}\big(a(u-v)\big)\, \lambda(du)\lambda(dv) \right)^{-1}$$

$$= \frac{1}{2\int_0^\infty R(t)\,dt}\,.$$

In the long memory case the uniform measure is no longer asymptotically optimal. We will assume that the covariance function of the process is regularly varying at infinity:

$$R_{\mathbf{X}}(t) = \frac{L(t)}{|t|^{\beta}}, \qquad 0 < \beta < 1,$$

where L is slowly varying at infinity.

Consider the minimization problem with respect to Riesz kernel,

$$\min_{\mu \in M_1^+([0,1])} \int_0^1 \int_0^1 \frac{\mu(du)\mu(dv)}{|u-v|^{\beta}} \qquad 0 < \beta < 1.$$

An optimal measure μ_β exists, but it is different from the uniform measure.

Theorem 5 Assume that $R_{\mathbf{X}}$ is positive and regularly varying. Then for any $\mu_\beta \in \mathcal{W}_\beta$, the set of optimal measures for the Riesz kernel,

$$\lim_{a\to\infty} R_{\mathbf{X}}(a)\mathcal{C}_{\mathbf{X}}(\mathbf{a}) = \left(\int_0^1 \int_0^1 \frac{\mu_\beta(du)\mu_\beta(dv)}{|u-v|^\beta}\right)^{-1}.$$

In particular, $\mathcal{C}_{\mathbf{X}}(\mathbf{a})$ is regularly varying with exponent β.