# MA 312 Commutative Algebra / Aug–Dec 2017

(Int PhD. and Ph. D. Programmes)

Download from : http://www.math.iisc.ernet.in/patil/courses/courses/Current Courses/...

**Tel :** +91-(0)80-2293 3212/09449076304 **E-mails :** patil@math.iisc.ernet.in

**Lectures :** Wednesday and Friday ; 14:00–15:30 **Venue:** MA LH-2 ( **if LH-1 is not free** ) / LH-1

**Seminars :** Sat, Nov 18 (10:30–12:45) ; Sat, Nov 25 (10:30-12:45)

**Final Examination :** Tuesday, December 05, 2017, 09:00–12:00

**Evaluation Weightage : Assignments :** 20% **Seminars :** 30% **Final Examination :** 50%

| Range of Marks for Grades (Total 100 Marks) | | | | | | |
|---|---|---|---|---|---|---|
| | **Grade S** | **Grade A** | **Grade B** | **Grade C** | **Grade D** | **Grade F** |
| **Marks-Range** | $> 90$ | 76 — 90 | 61 — 75 | 46 — 60 | 35 — 45 | $< 35$ |

| | **Grade $A^+$** | **Grade A** | **Grade $B^+$** | **Grade B** | **Grade C** | **Grade D** | **Grade F** |
|---|---|---|---|---|---|---|---|
| **Marks-Range** | $> 90$ | 81 — 90 | 71 — 80 | 61 — 70 | 51 — 60 | 40 — 50 | $< 40$ |

## 4. Noetherian and Artinian modules — Continued

**Submit a solutions of ∗-Exercises ONLY.** **Due Date : Friday, 13-10-2017**

**4.1** Let $A$ be a commutative ring, $V$ a finite $A$-module and $W$ an arbitrary $A$-module. If $V \cong V \oplus W$ (as $A$-modules) then $W = 0$. (**Hint :** Use : Every surjective endomorphism $f : V \to V$ of a finite module over a *commutative* ring is bijective.)

**4.2** **(a)** Every artinian module is a direct sum of finitely many indecomposable modules.

**(b)** Every noetherian module is a direct sum of finitely many indecomposable modules. (**Hint :** Suppose not, then construct an infinite strict decreasing sequence $V_0 \supset V_1 \supset \cdots$ of direct summands in the module and hence construct an infinite strict increasing sequence of direct summands.)

**4.3** Let $A$ be a ring and be $V$ an $A$-module which is a direct sum of submodules $V_1, \ldots, V_n$. Suppose that the endomorphism rings of $V_i$, $1 \leq i \leq n$, are local. If $V$ is a direct sum of the indecomposable submodules $W_1, \ldots, W_m$, then $m = n$ and there exists a permutation $\sigma \in \mathfrak{S}_n$ with $V_i \cong W_{\sigma(i)}$. (**Hint :** Proof by induction on $n$. Let $P_1, \ldots, P_n$ resp. $Q_1, \ldots Q_m$ be the families of projections corresponding to the decompositions $V = V_1 \oplus \cdots \oplus V_n$ resp. $V = W_1 \oplus \cdots \oplus W_m$. Let $P_{1j}$ be the restriction $P_1|W_j$ into the image $V_1$ and $Q_{j1}$ be the restriction $Q_j|V_1$ into the image $W_j$. Then $\mathrm{id}_{V_1} = \sum_{j=1}^{m} P_{1j} Q_{j1}$. Since $\mathrm{End}_A V_1$ is local, there exists $r$ such that $P_{1r} Q_{r1}$ is an isomorphism. Now, it follows from the analog Exercise 7.1, 2016 CSA-E0 219 Linear Algebra and Applications[1] of for a general (commutative) base ring, that $Q_{r1} : V_1 \to W_r$ is an isomorphism.)

**4.4** Let $A$ be a ring and $V$ be an indecomposable $A$-module which is artinian as well as noetherian. Then $\mathrm{End}_A V$ is a local ring whose Jacobson-radical is a nilideal. (**Hint :** Let $f \in \mathrm{End}_A V$. There exists a $m \in \mathbb{N}$ with $\mathrm{Ker} f^n = \mathrm{Ker} f^m$ and $\mathrm{Img} f^n = \mathrm{Img} f^m$ for all $n \geq m$. Then $V = \mathrm{Ker} f^m \oplus \mathrm{Img} f^m$ and it follows that $f$ is nilpotent or bijective.)

**4.5** (Theorem of Krull–Schmidt) Let $A$ be a ring and $V$ be an $A$-module which is artinian as well as noetherian. Then : $V$ is a direct sum of indecomposable submodules $V_1, \ldots, V_n$. If $V = W_1 \oplus \cdots \oplus W_m$ is another direct sum decomposition of $V$ into indecomposable submodules, then $m = n$, and there exists a permutation $\sigma \in \mathfrak{S}_n$ with $V_i \cong W_{\sigma(i)}$.

**4.6** Let $H$ be a finitely generated abelian group which is a homomorphic image of a torsion-free abelain group of the finite rank $n$. Then $H$ is a direct sum of $\leq n$ cyclic groups. (**Hint :** From the hypothesis it follows that $H$ is also homomorphic image of a finitely generated torsion-free group of the rank $\leq n$. For the concept of rank, see Supplements S1A.19 and S1A.24.)

**4.7** A finitely generated abelian group with commutative automorphism group is either cyclic or isomorphic to $\mathbb{Z} \times \mathbb{Z}_2$. (**Hint :** The endomorphism ring of $\mathbb{Z} \times \mathbb{Z}_2$ ist not commutative. Therefore : The endomorphism ring of a finitly generated abelian group $H$ is commutative if and only if $H$ is cyclic.)

**4.8** Let $V$ be an $A$-module. We say that $V$ is decomposable of bounded (type $\leq m$, $m \in \mathbb{N}$) if every direct sum decomposition of $V$ has at most $m$ non-trivial summands.

**(a)** Let $A$ be a noetherian commutative ring. Then $A$ (as an $A$-module) is decomposable of bounded type. If $A$ is decomposable of bounded type $\leq m$, but not of type $\leq m-1$, then the number of idempotents elements in $A$ is $2^m$ and $A$ is isomorphic to the product ring $A_1, \ldots, A_m$ with indecomposable rings $A_1, \ldots, A_m$.

**(b)** An $A$-module $V$ is decomposable of bounded type $\leq m$ if and only if every set (subset of $\mathrm{End}_A V$) of pairwise commuting $A$-linear projections have at most $2^m$ elements. (**Hint :** If $\mathrm{End}_A V$ has pairwise distinct commuting $A$-linear projections $P_1, \ldots, P_s$, $s > 2^m$, then by (a) the (noetherian) commutative ring $C := \mathbb{Z}[P_1, \ldots, P_s] \subseteq \mathrm{End}_A V$ is isomorphic to a product ring $C_1 \times \cdots \times C_n$ with $n > m$.)

[1] Let $f : V \to W$ and $g : W \to X$ be homomorphisms of modules over a ring. If the composition $gf$ is an isomorphism, then $f$ is injective and $W = \mathrm{Img} f \oplus \mathrm{Ker} g$.

**(c)** From part (b) deduce that: If $V$ is decomposable of bounded type $\leq m$ and if the homothecy $\vartheta_2 : V \to V$ of $V$ by 2 is bijective, then $V$ also has at most $2^m$ $A$-linear involutions. (— Recall that: An element $a \in M$ of a multiplicative monoid $M$ is called an involution if $a^2 = e_M (=$ the neural element of $M)$. The involutions are invertible elements which are self inverses. The product of two involutions in $M$ is again involution if and only if both these elements commute. If $M$ is a commutative monoid, then the set $\operatorname{Inv} M$ of all involutions in $M$ is a subgroup of the unit group $M^\times$ of $M$. — **Hint:** For a ring $A$, the map $\gamma : \operatorname{Idp} A \to \operatorname{Inv} A$, $a \mapsto 1 - 2a$, is injective if $2 \cdot 1_A$ is a non-zero divisor in $A$ and is bijective if $2 \cdot 1_A$ is a unit in $A$. Moreover, if $A$ is commutative, then $\gamma$ is even a group homomorphism of the additive group $\operatorname{Idp} A$ (with the addition $a \triangle b := (a-b)^2$) in the multiplicative group $\operatorname{Inv} A$. — For a commutative ring $A$, the set $\operatorname{Idp} A$ of idempotent elements in $A$ with the addition $\triangle$ defined above and the multiplication induced from $A$ is a Boolean ring. It coincides with $A$ if $A$ itself is Boolean.)

**(d)** Let $A$ be a local ring and $V$ be a finite $A$-module, then $V$ is decomposable of bounded type $\leq \operatorname{Dim}_{A/\mathfrak{m}_A} V/\mathfrak{m}_A V$. (**Hint:** Use the Lemma von K r u l l – N a k a y a m a, see Supplements S1A.19 and S1A.31.)

**(e)** If $V$ ia artinian and noetherian, then $V$ is decomposable of bounded type.

**(f)** Let $A$ be a commutative ring and $V$ be a noetherian $A$-module. Then $V$ is decomposable of bounded type. (— Recall the N o e t h e r i a n I n d u c t i o n: *Let $(X, \leq)$ be a noetherian ordered set. Suppose that a statement $A(x)$ is associated to each element $x \in X$. Assume that the following condition holds: for every $x \in X$, $A(y)$ holds for all $x < y$, then $A(x)$ also holds. Then $A(x)$ holds for every $x \in X$.* **Proof:** Let $Z := \{z \in X \mid A(z) \text{ does not hold}\}$. If $Z \neq \emptyset$, then $Z$ has a maximal element, say $x \in Z$. For every $y \in X$ with $x < y$, $y \notin Z$ and hence $A(x)$ holds for such $y$. But, then by hypothesis, $A(x)$ also holds, a contradiction! Therefore $Z = \emptyset$. •

— **Hint:** We may also assume that $A$ is noetherian. Now use noetherian induction on $\operatorname{Ann}_A V$ to assume that the assertion is true for all residue class rings of $A$. If there exist elements $a, b \in A$, $a \neq 0$, $b \neq 0$ and $ab = 0$, then consider $V/aV$ and $V/bV$. But, if $A$ is an integral domain, then $V$ is decomposable of bounded type $\leq m+n$, if $V$ is of rank $m$ and if the torsion submodule $t_A V$ (whose annihilator is $\neq 0$) is decomposable of bounded type $\leq m$. — **Remark:** ( P r i n c i p a l i d e m p o t e n t s ) Direct decompositions of rings can be described canonically by idempotent elements, see Supplement S4.1, Theorem 4.S.4. The indecomposability (connectedness) of a commutative ring $A$ is equivalent (see Supplement S4.1, Corollary 4.S.5) to the condition that $A$ has no idempotents other than 0 and 1. In case of a local ring this condition is satisfied as one can see it from an equation of the form $0 = e - e^2 = e(1-e)$ ; since if $e$ is not a unit, $e$ belongs to the Jacobson-radical, and so $1-e$ is a unit.

Now, let $A$ be an artinian commutative ring. Then by the decomposition theorem for artinian commutative rings (see Supplement S4.5, Theorem 4.S.17) there exists a direct decomposition of $A$ into local rings $A_i$, $i = 1, \ldots, s$, corresponding to a decomposition $1 = e_1 + \cdots + e_s$ into pairwise orthogonal idempotent elements $e_i \neq 0$ such that $A_i + a/A(1-e_i)$. These idempotent elements are uniquelly determined. Namely, if $e \in A$ is idempotent, then the homomorphic image of $e$ in $A_i$ and hence coincides with either 0 or 1 in $A_i$. It follows that $e$ is sum of some of the $e_i$. Every direct factor of $A$ is therefore direct product of some of the local rings $A_i$. This also proves once again the uniqueness assertion in Supplement S4.5, Theorem 4.S.17). The elements $e_1, \ldots, e_r$ are called p r i n c i p a l i d e m p o t e n t s of $A$.

The principal idempotents of $A$ are obviously distinguished idempotent elements which are $\neq 0$ and not representable as sum of two $\neq 0$ orthogonal idempotent elements. Therefore they are in this sense irreducible. An automorphism of $A$ permutes the principal idempotents of $A$.)