# MA 312 Commutative Algebra / Aug–Dec 2017

(Int PhD. and Ph. D. Programmes)

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**Tel :** +91-(0)80-2293 3212/09449076304 **E-mails :** patil@math.iisc.ernet.in

**Lectures :** Wednesday and Friday ; 14:00–15:30 **Venue:** MA LH-2 ( **if LH-1 is not free** ) / LH-1

**Seminars :** Sat, Nov 18 (10:30–12:45) ; Sat, Nov 25 (10:30-12:45)

**Final Examination :** Tuesday, December 05, 2017, 09:00–12:00

**Evaluation Weightage : Assignments :** 20% **Seminars :** 30% **Final Examination :** 50%

| Range of Marks for Grades (Total 100 Marks) | | | | | | |
|---|---|---|---|---|---|---|
| | **Grade S** | **Grade A** | **Grade B** | **Grade C** | **Grade D** | **Grade F** |
| **Marks-Range** | $>90$ | $76-90$ | $61-75$ | $46-60$ | $35-45$ | $<35$ |

| | **Grade A$^+$** | **Grade A** | **Grade B$^+$** | **Grade B** | **Grade C** | **Grade D** | **Grade F** |
|---|---|---|---|---|---|---|---|
| **Marks-Range** | $>90$ | $81-90$ | $71-80$ | $61-70$ | $51-60$ | $40-50$ | $<40$ |

## 2. Prime and Maximal Ideals

**Submit a solutions of $*$-Exercises ONLY.** **Due Date : Wednesday, 13-09-2017**

**Strongly Recommended to attempt the $*$-Exercise 2.11.**

All rings considered are commutative with unity. For a ring $A$, the set $A^*$ denote the set of all non-zero divisors in $A$ and $Z(R) := A \smallsetminus A^*$ denote the set of all zero divisors in $A$.

**2.1** Let $\mathcal{I}(A)$ denote the set of all ideals in a ring $A$.

**(a)** The operations sum, intersection and product on $\mathcal{I}(A)$ are commutative and associative. Moreover, for all $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c} \in \mathcal{I}(A)$, we have :

(i) (Distributive law) $\mathfrak{a}(\mathfrak{b}+\mathfrak{c}) = \mathfrak{a}\mathfrak{b}+\mathfrak{a}\mathfrak{c}$.

(ii) (Modular law) If $\mathfrak{a} \supseteq \mathfrak{b}$ or $\mathfrak{a} \supseteq \mathfrak{c}$, then $\mathfrak{a}\cap(\mathfrak{b}+\mathfrak{c}) = \mathfrak{a}\cap\mathfrak{b}+\mathfrak{a}\cap\mathfrak{c}$.

(iii) $(\mathfrak{a}+\mathfrak{b})(\mathfrak{a}\cap\mathfrak{b}) \subseteq \mathfrak{a}\mathfrak{b}$.

**0..1 Remark** In the ring $\mathbb{Z}$ the equality $(\mathfrak{a}+\mathfrak{b})(\mathfrak{a}\cap\mathfrak{b}) = \mathfrak{a}\mathfrak{b}$ holds.

(iv) $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a}\cap\mathfrak{b}$ and the equality $\mathfrak{a}\cap\mathfrak{b} = \mathfrak{a}\mathfrak{b}$ holds if $\mathfrak{a}$ and $\mathfrak{b}$ are comaximal, i.e. $\mathfrak{a}+\mathfrak{b} = A$.

(**Remark :** For a ring $\mathcal{I}(A)$ is a (multiplicative and additive) monoid (with binary operations product and sum of ideals, respectively) and also an ordered set (with respect to the natural inclusion) which is compatible with the multiplication. Therefore $\mathcal{I}(A)$ is an ordered monoid. Moreover, it is a lattice, i.e. for any two elements $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}(A)$, both $\operatorname{Sup}\{\mathfrak{a},\mathfrak{b}\}$ and $\operatorname{Inf}\{\mathfrak{a},\mathfrak{b}\}$ exist.)

**(b)** (Ideal quotient) For $\mathfrak{a}$, $\mathfrak{b} \in \mathcal{I}(A)$, the ideal quotient of $\mathfrak{a}$ by $\mathfrak{b}$ is the ideal $(\mathfrak{a}:\mathfrak{b}) := \{a \in A \mid a\mathfrak{b} \subseteq \mathfrak{a}\}$. In particular, $(0:\mathfrak{b})$ is $\{a \in A \mid a\mathfrak{b} = 0\}$ is the annihilator of $\mathfrak{b}$ and is also denoted by $\operatorname{Ann}_A(\mathfrak{b})$. If $\mathfrak{b} = Ab$, then we simply write $(\mathfrak{a}:b)$ for $(\mathfrak{a}:\mathfrak{b})$. (In the ring $A = \mathbb{Z}$, let $\mathfrak{a} = \mathbb{Z}m$, $\mathfrak{b} = \mathbb{Z}n$. Then $(\mathfrak{a}:\mathfrak{b}) = \mathbb{Z}q$, where $q = \prod_{p \text{ prime}} p^{r_p}$, $r_p := \max(v_p(m)-v_p(n),0) = v_p(m) - \min(v_p(m)-v_p(n))$. Therefore $q = m/\gcd(m,n)$.)

For ideals $\mathfrak{a}$, $\mathfrak{a}_i, i \in I$, $\mathfrak{b}$, $\mathfrak{b}_i, i \in I$, $\mathfrak{c} \in \mathcal{I}(A)$, the following computational rules are easy to verify :

(i) $\mathfrak{a} \subseteq (\mathfrak{a}:\mathfrak{b})$. (ii) $(\mathfrak{a}:\mathfrak{b})\mathfrak{b} \subseteq \mathfrak{a}$. (iii) $\big((\mathfrak{a}:\mathfrak{b}):\mathfrak{c}\big) = (\mathfrak{a}:\mathfrak{b}\mathfrak{c}) = \big((\mathfrak{a}:\mathfrak{c}):\mathfrak{b}\big)$.

(iv) $\big(\cap_{i\in I}\mathfrak{a}_i : \mathfrak{b}\big) = \cap_{i\in I}(\mathfrak{a}_i:\mathfrak{b})$. (v) $\big(\mathfrak{a} : \sum_{i\in I}\mathfrak{b}_i\big) = \cap_{i\in I}(\mathfrak{a}:\mathfrak{b}_i)$.

**(c)** (Radical of an ideal) For $\mathfrak{a} \in \mathcal{I}(A)$, the radical of $\mathfrak{a}$ is the ideal $\mathfrak{r}(\mathfrak{a}) = \sqrt{\mathfrak{a}} := \{a \in A \mid a^n \in \mathfrak{a} \text{ for some } n \in \mathbb{N}^+\}$. For ideals $\mathfrak{a}$, $\mathfrak{b} \in \mathcal{I}(A)$, the following computational rules are easy to verify :

(i) $\mathfrak{a} \subseteq \sqrt{\mathfrak{a}}$. (ii) $\sqrt{\sqrt{\mathfrak{a}}} = \sqrt{\mathfrak{a}}$. (iii) $\sqrt{\mathfrak{a}\mathfrak{b}} = \sqrt{\mathfrak{a}\cap\mathfrak{b}} = \sqrt{\mathfrak{a}}\cap\sqrt{\mathfrak{b}}$ (iv) $\sqrt{\mathfrak{a}+\mathfrak{b}} = \sqrt{\sqrt{\mathfrak{a}}+\sqrt{\mathfrak{b}}}$.

(v) $\sqrt{\mathfrak{a}} = A$ if and only if $\mathfrak{a} = A$. (vi) If $\mathfrak{p}$ is a prime ideal in $A$, then $\sqrt{\mathfrak{p}^n} = \mathfrak{p}$ for all $n \in \mathbb{N}^+$.

**2.2** (Extensions and Contractions of ideals) Let $\varphi : A \to B$ be a ring homomorphism. We can use $\varphi$ to transport ideals from $A$ to $B$ and also to transport ideals from $B$ to $A$. More precisely :

If $\mathfrak{a}$ is an ideal in $A$, then the set $\varphi(\mathfrak{a})$ need not be an ideal in $B$. The ideal $B\varphi(\mathfrak{a})$ generated by $\varphi(\mathfrak{a})$ is called the extension or the pushforward of $\mathfrak{a}$ in $B$. Similarly, if $\mathfrak{b}$ is an ideal in $B$, then $\varphi^{-1}(\mathfrak{b})$ is always an ideal in $A$ which is called the contraction or the pullback of $\mathfrak{b}$ in $A$. Therefore, we have the maps :

$\varphi_* : \mathcal{I}(A) \to \mathcal{I}(B)$, $\mathfrak{a} \mapsto \varphi_*(\mathfrak{a}) := B\varphi(\mathfrak{a})$ and $\varphi^* : \mathcal{I}(B) \to \mathcal{I}(A)$, $\mathfrak{b} \mapsto \varphi^*(\mathfrak{b}) := \varphi^{-1}(\mathfrak{b})$, which are obviously homomorphisms of ordered sets. It is extremely useful to ask about properties of $\varphi_*$ and $\varphi^*$, in particular, when is $\varphi^*$ is injective or surjective.

**(a)** $\varphi^*(\operatorname{Spec} B) \subseteq \operatorname{Spec} A$, in other words, *contraction of a prime ideal is always a prime ideal.* But, in general, $\varphi^*(\operatorname{Spm} B) \subseteq \operatorname{Spm} A$, i.e. contraction of a maximal ideal need not be a maximal ideal. ( — **Remark :** However, the behavior of prime ideals under $\varphi_*$ under the ring extensions $\iota : \mathbb{Z} \to B$, where $B$ is the ring of algebraic integers in a number field, is one of the central problems of algebraic number theory.)

**(b)**

Moreover, (Push-pull formula) $\varphi^*\varphi_*\mathfrak{a} = \mathfrak{a}$ for all $\mathfrak{a} \in \mathcal{I}(A)$.

**2.3** (Nil-radical) The set $\mathfrak{n}_A$ of all nilpotent elements in a ring $A$ is an ideal. (The ideal $\mathfrak{n}_A$ is called the nil-radical of $A$. The ring $A$ is called reduced if $\mathfrak{n}_A = 0$. For example, integral domains are reduced.)

**(a)** The nil-radical of $A$ is the intersection of all the prime ideal in $A$. i.e. $\mathfrak{n}_A = \cap_{\mathfrak{p}\in\operatorname{Spec}A}\,\mathfrak{p}$.
(**Hint:** For the difficult inclusion use the following observation (with $\mathfrak{a}=0$ and $S:=\{s^n \mid n\in\mathbb{N}\}$, where $s\in A\smallsetminus\mathfrak{n}_A$): Let $\mathfrak{a}\subseteq A$ be an ideal and $S\subseteq A$ be a multiplicative subset of $A$, i.e. a submonoid of the multiplicative monoid $(A,\cdot)$ of $A$. Then the set $\mathcal{P}:=\{\mathfrak{b}\in\mathfrak{I}(A)\mid \mathfrak{a}\subseteq\mathfrak{b}\subseteq A\smallsetminus S\}$ has maximal elements with respect to the natural inclusion and every such a maximal element in $\mathcal{P}$ is a prime ideal in $A$. By the way this also proves that $\operatorname{Spec}A\neq\emptyset$ if and only if $A\neq 0$.)

**(b)** For an ideal $\mathfrak{a}$ in $A$, $\sqrt{\mathfrak{a}}=\cap_{\mathfrak{p}\in\mathrm{V}(\mathfrak{a})}\,\mathfrak{p}$, where $\mathrm{V}(\mathfrak{a}):=\{\mathfrak{p}\in\operatorname{Spec}A\mid\mathfrak{a}\subseteq\mathfrak{p}\}$.
(**Hint:** Let $\pi: A\to A/\mathfrak{a}$ be the canonical projection. Then $\sqrt{\mathfrak{a}}=\pi^{-1}(\mathfrak{n}_{A/\mathfrak{a}})$. Now, apply (a).)

**(c)** In a ring $A$ the set of zero-divisors $\mathrm{Z}(A)$ is a union of certain prime ideals. (**Hint:** We may assume that $A\neq 0$. Since $S=A^*$ is a multiplicative subset of $A$ with $\{0\}\cap S=\emptyset$, by the obeservation in the hint of (a), there are prime ideal $\mathfrak{p}$ in $A$ with $\mathfrak{p}\cap S=\emptyset$. Show that $\mathrm{Z}(A)=\cup_{\mathfrak{p}\in\mathcal{M}}\mathfrak{p}$, where $\mathcal{M}:=\{\mathfrak{p}\in\operatorname{Spec}A\mid\mathfrak{p}\cap S=\emptyset\}$. See also Exercises 2.9 and 2.10)

**2.4** For the polynomial ring $A[X]$ over a ring $A$, show that $\mathfrak{n}_{A[X]}=(\mathfrak{n}_A)[X]$ and $\mathrm{Z}(A[X])=\mathrm{Z}(A)+(\mathfrak{n}_A)[X]$.
(**Hint:** Use Exercise 1.3 a) and c).

**2.5** The intersection $\mathfrak{m}_A:=\cap_{\mathfrak{m}\in\operatorname{Spm}A}\mathfrak{m}$ of all maximal ideals in a ring $A$ is called the Jacobson-radical of $A$.

**(a)** For an element $x\in A$, the following statements are equivalent :
(i) $x\in\mathfrak{m}_A$. (ii) $1-xy\in A^\times$ for every $y\in A$.

**(b)** Let $\mathrm{P}:=A[X_i]_{i\in I}$ with $I\neq\emptyset$. Then the Jacobson-radical $\mathfrak{m}_\mathrm{P}$ and the nil-radical $\mathfrak{n}_\mathrm{P}$ of P are equal. **Hint:** $1+X_i\mathfrak{m}_\mathrm{P}\subseteq\mathrm{P}^\times$.

**2.6** **(a)** Compute $\operatorname{Spm}\mathbf{A}_m$, $\mathfrak{m}_{\mathbf{A}_m}$, $\mathfrak{n}_{\mathbf{A}_m}$ for a *minimal ring* $\mathbf{A}_m$ of positive characteristic $m\in\mathbb{N}^*$. What are $\operatorname{Spec}\mathbf{A}_m$, $\operatorname{Spm}\mathbf{A}_m$ and their cardinalities? When exactly $\mathbf{A}_m$ is reduced? (**Some Definitions:** Let $A$ be a ring. Then the additive subgroup $\mathbb{Z}\cdot 1_A$ generated by $1_A$ is the smallest subring of $A$ (since every subring of $A$ contains the identity element $1_A$ of $A$ and the cyclic subgroup generated by $1_A$ is already a subring of $A$). It is called the minimal ring of $A$. The minimal ring of $A$ is also the minimal ring of every subring of $A$. For example, the minimal ring of $\mathbb{Z}$ is $\mathbb{Z}$ itself. In particular, $\mathbb{Z}$ has no subrings different from $\mathbb{Z}$. A ring which coincides with its minimal ring is called a minimal ring per se. The order of the identity element $1_A$ of $A$ in the additive group of $A$ is called the characteristic of $A$ and is denoted by $\operatorname{char}A$. A ring $A$ is of characteristic 0 if and only if its minimal ring $\mathbb{Z}\cdot 1_A$ is infinite. In this case all the multiples $n\cdot 1_A$, $n\in\mathbb{Z}$, are pairwise distinct. $A$ is of positive characteristic $m\in\mathbb{N}^*$ if and only if the minimal ring $\mathbb{Z}\cdot 1_A$ is finite and consists of the $m$ pairwise different elements $r\cdot 1_A$, $r=0,\ldots,m-1$. If $n\in\mathbb{Z}$ is a multiple of $\operatorname{char}A$, then $na=0$ for all $a\in A$ because of $na=(n\cdot 1_A)\cdot a=0\cdot a=0$. In other words, *the characteristic of $A$ is the exponent of the additive group of $A$*. It follows, that the order of a finite ring and its characteristic have the same prime divisors, cf. the Theorem of Cauchy (which is easy to prove for finite *abelian* groups). *All subrings of a ring $A$ have the same characteristic as $A$.*)

**(b)** For a family $A_i$, $i\in I$, of rings and its product $A:=\prod_{i\in I}A_i$, show that $\mathfrak{m}_A=\prod_{i\in I}\mathfrak{m}_{A_i}$ and $\mathfrak{n}_A\subseteq\prod_{i\in I}\mathfrak{n}_{A_i}$. Give examples that the inclusion for the nil radical may be strict.

**2.7** Let $\mathrm{R}:=A[\![X]\!]$ be the formal power series ring one indeterminate $X$ over $A$. Then :

**(a)** The nil-radical $\mathfrak{n}_\mathrm{R}=\{f\in\mathrm{R}\mid$ all coefficients of $f\subseteq\mathfrak{n}_A\}$ and the Jacobson-radical $\mathfrak{m}_\mathrm{R}=\{f\in\mathrm{R}\mid f(0)\in\mathfrak{m}_A\}$. (**Hint:** Use the analog of the Exercise 1.3 to the power series ring $\mathrm{R}=A[\![X]\!]$: If $f\in\mathrm{R}$ is nilpotent, then all the coefficients of $f$ are nilpotent. Is the converse true?. Further, $f\in\mathrm{R}^\times$ if and only if $f(0)\in A^\times$.)

**(b)** Show that each prime ideal of $A$ is a contraction of a prime ideal of R.

**(c)** If $\mathfrak{M}\in\operatorname{Spm}\mathrm{R}$, then $\mathfrak{M}$ is generated by $(\mathfrak{M}\cap A)\cup\{X\}$ and the contraction $\mathfrak{M}\cap A$ of $\mathfrak{M}$ is a maximal ideal of $A$.

**2.8** Let $A$ be a ring and $\mathfrak{m}_A$ be its Jacobson-radical.

**(a)** Let $\mathfrak{a}$ be an ideal in $A$ with $\mathfrak{a}\subseteq\mathfrak{m}_A$. Then the group homomorphism $\pi^\times: A^\times\to(A/\mathfrak{a})^\times$ of unit groups induced by the canonical projection $\pi: A\to A/\mathfrak{a}$ is surjective with kernel $\operatorname{Ker}\pi^\times=1+\mathfrak{a}$. In particular, $A^\times/(1+\mathfrak{a})\cong(A/\mathfrak{a})^\times$. If $\mathfrak{a}^2=0$, then the map $\mathfrak{a}\to 1+\mathfrak{a}$ with $a\mapsto 1+a$ is an isomorphism of the additive group $\mathfrak{a}$ onto the multiplicative group $1+\mathfrak{a}$. — Deduce that: If $A$ is a ring with finitely many elements, then

$$\text{(Euler's Formula)}\quad |A^\times|=|A|\cdot\prod_{\mathfrak{m}\in\operatorname{Spm}A}\left(1-\frac{1}{|A/\mathfrak{m}|}\right).$$

**(b)** Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals in $A$ with $\mathfrak{a}^2\subseteq\mathfrak{b}\subseteq\mathfrak{a}\subseteq\mathfrak{m}_A$. Then there exists a canonical isomorphism from the additive group $\mathfrak{a}/\mathfrak{b}$ onto the multiplicative group $(1+\mathfrak{a})/(1+\mathfrak{b})$ with $\overline{a}\mapsto\overline{1+a}$, where $^{-}$ denote the residue-class map into $A/\mathfrak{b}$ resp. in $A^\times/(1+\mathfrak{b})$.

**2.9** (Local rings) For a ring $A$ the following five conditions are equivalent: (i) $A$ contains exactly one maximal left ideal. (ii) $A$ contains exactly one maximal right ideal. (iii) $\mathfrak{m}_A=A\smallsetminus A^\times$. (iv) $A\smallsetminus A^\times$ is a (two-sided) ideal in $A$. (v) $A\smallsetminus A^\times$ is a subgroup of $(A,+)$. (v′) $A\neq 0$ and, if $a,b\in A$ and $a+b\in A^\times$, then $a\in A^\times$ or $b\in A^\times$. (v″) For every $n\in\mathbb{N}$, if $a_1,\ldots,a_n\in A$ and $a_1+\cdots+a_n\in A^\times$, then $a_i\in A^\times$ for some $i$. (A ring $A$ satisfying these conditions is called a local ring.) Using the residue-class ring $A/\mathfrak{m}_A$, the above conditions are also equivalent to the following condition: (vi) $A/\mathfrak{m}_A$ is a division domain. — For which $m \in \mathbb{N}^*$ is the minimal ring $\mathbf{A}_m$ a local ring? A non-zero ring in which every element is either a unit or nilpotent is a local ring and its Jacobson-radical is a nil-ideal.

*__2.10__ (Prime Avoidance Theorem) Let $\mathfrak{p}_1,\ldots,\mathfrak{p}_n$, $n \geq 2$, be ideals in $A$ such that at most two of $\mathfrak{p}_1,\ldots,\mathfrak{p}_n$ are not prime and let $R$ be an additive subgroup of $A$ which is closed under multiplication. (for example, $R$ could be an ideal of $A$ or a subring of $A$.) Suppose that $R \subseteq \cup_{i=1}^n \mathfrak{p}_i$. Then $R \subseteq \mathfrak{p}_j$ for some $j$ with $1 \leq j \leq n$. (**Hint :** We use induction on $n$. For the beginning of induction at $n=2$, note that we assume merely that $\mathfrak{a} := \mathfrak{p}_1$ and $\mathfrak{b} := \mathfrak{p}_2$ are ideals. If $R \not\subseteq \mathfrak{a}$ and $R \not\subseteq \mathfrak{b}$, then choose $a \in R \smallsetminus \mathfrak{a}$ and $b \in R \smallsetminus \mathfrak{b}$. Then $a \in \mathfrak{b}$ and $b \in \mathfrak{a}$ by hypothesis $R \subseteq \mathfrak{a} \cup \mathfrak{b}$, but then $a+b \in R \smallsetminus (\mathfrak{a} \cup \mathfrak{b})$ a contradiction. For induction step we may assume (by renumbering) that $\mathfrak{p}_{n+1}$ is prime. By induction hypothesis, for each $j = 1,\ldots,n+1$, there is an element $a_j \in R \smallsetminus \cup_{i=1, i\neq j}^{n+1} \mathfrak{p}_i$. Then $a_j \in \mathfrak{p}_j$ for all $j = 1,\ldots,n+1$ by hypothesis $R \subseteq \cup_{i=1}^{n+1} \mathfrak{p}_i$, and $a_1 \cdots a_n \notin \mathfrak{p}_{n+1}$, since $\mathfrak{p}_{n+1}$ is prime. Now, consider the element $b := a_1 \cdots a_n + a_{n+1} \in R$. — **Remark :** The Prime Avoidance Theorem is most frequently used in situations where $R$ is actually an ideal of $A$ and $\mathfrak{p}_1,\ldots,\mathfrak{p}_n$ are all prime ideals of $A$. However, there are some occasions when general statement useful. The name "Prime Avoidance Theorem" is clear from its reformulation : *If $\mathfrak{p}_1,\ldots,\mathfrak{p}_n$, $n \geq 2$, be ideals in $A$ and at most two of $\mathfrak{p}_1,\ldots,\mathfrak{p}_n$ are not prime and if $R \not\subseteq \mathfrak{p}_i$ for every $i = 1,\ldots,n$, then there exists $c \in S \setminus \cup_{i=1}^n \mathfrak{p}_i$, i.e. $c$ "avoids" all the ideals $\mathfrak{p}_1,\ldots,\mathfrak{p}_n$, "most" of which are prime.*)

The following refinements of the Prime Avoidance Theorem are extremely useful :

**(a)** Let $\mathfrak{p}_1,\ldots,\mathfrak{p}_n$ be prime ideals in $A$, $\mathfrak{a}$ be an ideal in $A$ and let $a \in A$ be such that $Aa + \mathfrak{a} \not\subseteq \bigcup_{i=1}^n \mathfrak{p}_i$. Then show that there exists $c \in \mathfrak{a}$ such that $a + c \notin \cup_{i=1}^n \mathfrak{p}_1$. (**Hint :** We may assume that $\mathfrak{p}_1,\ldots,\mathfrak{p}_n$ are not contained in another of them. Further, we may assume that $a \in \cup_{i=1}^n \mathfrak{p}_i$ (otherwise take $c = 0$) and $\mathfrak{a} \not\subseteq \cup_{i=1}^n \mathfrak{p}_i$ by Prime avoidance. Renumber $\mathfrak{p}_1,\ldots,\mathfrak{p}_n$ so that $a \in \mathfrak{p}_i$ for $i = 1,\ldots,k$ and $a \notin \mathfrak{p}_j$ for $j = k+1,\ldots,n$. Now choose $b \in \mathfrak{a} \smallsetminus \cup_{i=1}^n \mathfrak{p}_i$ by assumption and $b' \in \mathfrak{p}_{k+1} \cap \cdots \cap \mathfrak{p}_n \smallsetminus \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_k$ by prime avoidance (clear for $k = n$ and for $k < n$, otherwise $\mathfrak{p}_{k+1} \cap \cdots \cap \mathfrak{p}_n \subseteq \mathfrak{p}_i$ for some $1 \leq i \leq k$ and hence[1] $\mathfrak{p}_j \subseteq \mathfrak{p}_i$ for some $1 \leq i \leq k$ and some $k+1 \leq j \leq n$ a contradiction of the assumption). Now check that $c := bb' \in \mathfrak{a}$ and $a + c \notin \cup_{i=1}^n \mathfrak{p}_i$.)

**(b)** Let $A$ be a ring which contain an *infinite* field as subring. and let $\mathfrak{a}, \mathfrak{b}_1,\ldots,\mathfrak{b}_n$, $n \geq 2$, be ideals in $A$ such that $\mathfrak{a} \subseteq \cup_{i=1}^n \mathfrak{b}_i$, then prove that $\mathfrak{a} \subseteq \mathfrak{b}_j$ for some $j$ with $1 \leq j \leq n$. (**Hint :** Use : Let $V_0,\ldots,V_n$ be subspaces of a vector space $V$. If $V_0 \not\subseteq V_j$ for all $i = 1,\ldots,n$, then $V_0 \not\subseteq V_1 \cup \ldots \cup V_n$, which is a consequence of the Exercise 2.2, 2016 CSA-E0 219 Linear Algebra and Applications ($V = V_0$ with subspaces $V_1 \cap V_0,\ldots,V_n \cap V_0$))

*__2.11__ (Minimal prime ideals) Let $A$ be a ring and let $\mathfrak{a}$ be an ideal in $A$. A minimal element in the set $\mathrm{V}(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{a} \subseteq \mathfrak{p}\}$ (partially ordered by the inclusion) is called a minimal prime ideal of $\mathfrak{a}$. If $A \neq 0$, then a minimal prime ideal of the zero ideal $0$ in $A$ is called a minimal prime ideal in $A$. The set of minimal prime ideals of $\mathfrak{a}$ is denoted by $\operatorname{Min}(\mathfrak{a})$.

**(a)** Every prime ideal in $A$ containing the ideal $\mathfrak{a}$ in $A$ contains a minimal prime ideal of $\mathfrak{a}$. (**Hint :** For $\mathfrak{p} \in \mathrm{V}(\mathfrak{a})$, the set $\{\mathfrak{p}' \in \operatorname{Spec} A \mid \mathfrak{a} \supseteq \mathfrak{p}' \supseteq \mathfrak{p}\}$ is inductively ordered with respect to the reverse inclusion and hence by Zorn's lemma has a maximal elements with respect to the reverse inclusion, i. e., has a minimal element with respect to the inclusion.)

**(b)** The radical of the ideal $\mathfrak{a}$ is the intersection of the minimal prime ideals of $\mathfrak{a}$, i.e. $\sqrt{\mathfrak{a}} = \cap_{\mathfrak{p} \in \operatorname{Min}(\mathfrak{a})} \mathfrak{p}$. In particular, *the nil-radical of $A$ is the intersection of the minimal prime ideals of $A$.*

**(c)** If $\mathfrak{a}$ is a radical ideal, i. e. $\mathfrak{a} = \sqrt{\mathfrak{a}}$, then the set

$$\mathrm{Z}_A(A/\mathfrak{a}) := \{a \in A \mid \vartheta_a : A/\mathfrak{a} \to A/\mathfrak{a} \text{ is not injective}\}$$

of zero-divisors for the $A$-module $A/\mathfrak{a}$ is the union of the minimal prime ideals of $\mathfrak{a}$, i. e, $\mathrm{Z}_A(A/\mathfrak{a}) == \cup_{\mathfrak{p} \in \operatorname{Min}(\mathfrak{a})} \mathfrak{p}$. In particular, *the set of zero-divisors in $A$ is the union of the minimal prime ideals of $A$ and hence all elements of a minimal prime ideals of $A$ are zero-divisors.*

**(d)** Suppose that $A$ is noetherian. Then the set of minimal prime ideals of $\mathfrak{a}$ is finite. In particular, *in a noetherian reduced ring $A$, the set of zero divisors in $A$ is a finite union (the minimal) prime ideals in $A$.* (**Hint :** Let $\mathfrak{a}$ be a maximal in the set of ideals $\{\mathfrak{b} \mid \operatorname{Min}(\mathfrak{b}) \text{ is not finite}\}$ in $A$. Then there exist $a, b \in A$ such that $a \notin \mathfrak{a}$, $b \notin \mathfrak{a}$, $ab \in \mathfrak{a}$. Now, consider the minimal prime ideals of $\mathfrak{a} + Aa$, $\mathfrak{a} + Ab$.)

*__2.12__ (Associated Prime Ideals) In this Exercise another proof of the important assertion about the set $\mathrm{Z}(A)$ of zero divisors in noetherian ring $A$ (see Exercise 2.9 (d) by using an idea of I. KAPLANSKY), namely : *The set of zero-divisors in a noetherian ring is a finite union of prime ideals*
— Let $A$ be an arbitrary ring. The set $\mathrm{Z}(A)$ of zero-divisors in $A$ is the union of the annihilators $(0 :_A a) = \operatorname{Ann}_A a := \{b \in A \mid ba = 0\}$, $a \in A$, $a \neq 0$.

**(a)** Every maximal element in set of ideals $\{\operatorname{Ann}_A a \mid a \in A, a \neq 0\}$ (with respect to the natural inclusion) is a prime ideal. — (**Remark :** The prime ideals of the form $\operatorname{Ann}_A a$ $a \in A, a \neq 0$ are called the associated prime ideals of the ring $A$ and their subset is denoted by $\operatorname{Ass} A$.)

**(b)** If $A$ is noetherian, then the set $\{\operatorname{Ann}_A a \mid a \in A, a \neq 0\}$ has only finitely many maximal elements (with respect to the natural inclusion). In particular, the set of zero-divisors in $A$ is a finite union of prime ideals

[1] Let $\mathfrak{p}$ be a prime ideal and $\mathfrak{a}_1,\ldots,\mathfrak{a}_n$ are arbitrary ideals in a ring $A$. Then the following are equivalent : (i) $\mathfrak{a}_i \subseteq \mathfrak{p}$ for some $1 \leq i \leq n$. (ii) $\cap_{i=1}^n \mathfrak{a}_i \subseteq \mathfrak{p}$. (iii) $\prod_{i=1}^n \mathfrak{a}_i \subseteq \mathfrak{p}$. In particular, if $\mathfrak{p} = \cap_{i=1}^n \mathfrak{a}_i$, then $\mathfrak{p} = \mathfrak{a}_i$ for some $i$ with $1 \leq i \leq n$.

which are precisely the annihilators of elements of $A$. (**Hint :** Let $\operatorname{Ann}_A a_i$, $i \in I$ be the maximal elements and let $a_{i_1}, \ldots, a_{i_n}$ be a finite generating system for the ideal $\sum_{i\in I} Aa_i$ and $\mathfrak{p}_\nu := \operatorname{Ann}_A a_{i_\nu}$ for $\nu = 1, \ldots, n$. Then from $\cap_{\nu=1}^n \mathfrak{p}_\nu \subseteq \operatorname{Ann}_A a_i$ it follows that $\mathfrak{p}_{\nu_0} \subseteq \operatorname{Ann}_A a_i$ and hence (see the Footnote 1) $\mathfrak{p}_{\nu_0} = \operatorname{Ann}_A a_i$ for some $\nu_0 \in \{1, \ldots, n\}$.)

**(c)** Let $\mathfrak{a}$ be an ideal in a noetherian ring $A$. Then $\mathfrak{a}$ contains a non-zero divisor if and only if $\operatorname{Ann}_A \mathfrak{a} = 0$. **Hint:** Use the part b) and the Prime Avoidance Theorem, See Exercise 2.8.

**2.13** Let $A := \mathrm{C}_{\mathbb{R}}([0,1])$ be the $\mathbb{R}$-algebra of continuous real valued functions on the closed interval $[0,1] \subseteq \mathbb{R}$. For $f \in A$, let $\mathrm{V}(f) := \{t \in [0,1] \mid f(t) = 0\}$ denote the set of zeros of $f$ in $[0,1]$ and $\mathrm{U}(f) := [0,1] \smallsetminus \mathrm{V}(f)$. For $f \in A$, prove that :

**(a)** $f \in A^\times$ if and only if $\mathrm{V}(f) = \emptyset$.

**(b)** $f \in A$ is a non-zero divisor in $A$ if and only if $\mathrm{V}(f)$ is nowhere dense in $[0,1]$, i.e. the complement $\mathrm{U}(f)$ of $\mathrm{V}(f)$ is dense in $[0,1]$. (**Hint :** ($\Rightarrow$) Let $\mathrm{U} := \mathrm{U}(f)$. If $\overline{\mathrm{U}} \subsetneq [0,1]$, then $\mathrm{U} \cap V = \emptyset$ for some non-empty subset $V \subseteq [0,1]$, i.e. $V \subseteq \mathrm{V}(f)$. By Exercise[2] there exists $g \in A$ with $\mathrm{V}(f) = [0,1] \smallsetminus V$. But, then $fg = 0$ and $g \neq 0$, i.e. $f$ is a zero-divisor in $A$. ($\Leftarrow$) Suppose that $fg = 0$ and $g \neq 0$. Then $g = 0$ on $\mathrm{U}$ which is dense in $[0,1]$ and hence $g = 0$ on $[0,1]$ by continuity of $g$. )

**(c)** $A^\times \subsetneq \mathrm{S}_0 := A \smallsetminus \mathrm{Z}(A)$, i.e. there are non-zero divisors which are non-units in $A$. (**Hint :** Consider $f \in A$ with $\mathrm{V}(f) = \{x_1, \ldots, x_r\}$, $\mathrm{U}(f) := [0,1] \smallsetminus \{x_1, \ldots, x_r\} = \cap_{i=1}^r ([0,1] \smallsetminus \{x_i\})$ is dense in $[0,1]$.)

**(d)** There exists a prime ideal in $A$ which is not maximal in $A$. (**Hint :** The set of zero divisors $\mathrm{Z}(A) = \cup_{\mathfrak{p}\in \mathrm{Min}(A)}$, see Exercise 2.9 (c). If $\operatorname{Spec} A = \operatorname{Spm} A$, then $A \smallsetminus \mathrm{S}_0 = \mathrm{Z}(A) = \cup_{\mathfrak{m}\in \operatorname{Spm} A} \mathfrak{m} = A \smallsetminus A^\times$, i.e. $\mathrm{S}_0 = A^\times$ which contracdicts (b).)

**(e)** For a subset $Y \subset [0,1]$, let $\mathrm{I}(Y) := \{f \in A \mid f(y) = 0 \text{ for all } y \in Y\}$. For example, $\mathrm{I}(\{t\}) = \mathfrak{m}_t := \{f \in A \mid f(t) = 0\}$ is a maximal ideal in $A$. Show that $\mathrm{I}(Y)$ is an ideal in $A$ and $\mathrm{I}(Y) \in \operatorname{Spm} A$ if and only if $Y$ is singleton. (**Hint :** Note that if $Y' \subseteq Y \subseteq [0,1]$, then $\mathrm{I}(Y) \subseteq \mathrm{I}(Y')$.)

[2] **Exercise :** For every closed subset $Z \subseteq \mathbb{R}$, there exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $Z = \mathrm{Z}(f) = \{t \in \mathbb{R} \mid f(t) = 0\}$. (**Hint :** Consider the distance function $t \mapsto d(t, Z)$.)