Topology of some random complexes.

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Technion, Haifa.

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Topological Data Analysis

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OUR GOAL : Simplicial complexes on different point processes

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Graphs are 1-Dimensional complexes.

Geometric graphs

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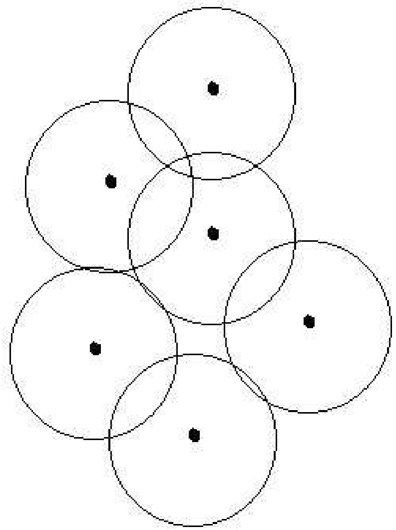

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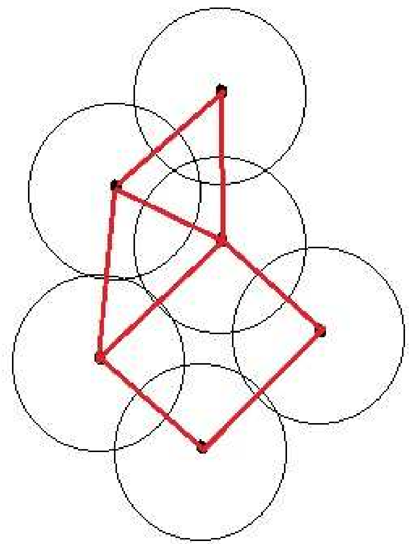

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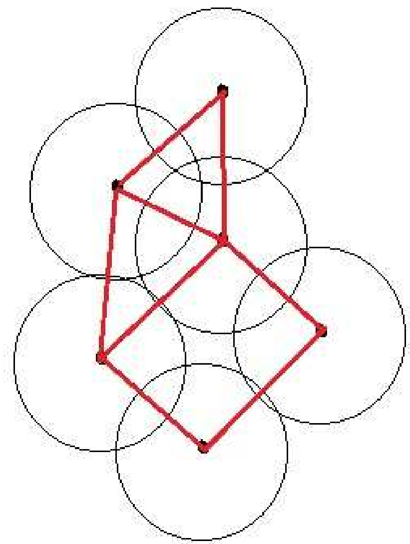

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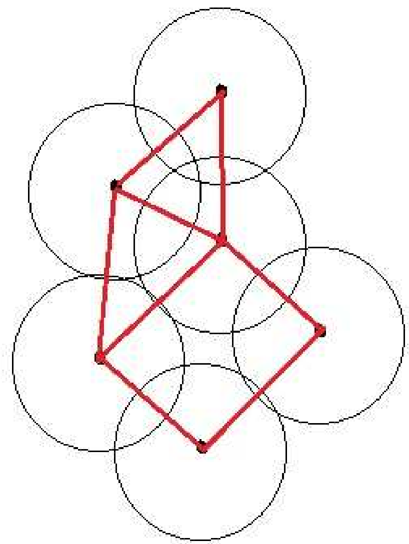

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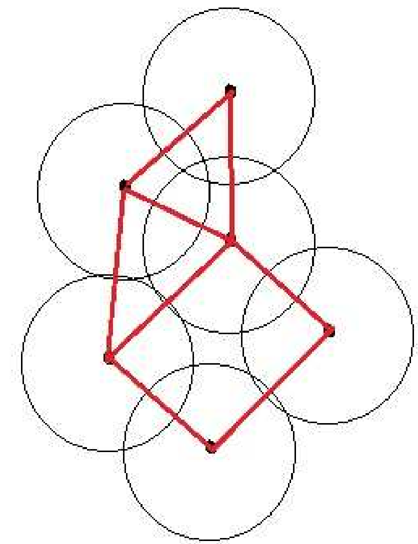

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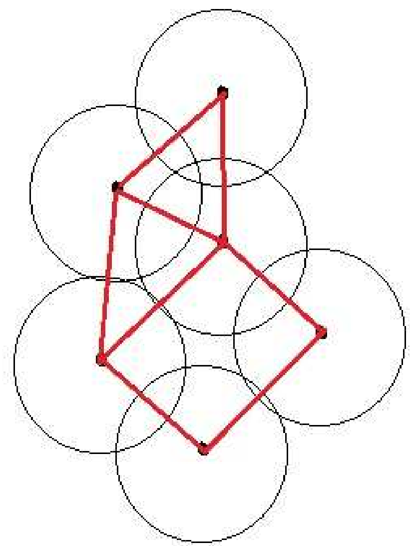

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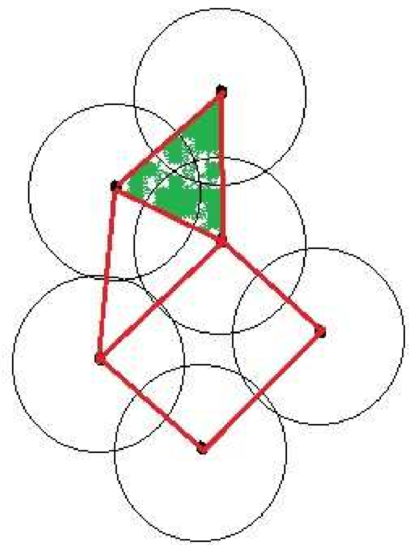

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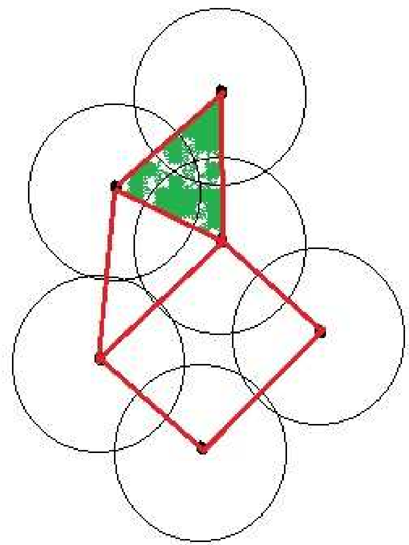

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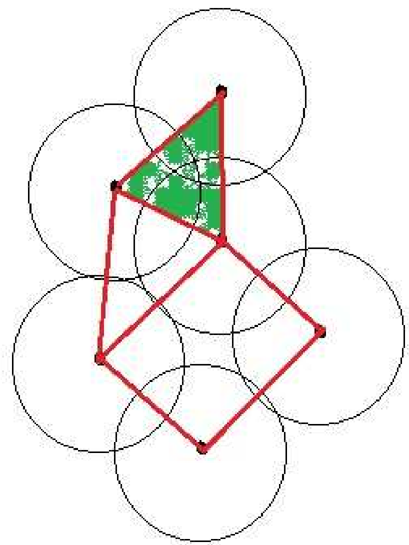

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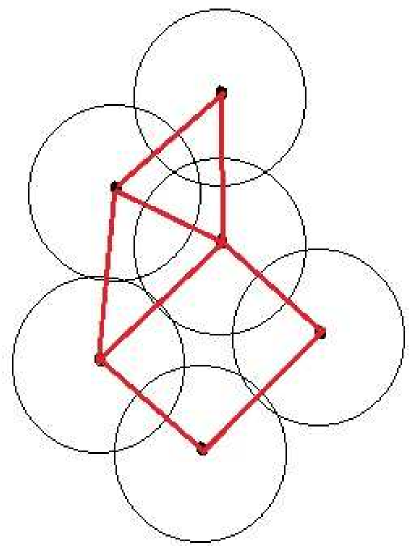

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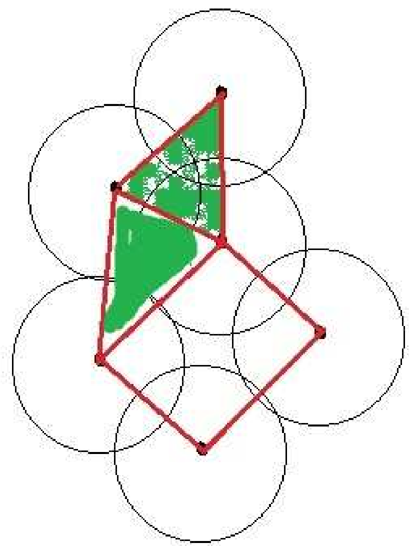

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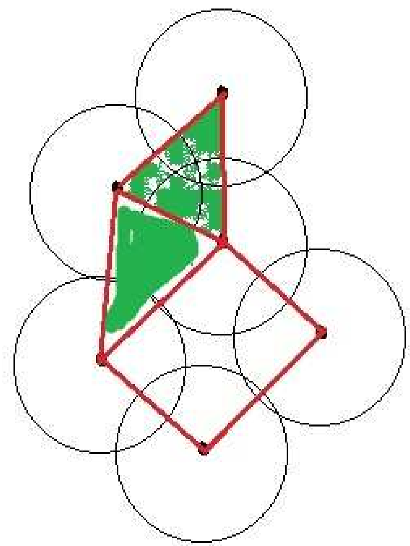

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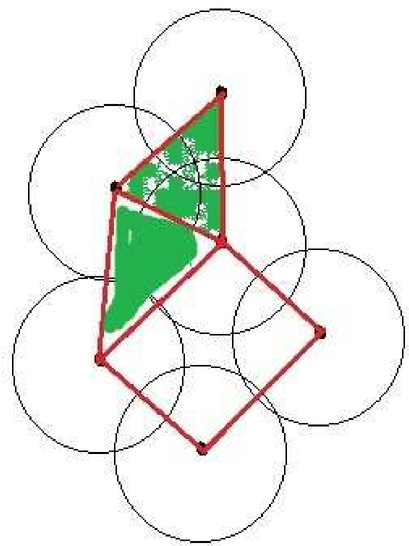

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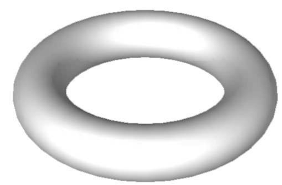

Figure 9: $\beta_0(T) = 1, \beta_1(T) = 2, \beta_2(T) = 1.$

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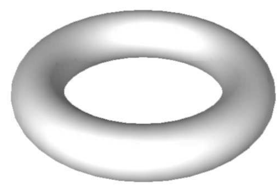

Figure 11: $\beta_0(T) = 1, \beta_1(T) = 2, \beta_2(T) = 1.$

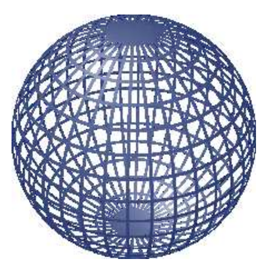

Figure 12: $\beta_0(T) = 1, \beta_1(T) = 0, \beta_2(T) = 1.$

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First term in upper bound is of smaller order (i.e, number of vertices) than the second term.

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Ginibre point process - Eigenvalues of $n \times n$ matrix with i.i.d. $N_{\mathbb{C}}(0,1)$ entries as $n \to \infty$.

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If $\rho^{(k)}(0, \ldots, 0) = 0$, then assume that for $y = (0, y_2, \ldots, y_k)$,

$$\rho^{(k)}(ry) = \Theta(f^k(r)) \ ; \ \ f^k(r) \overset{r \to 0}{\longrightarrow} 0.$$

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$$\lim_{n\to\infty} \mathsf{P}(\beta_k(C(\Phi_n, r_n) \geq 1) = \begin{cases} 0 & \text{if } n r_n^{d(k+1)} f^{k+2}(r_n) \to 0. \\ 1 & \text{if } n r_n^{d(k+1)} f^{k+2}(r_n) \to \infty + \\ & \rho^{(k+l)}(.) \leq \rho^{(k)}(.)\rho^{(l)}(.). \end{cases}$$

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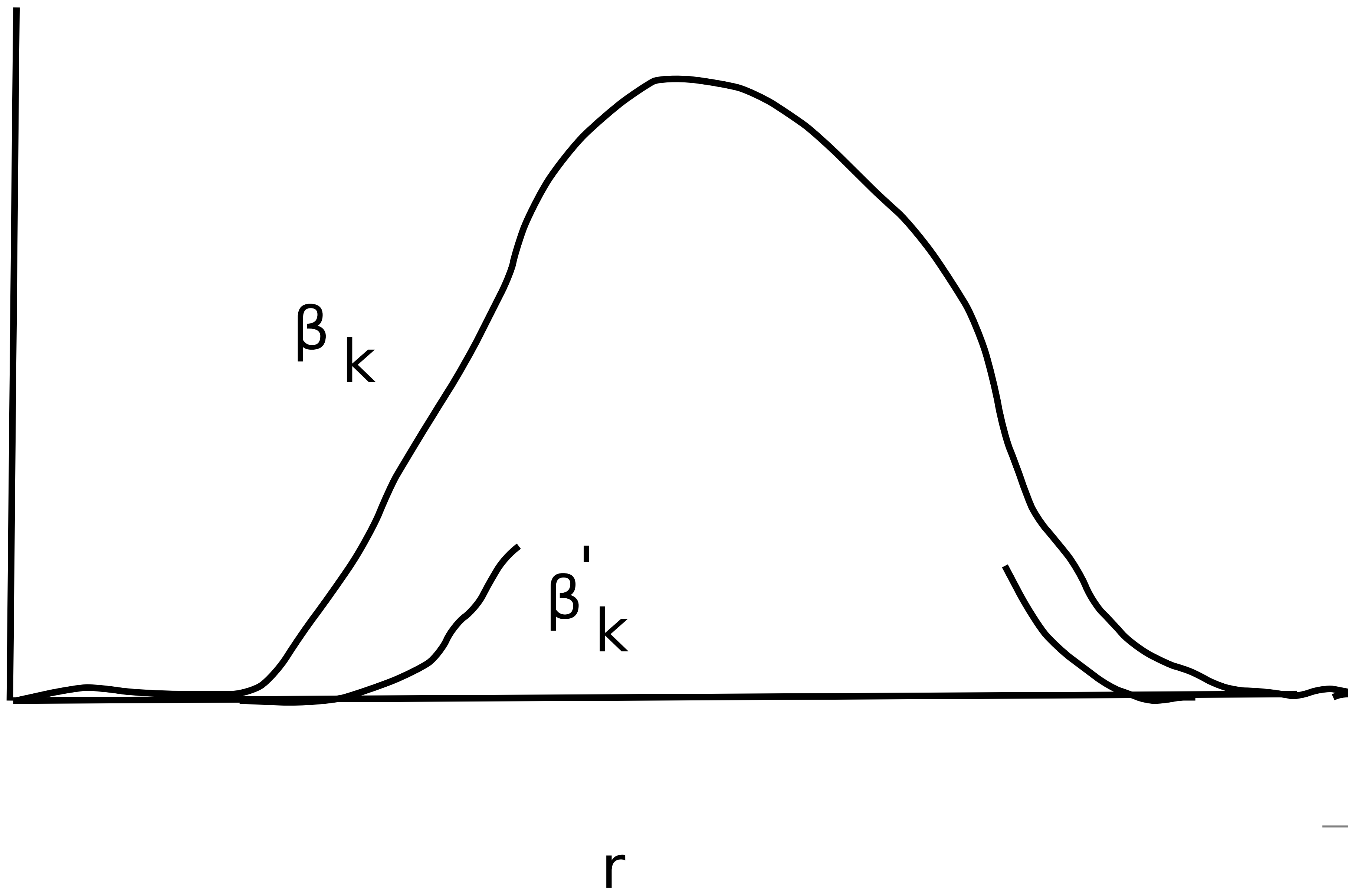
For weakly sub-Poisson, Betti numbers are formed later and end earlier.

In Picture



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Morse critical points : Link between differential and algebraic topology.

References :

- D. Yogeshwaran and R. J. Adler : On topology of some random complexes over stationary point processes. arXiv:1211.0061

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- On Erdos-Renyi-type random complexes : Mathew Kahle (Ohio).
- More on Random Topology : Gunnar Carlsson(Stanford), Michael Farber(Warwick) and John Harer(Duke).


Thank You
Mahalo
Kiitos
Thanks
Tack
Toda
Grazie
Obrigado
Takk
Merci
Gracias
நன்றி
Dziękuję