MA 312 Commutative Algebra / Aug–Dec 2017

(Int PhD. and Ph. D. Programmes)

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Lectures : Wednesday and Friday ; 14:00–15:30 **Venue:** MA LH-2 (**if LH-1 is not free**) / LH-1

Seminars : Sat, Nov 18 (10:30–12:45) ; Sat, Nov 25 (10:30-12:45)

Final Examination : Tuesday, December 05, 2017, 09:00–12:00

Supplement 6

Galois Connections*

* In this supplement is we discuss an abstraction motivated by the celebrated Fundamental theorem of Galois Theory (named after the French mathematician Évariste Galois[1] (1811-1832)). The notion of Galois connection can be defined on arbitrary ordered sets which is a generalisation of the Galois correspondence from Galois theory which investigate the correspondence between subgroups of the Galois group of a field extension and its intermediary subfields.

A *Galois connection* is a particular correspondence between ordered sets and is rather weak compared to an order isomorphism, but every Galois connection induces an isomorphism of certain sub-ordered sets, see ???. We use the term *Galois correspondence* for bijective Galois connection.

S6.1 Let $(X,\leq)$, $(Y,\leq)$ and $(Z,\leq)$ be ordered sets.

(a) (Isotone and Antitone) Let $(X,\leq)$ and $(Y,\leq)$ be ordered sets. A map $f:X\to Y$ is called order-preserving or isotone if for all $x,x'\in X$, $x\leq x'$ implies $f(x)\leq f(x')$. Similarly, f is called order-reversing or antitone if for all $x,x'\in X$, $x\leq x'$ implies $f(x)\geq f(x')$.

(b) Let $f:X\to Y$ and $g:Y\to Z$ be maps. **(1)** If f and g are isotone, then $g\circ f$ is isotone. **(2)** If f and g are antitone, then $g\circ f$ is isotone. **(3)** If one of f and g is isotone and the other is antitone, then $g\circ f$ is antitone.

(c) (Galois Connections) An isotone (resp. antitone) Galois connection between X and Y is a pair of maps (Φ,Ψ), $\Phi:X\to Y$, $\Psi:Y\to X$ such that

(IGC1 (resp. GC1)) Φ and Ψ are both isotone (resp, antitone) maps, and

(IGC2 (resp. GC2)) For all $x\in X$ and all $y\in Y$, $\Phi(x)\leq y \Longleftrightarrow x\leq\Psi(y)$ (resp. $y\leq\Phi(x)\Longleftrightarrow x\leq\Psi(y)$).

(**Remarks :** (1) Note that the condition (IGC2 (resp. GC2)) is equivalent to the condition :

(IGC2$'$ (resp. GC2$'$)) For all $x\in X$ and all $y\in Y$, both inequalities $\Psi(\Phi(x)\leq x$ and $y\leq\Phi(\Psi(y))$ (resp. $x\leq\Psi(\Phi(x)$ and $y\leq\Phi(\Psi(y))$) hold.

(2) In the antitone case there is a symmetry in the defintion : If (Φ,Ψ) is an antitone Galois connection between X and Y, then (Ψ,Φ) is an antitone Galois connection between Y and X. — In contrast to the antitone case, there is a *asymmetry* in the isotone case, more precisely, see the part (d) below.

(3) In the isotone case Φ is called the lower adjoint of Ψ and Ψ is called the upper adjoint of Φ. The essential property of a isotone Galois connection is that its upper and lower adjoint determine each other : $\Phi(x)=\mathrm{LUB}_Y\{y'\in Y\mid x\leq\Psi(y')\}$ and $\Psi(y)=\mathrm{GLB}_X\{x'\in X\mid\Phi(x')\leq y\}$. In particular, if Φ or Ψ is invertible, then each is the inverse of the other, i.e. $\Phi=\Psi^{-1}$.

(4) In the antitone case the symmetry erases the distinction between upper and lower and the two maps Φ and Ψ are called polarities. Each polarity uniquely determines the other, since $\Phi(x)=\mathrm{GLB}_Y\{y\in Y\mid x\leq\Psi(y)\}$ and $\Psi(y)=\mathrm{GLB}_X\{x\in X\mid y\leq\Phi(x)\}$.)

(5) The composition $\Psi\Phi:X\to X$ (resp. $\Phi\Psi:Y\to Y$) is called the closure operator (resp. kernel operator) of the Galois connection (Φ,Ψ). Both these are isotone idempotent maps with the property : $x\leq\Psi(\Phi(x))$ for all $x\in X$ and $\Phi(\Psi(y))\leq$ for all $y\in Y$ (resp. $x\leq\Psi(\Phi(x))$ for all $x\in X$ and $y\leq\Phi(\Psi(y))$ for all $y\in Y$ if (Φ,Ψ) is isotone (resp. antitone), see details in Supplement S6.3.)

(d) The concepts of isotone and antitone Galois connections are equivalent. More precisely : For ordered sets X,Y (with order duals[2] $X^\vee, Y^\vee$) and the maps $\Phi:X\to Y$ and $\Psi:Y\to X$, the following statements are equivalent : (i) (Φ,Ψ) is an antitone Galois connections between X and Y. (ii) (Φ,Ψ) is an isotone Galois connections between $X^\vee$ and Y. (iii) (Ψ,Φ) is an isotone Galois connections between $Y^\vee$ and X. (**Remark :** With this equivalence the study of Galois connections one can always assume that the given Galois connection is antitone. Antitone Galois connections are more common in many applications and prominent in Lattice Theory.)

[1] While still in his teens, he was able to determine a necessary and sufficient condition for a polynomial to be solvable by radicals, thereby solving a problem standing for 350 years. His work laid the foundations for Galois theory and group theory, two major branches of abstract algebra, and the subfield of Galois connections. He died at age 20 from wounds suffered in a *duel*.— A duel is an arranged engagement in combat between two people, with matched weapons, in accordance with agreed-upon rules. Duels in this form were chiefly practiced in early modern Europe with precedents in the medieval code of chivalry, and continued into the modern period (19th to early 20th centuries) especially among military officers. For example, in an ancient epic *Mahabharata* records that hitting below the waist is forbidden in mace duels.

[2] An order dual of an ordered set $(X,\leq)$ is the ordered set $X^\vee:=(X,\geq)$, where $\geq$ is the inverse order relation : $x\geq y$ if and only if $y\leq x$.

S6.2 Let A be any set and $X := (\mathfrak{P}(A), \subseteq)$.

(a) Let $Y := (\mathfrak{P}(A), \subseteq)$. For a fixed subset $B \in \mathfrak{P}(A)$, the maps $\Phi : X \to Y$, $A' \mapsto B \cap A'$ and $\Psi : Y \to X$, $C \mapsto C \cup (A \smallsetminus B)$ form an isotone Galois connection.

(**Remark :** A similar Galois connection can be found in any *Heyting algebra*. — Recall that a Heyting algebra[3] is a bounded lattice (with operations $\vee$ and $\wedge$ and with least element 0 and greatest element 1) equipped with a binary operation $a \Rightarrow b$ of implication such that $c \wedge a \leq b$ is equivalent to $c \leq a \Rightarrow b$. In particular, in any Boolean algebra the maps $\Phi(x) = (a \wedge x)$ and $\Phi(y) = (y \vee \neg a) = (a \Rightarrow y)$ form an isotone Galois connection. In logical terms "implication from a" is the upper adjoint (see Remarks in Supplement S6.1(c)) of "conjunction with a".)

(b) (Galois connection of a map) Let $f : A \to B$ be a map of sets. The maps $f_* : \mathfrak{P}(A) \to \mathfrak{P}(B)$, $A' \mapsto f(A')$ and $f^* : \mathfrak{P}(B) \to \mathfrak{P}(A)$, $B' \mapsto f^{-1}(B')$, form an isotone Galois connection. Further, (1) The interior operator is $f_* \circ f_* : \mathfrak{P}(B) \to \mathfrak{P}(A)$, $B' \mapsto B' \cap f(A)$. In particular, the Galois connection is left-perfect if and only if f is surjective. (2) The Galois connection is right-perfect, i.e. $f^* f_*(A') = A'$ for all $A' \in \mathfrak{P}(A)$ if and only if f is injective. (3) Interpret this isotone Galois connection in terms of the universal antitone Galois connection of Supplement S6.7.

(c) (Galois connection of a ring homomorphism) Let $f : A \to B$ be a homomorphism of commutative rings and let $\mathfrak{I}(A)$ and $\mathfrak{I}(B)$ the lattices of ideals of A and B, respectively. The push-forward map $f_* : \mathfrak{I}(A) \to \mathfrak{I}(B)$, $\mathfrak{a} \mapsto \mathfrak{a}B$ and the pull-back map $f^* : \mathfrak{I}(B) \to \mathfrak{I}(A)$, $\mathfrak{b} \mapsto f^{-1}(\mathfrak{b})$ form an isotone Galois connection. (**Remarks :** In general, it is fruitful to describe the closure operators $f^* f_*$ and $f_* f^*$ and to ask about properties of f_* and f^*, in particular, are they injective or surjective? The most satisfying and important answers can be given in the case when f is surjective, $f = \iota_S : A \to A_S$ is the localisation map and f is an integral. For instance :

(1) If f is surjective with $\mathfrak{a} := \operatorname{Ker} f$, then f^* is injective homomorphism of lattices and $\operatorname{Img} f^* = \{\mathfrak{a} \in \mathfrak{I}(A) \mid \operatorname{Ker} f \subseteq \mathfrak{a}\}$. Moreover, (pull-push formula) $f_* f^*(\mathfrak{b}) = \mathfrak{b}$ for all ideals $\mathfrak{b} \in \mathfrak{I}(B)$, i.e. $f_* f^* = \mathrm{id}_{\mathfrak{I}(B)}$ and (push-pull formula) $f^* f_*(\mathfrak{a}) = \mathfrak{a} + \operatorname{Ker} f$ for all ideals $\mathfrak{a} \in \mathfrak{I}(A)$. In particular, there is a bijective isotone Galois connection (which is also an isomorphism of lattices) between $\mathfrak{I}(B)$ and $\operatorname{Img} f^* = \{\mathfrak{a} \in \mathfrak{I}(A) \mid \operatorname{Ker} f \subseteq \mathfrak{a}\}$.

(2) If $S \subseteq (A, \cdot)$ is a submonoid of the multiplicative monoid $(A, \cdot)$ of A and $\iota = \iota_S : A \to A_S$ is the localisation map, then $\iota_* \iota^* = \mathrm{id}_{\mathfrak{I}(A_S)}$ and $\iota^* \iota_*(\mathfrak{a}) = \{x \in A \mid sx \in \mathfrak{a} \text{ for some } s \in S\}$. Moreover, the restriction of the map ι^* induces the injective map $\iota^* : \operatorname{Spec} A_S \to \operatorname{Spec} A$ image $\operatorname{Img} \iota^* = \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \cap S = \emptyset\}$. In particular, there is a bijective isotone Galois connection between $\operatorname{Spec} A_S$ and $\operatorname{Img} \iota^* = \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \cap S = \emptyset\}$.)

(3) If f is an integral, i.e. B is integral over the image $f(A)$ of A, then the restriction of f^* to $\operatorname{Spec} B$ (resp. $\operatorname{Spm} B$) induces the surjective (lying over Theorem) map $f^* : \operatorname{Spec} B \to \operatorname{Spec} A$ (resp. $f^* : \operatorname{Spm} B \to \operatorname{Spm} A$).

(d) (Span and Closure) Let $A \in \operatorname{Obj} \mathcal{C}$ be a object is in the category $\mathcal{C}$, where $\mathcal{C}$ is one of the category : the category of groups, Rings, K-vector spaces, R-modules, R-algebras and $\mathcal{S}_{\mathcal{C}}(A)$ ($\subseteq \mathfrak{P}(A)$) be the subset of all subobjects to A. The maps $\Phi : \mathfrak{P}(A) \to \mathcal{S}_{\mathcal{C}}(A)$, $X \mapsto \Psi(X) :=$ the smallest subobject of A containing X, and $\Psi : \mathcal{S}_{\mathcal{C}}(A) \to \mathfrak{P}(A)$, $B \to B$ (the natural inclusion) form an isotone Galois connection. — For a topological space X, one may take $\mathcal{S}_{\mathfrak{T}_X}(X) :=$ the set of all closed sets in X and $\Phi : \mathfrak{P}(X) \to \mathcal{S}_{\mathfrak{T}_X}(X)$, $Y \mapsto \overline{Y} :=$ the closure of Y in X.

S6.3 In the examples below $(X, \leq)$, $(Y, \leq)$ are ordered sets.

(a) (Indicretion) Suppose that $x_0 := \operatorname{Max} X$ and $y_0 := \operatorname{Max} Y$ exist. Let $\Phi : X \to Y$, $x \mapsto y_0$ and $\Psi : Y \to X$, $y \mapsto x_0$. Then (X, Y, Φ, Ψ) is an antitone Galois connection. In this case, the induced closure operators (see Supplement S6.6) are the constant maps $X \to X$, $x \mapsto x_0$ and $Y \to Y$, $y \mapsto y_0$. Further, $X_0 = \{x_0\}$ and $Y_0 = \{y_0\}$ are singletons.

(b) (Perfection) Suppose that X and Y are anti-isomorohic ordered sets, i.e. there exists a bijection $\Phi : X \to Y$ such that for all $x, x' \in X$, $x \leq x'$ if and only if $\Phi(x') \leq \Phi(x)$. Then the inverse map $\Psi := \Phi^{-1} : Y \to X$ satisfies for all $y, y' \in Y$, $y \leq y'$ if and only if $\Psi(y') \leq \Psi(y)$. Moreover, for all $x \in X$, $y \in Y$, $x \leq \Psi(y) \iff y = \Psi(\Phi(x)) \leq \Phi(x)$. Therefore (X, Y, Φ, Ψ) is an antitone Galois connection. In this case, $X_0 = X$ and $Y_0 = Y$. Further, the converse also holds : If $X_0 = X$ and $Y_0 = Y$, then Φ and Ψ are inverses of each other, see Supplement S6.6 (d). Such a Galois connection is called perfect.

S6.4 Let $\mathcal{G} = (X, Y, \Phi, \Psi)$ be an antitone Galois connection.

(a) If both X and Y are lattices, then for all $x, x' \in X$, $\Phi(x \wedge x') = \Phi(x) \vee \Phi(x')$ and $\Phi(x \vee x') = \Phi(x) \wedge \Phi(x')$.

(b) If both X and Y are complete lattices, then for all $A \in \mathfrak{P}(X)$, $\Phi(\wedge A) = \vee \Phi(A)$ and $\Phi(\vee A) = \wedge \Phi(A)$.

[3] Heyting algebras were introduced in 1930 by a Dutch mathematician and logician Arend Heyting (1898-1980) to formalize intuitionistic logic. Heyting algebras serve as the algebraic models of propositional intuitionistic logic in the same way Boolean algebras model propositional classical logic. Intuitionistic logic is weaker than classical logic. Each theorem of intuitionistic logic is a theorem in classical logic. Many tautologies in classical logic are not theorems in intuitionistic logic. Examples include the law of excluded middle $p \vee \neg p$, Peirce's law $((p \Rightarrow q) \Rightarrow p) \Rightarrow p$, and double negation elimination $\neg\neg p \Rightarrow p$. But double negation introduction $p \Rightarrow \neg\neg p$ is a theorem.

(Recall that an ordered set $(X,\leq)$ is a lattice if for all $x,x'\in X$, there is a greatest lower bound (GLB) $x\wedge x'$ and a least upper bound (LUB) $x\vee x'$. Moreover, it is a complete lattice if for every subset $A\subseteq X$, both the greatest lower bound $\wedge A$ and the least upper bound $\vee A$ exist.)

(c) Let $c:(\mathfrak{P}(A),\subseteq)\to(\mathfrak{P}(A),\subseteq)$ be a closure operator on the power set of a set A. Then the collection $c(\mathfrak{P}(A))$ of closed subsets of A forms a complete lattice with $\wedge\mathfrak{B}=\cap_{B\in\mathfrak{B}}B$ and $\vee\mathfrak{B}=c(\cup_{B\in\mathfrak{B}}B)$ for every $\mathfrak{B}\subseteq\mathfrak{P}(A)$. (— For definitions see Supplement S6.6 (a).)

S6.5 (Lattice Theory) Let $(A,\leq)$ be a lattice. Then the maps $\Phi:\mathfrak{P}(A)\to\mathfrak{P}(A)$, $B\mapsto \mathrm{UB}_A(B):=\{x\in A\mid b\leq x \text{ for all } b\in B\}$ and $\Psi:\mathfrak{P}(A)\to\mathfrak{P}(A)$, $A'\mapsto \mathrm{LB}_A(B):=\{x\in A\mid x\leq b \text{ for all } b\in B\}$ form an antitone Galois connection. The subset $\mathrm{M}(A):=\{B\in\mathfrak{P}(A)\mid B=(\Psi\Phi)(B)=\mathrm{LB}_A(\mathrm{UB}_A(B))$ with inclusion is a complete lattice with the operations joins and meets are given by $\vee_{i\in I}B_i=\mathrm{LB}_A(\mathrm{UB}_A(\cup_{i\in I}B_i))$ and $\wedge_{i\in I}B_i=\mathrm{LB}_A(\mathrm{UB}_A(\cap_{i\in I}B_i)$. Further, A and $\emptyset$ are the greatest and least elements in $\mathrm{M}(A)$. If $A\in\mathrm{M}(A)$, then $B=\vee_{b\in B}]\to,b]$ and hence every element of $\mathrm{M}(A)$ is a join of elements of A. By means of the canonical injective map $\iota_A:A\to\mathrm{M}(A)$, $a\mapsto]\to,a]:=\{x\in A\mid x\leq a\}$, $(\mathrm{M}(A),\subseteq)$ may be considered as a complete extension of $(A,\leq)$. It is called the Dedekind-MacNeille completion of A. Note that $A=\mathrm{M}(A)$ if and only if A is already complete. Prove that the Dedekind-Mac Neille completion of A is its smallest completion in the following sense: If the ordered set A is an ordered subset of the complete lattice C, then the map $B\mapsto\sup_C B$ is strictly increasing and induces an order isomorphism of $\mathrm{M}(A)$ onto its image in C.

(**Remarks:** In general, the Dedekind-Mac Neille completion is much smaller than the completion $\mathcal{D}cl\,(A)\,(\supseteq\mathrm{M}(A))$ of downward closed subsets of A, see § A, A.4, Exercise 9 b) of the Appendix on Sets and Categories. For example, compare both completions for an anti-chain. The Dedekind-Mac Neille completion of $\mathbb{R}$ is the extended line $\overline{\mathbb{R}}=\mathbb{R}\uplus\{\pm\infty\}$, where the extremal elements $\pm\infty$ with $-\infty<x<+\infty$ for all $x\in\mathbb{R}$ are represented by the lower cuts $\emptyset,\mathbb{R}\in\mathrm{M}(\mathbb{R})$, respectively. More generally, determine the Dedekind-Mac Neille completion of an arbitrary conditionally complete ordered set. $\overline{\mathbb{R}}$ is also the Dedekind-Mac Neille completion of $\mathbb{Q}$. Why? Dedekind used the ordered set $\mathrm{M}(\mathbb{Q})\setminus\{\emptyset,\mathbb{Q}\}$ as a model for the field $\mathbb{R}$ of real numbers. This construction can be generalised to arbitrary ordered division domains K. There we define so-called Dedekind cuts (= Dedekind sections). Besides the *open* initial segments $]\leftarrow,a[$, $a\in K$, these are *particular* lower cuts of K.)

S6.6 (Closure operators and Closed elements of a Galois connection) Let $\mathcal{G}=(X,Y,\Phi,\Psi)$ be an antitone Galois connection.

(a) The map $\Psi\Phi:X\to X$ (resp. $\Phi\Psi:Y\to Y$) is a *closure operator* (See Remarks below) on the ordered set X (resp. Y). (— Recall that a map $f:(X,\leq)\to(X,\leq)$ of an ordered set X is called a closure operator on X if it satisfies: (C1) For all $x\in X$, $x\leq f(x)$. (C2) For all $x,x'\in X$, $x\leq x'$ implies $f(x)\leq f(x')$. (C3) For all $x\in X$, $f(f(x))=f(x)$. Similarly, a map $f:(X,\leq)\to(X,\leq)$ of an ordered set X is called an interior operator on X if it satisfies: (I1) For all $x\in X$, $f(x)\leq x$. (C2) For all $x,x'\in X$, $x\leq x'$ implies $f(x)\leq f(x')$. (C3) For all $x\in X$, $f(f(x))=f(x)$. A map $f:X\to X$ of an ordered set X is a closure operator on X if and only if $f:X^\vee\to X^\vee$ is an interior operator on the order dual $X^\vee$ of X. For example, if X is a topological space, then the operator on the power set $\mathfrak{P}(A)\to\mathfrak{P}(X)$, $A\mapsto\overline{A}:=$ the closure of A in X (resp. $A\mapsto A^\circ:=$ the interior of A in X) is a closure (resp. an interior) operator on the ordered set $(\mathfrak{P}(X),\subseteq)$. — **Hint:** By symmetry it is enough to prove that the map $\Psi\circ\Phi$ is a closure operator on X. Since both Φ and Ψ are antitone, for all $x,x'\in X$ with $x\leq x'$, we have $\Phi(x')\leq\Phi(x)$ and so $\Psi(\Phi(x))\leq\Psi(\Phi(x'))$. For all $x\in X$, since $\Phi(x)\leq\Phi(x)$, by (GC2), see Supplement S6.1 (c), we have $x\leq\Psi(\Phi(x))$. This proves (C1). Finally, applying (C1) to the element $\Psi(\Phi(x))$, we get $\Psi(\Phi(x))\leq\Psi(\Phi(\Psi(\Phi(x))))$. For the reverse inequality, apply (GC2) to $\Psi(\Phi(x))\leq\Psi(\Phi(x))$ to get $\Phi(x)\leq\Phi(\Psi(\Phi(x)))$. Now, apply the order reversing map Ψ to get $\Psi(\Phi(\Psi(\Phi(x))))\leq\Psi(\Phi(x))$. — If (Φ,Ψ) is an isotone Galois connection, then the map $\Psi\circ\Phi:X\to X$ is an interior operator on X and the map $\Phi\circ\Psi:Y\to Y$ is a closure operator on Y. — **Hint:** By Supplement S6.1 (d) (Φ,Ψ) is an antitone Galois connection between $X^\vee$ and Y and hence $\Phi\circ\Psi$ is a closure operator on Y and $\Psi\circ\Phi$ is a closure operator on $X^\vee$. Therefore $\Psi\circ\Phi$ is an interior operator on X, see remark in (a).)

(b) $\Phi\Psi\Phi=\Phi$ and $\Psi\Phi\Psi=\Psi$. (**Hint:** By symmetry it is enough to prove the first equality. Since $\Phi\Psi$ is a closure operator by (a), $\Phi(x)\leq(\Phi\Psi)(\Phi(x))$ for every $x\in X$. Moreover, since $\Psi\Phi$ is a closure operator by (a), $x\leq(\Psi\Phi)(x)$, and since Φ is antitone, $\Phi(\Psi\Phi(x))\leq\Phi(x)$. This proves the equality $\Phi\Psi\Phi(x)=x$.)

(c) The subset $X_0:=\{x\in X\mid x=\Psi(\Phi(x))\}$ (resp. $Y_0:=\{y\in Y\mid y=\Phi(\Psi(y))\}$) of X (resp Y) is called the set of closed elements of X (resp. Y). The subsets of closed points are $X_0=\mathrm{Img}\,(\Psi)$ and $Y_0=\mathrm{Img}\,(\Phi)$. (**Hint:** If $x=\Psi(\Phi(x))$, then $x\in X_0$. Conversely, if $x\in X_0$, then $x=(\Psi\Phi)(x')$ for some $x'\in X$ and so by (b) $(\Psi\Phi)(x)=(\Psi\Phi)(\Psi\Phi(x'))=\Psi(\Phi\Psi\Phi(x'))=(\Psi\Phi)(x')=x$, i.e. $x\in X_0$ is a closed point in X.)

(d) The maps $\Phi:X_0\to Y_0$, $x\mapsto\Phi(x)$ and $\Psi:Y_0\to X_0$, $y\mapsto\Psi(y)$, are antitone bijective and are inverses of each other. (**Hint:** Note that if $x\in X_0$, then $x=(\Psi\Phi)(x)$ by definition and hence $\Phi(x)=\Phi((\Psi\Phi)(x))\in\mathrm{Img}\,\Phi=Y_0$, i.e. $\Phi(X_0)\subseteq Y_0$. For $x,x'\in X_0$ with $\Phi(x)=\Phi(x')$, applying Φ yields $x=(\Psi\Phi)(x)=(\Psi\Phi)(x')=x'$, i.e. Φ is injective on X_0. For $y\in Y_0=\mathrm{Img}\,(\Phi)$, there is $x\in X$ with $y=\Phi(x)=(\Phi\Psi\Phi)(x)=\Phi(\Psi(\Phi(x)))\in\Phi(\mathrm{Img}\,\Psi)\subseteq\Phi(X_0)$ by (b) and (c). This proves that Φ is a bijection from X_0 onto Y_0. Similarly, Ψ is a bijection from Y_0 onto X_0. Finally, For $x\in X_0$ and $y\in Y_0$, by definitions $x=(\Psi\Phi)(x)$ and $y=(\Phi\Psi)(y)$ and hence $\Phi:X_0\to Y_0$ and $\Psi:Y_0\to X_0$ are inverses of each other.)

S6.7 (Linear Algebra) Let V be a vector space over a field K and $\mathcal{S}(V)$ the set of all subspaces of V.

(a) Let $V^*:=\mathrm{Hom}_K V,K)$ be the dual space of of V.

The maps $\Phi : \mathcal{S}(V) \longrightarrow \mathcal{S}(V^*),\, U \longmapsto U^\circ := \{e \in V^* \mid e(x) = 0 \text{ for all } x \in U\}$, and
$\Psi : \mathcal{S}(V^*) \longrightarrow \mathcal{S}(V),\; W \longmapsto {}^\circ W := \{x \in V \mid e(x) = 0 \text{ for all } e \in W\}$
form an antitone Galois connection. What are the closure operators and closed points (see Supplement S6.6) of this Galois connection? See Lecture Notes, § 5G, 2016 CSA-E0 219 Linear Algebra and Applications

(b) Suppose that $\varphi : V \times V \to \mathbb{K}$ be either a symmetric bilinear form or a complex hermitian form on V, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then $\mathcal{G} = (\mathcal{S}(V), \mathcal{S}(V), \Phi, \Phi)$ is an antitone Galois connection, where $\Phi : \mathcal{S}(V) \longrightarrow \mathcal{S}(V)$ is the map defined by $U \longmapsto U^\perp := \{x \in V \mid \varphi(u,x) = 0 \text{ for all } u \in U\}$.
What are the closure operators and closed points (see Supplement S6.6) of this Galois connection?

S6.8 (Transitive Group Actions) Let G be a group which operates on a set X. A subset $Y \subseteq X$ is called a block if for every $\sigma \in G$, then either $\sigma Y \cap Y = \emptyset$, or $\sigma Y = Y$.

For $x \in X$, let $\mathcal{B}_x := \{Y \in \mathfrak{P}(X) \mid Y \text{ is a block with } x \in Y\}$ be the set of all blocks in X containing x and $\mathcal{S}_x := \{H \in \mathfrak{P}(G) \mid H \text{ is a subgroup of } G \text{ with } G_x \subseteq H\}$.

The maps $\Phi : \mathcal{S}_x \longrightarrow \mathcal{B}_x,\, H \longmapsto Hx := \{hx \mid h \in H\}$ and $\Psi : \mathcal{B}_x \longmapsto \mathcal{S}_x,\, Y \longmapsto H_Y := \{g \in G \mid gx \in Y\}$ form an isotone Galois connection. Moreover, *if G operates on X transitively, then they are inverses of each other.*

(— **Remarks :** As a consequence it follows that : *if G operates 2-transitive on X, then there are only trivial blocks in X,* or equivalently, *for every $x \in X$, the isotropy subgroups G_x is a maximal subgroup in G.* In this case, we say that the group G operates primitively on X. If $G \subseteq \mathfrak{S}(G)$ is a subgroup of the symmetric group on X, then we say that G is primitive if the natural operation of G on X is primitive. For example, the symmetric group is primitive for any set X. The alternating group $\mathfrak{A}(X)$ is primitive if X is finite. If $|X| = 4$, then the Klein's four group operates transitively, but not primitively on X.

This terminology was introduced by Évariste Galois in his last letter, in which he used the French term équation primitive for an equation whose Galois group is primitive.)

S6.9 (Galois Theory) This is the motivating example for the concept Galois connection.

Let $L|K$ be a field extension, $\mathrm{Gal}(L|K) := \mathrm{Aut}_{K\text{--alg}}(L)$ be the group of K-algebra automorphisms of L, $\mathcal{I}(L|K) := \{E \mid K \subseteq E \subseteq L,\, E \text{ subfield of } L\}$, and $\mathcal{S}(\mathrm{Gal}(L|K) := \{H \mid H \text{ subgroup of } \mathrm{Gal}(L|K)\}$.

(a) The maps $\Phi := \mathrm{Gal}(L|-) : \mathcal{I}(L|K) \longrightarrow \mathcal{S}(\mathrm{Gal}(L|K)),\, E \longmapsto \mathrm{Gal}(L|E)$ and
$\Psi := \mathrm{Fix}_{\_}\, L : \mathcal{S}(\mathrm{Gal}(L|K)) \longrightarrow \mathcal{I}(L|K),\, H \longmapsto \mathrm{Fix}_H\, L$
form an antitone Galois connection. (Immediate from the obvious inclusions $E = x \leq \Psi(\Phi(x)) = \mathrm{Fix}_{\mathrm{Gal}(L|E)}(L)$ for every $x = E \in X = \mathcal{I}(L|K)$ and $H = y \leq \Phi(\Psi(y)) = \mathrm{Gal}(L|\ \mathrm{Fix}_H\, L)$, see Remark (1) in the Supplement S6.1.)

(b) (Fundamental Theorem of Galois Theory — Finite Case) If Ł$|K$ is a finite *Galois extension*, then $X_0 = X$ and $Y_0 = Y$ and hence Φ and Ψ are antitone bijective which are inverses of each other, see Supplement S6.6. (— Recall that : (1) (Dedekind-Artin) For every finite field extension $L|K$, $|\mathrm{Gal}(L|K)| \leq [L:K] =: \mathrm{Dim}_K L$. (2) A finite field extension $L|K$ is called a Galois extension if the above inequality is an equality, i.e. if $|\mathrm{Gal}(L|K)| = [L:K]$, or equivalently, $\mathrm{Fix}_{\mathrm{Gal}(L|K)} L = K$. (3) An algebraic extension $L|K$ of fields is called a Galois extension if $\mathrm{Fix}_{\mathrm{Gal}(L|K)} L = K$ or equivalently, $L|K$ is normal and separable.)

(c) (Fundamental Theorem of Galois Theory — Infinite Case) Let Ł$|K$ be an algebraic *Galois extension* (but not necessarily finite). In this case, the set of closed points (see Supplement S6.6) $X_0 = X$, but Y_0 is rather difficult to describe. However, there is a topology on the Galois group $\mathrm{Gal}(L|K)$ called the *Krull topology* and the closed points Y_0 is precisely the subgroups of $\mathrm{Gal}(L|K)$ which are closed in the Krull toplogy of $\mathrm{Gal}(L|K)$. (— Recall that : An algebraic extension $L|K$ of fields is called a Galois extension if $\mathrm{Fix}_{\mathrm{Gal}(L|K)} L = K$ or equivalently, $L|K$ is normal and separable field extension.)

S6.10 (Commutative algebra/Modern Algebraic Geometry) Let A be a commutative ring, $\mathcal{I}(A)$ (resp. $\mathrm{Spec}\, A$, $\mathrm{Spm}\, A$) be the set of all ideals (resp. prime ideals, maximal ideals) in A.

(a) The maps $\Phi := \mathrm{V} : \mathcal{I}(A) \longrightarrow \mathfrak{P}(\mathrm{Spec}\, A),\; \mathfrak{a} \longmapsto \mathrm{V}(\mathfrak{a}) := \{\mathfrak{p} \in \mathrm{Spec}\, A \mid \mathfrak{a} \subseteq \mathfrak{p}\}$ and
$\Psi := \mathrm{I} : \mathfrak{P}(\mathrm{Spec}\, A) \longrightarrow \mathcal{I}(A),\;\; W \longmapsto \mathrm{I}(W) := \cap_{\mathfrak{p} \in W} \mathfrak{p}$
form an antitone Galois connection. (For $\mathfrak{a} \in \mathcal{I}(A)$ and $W \in \mathfrak{P}(\mathrm{Spec}\, A)$, $\mathfrak{a} \subseteq \Psi(W) = \cap_{\mathfrak{p} \in W} \mathfrak{p} \iff \mathfrak{a} \subseteq \mathfrak{p}$ for all $\mathfrak{p} \in W \iff W \subseteq \mathrm{V}(\mathfrak{a}) = \Phi(\mathfrak{a})$.)

(b) The set of closed points $X_0 := \mathrm{Img}\, \mathrm{I}$ in $X = \mathcal{I}(A)$ is the set of all ideals in A which can be written as the intersection of a family of prime ideals and hence are precisely the radical ideals in A, i.e. $X_0 = \mathrm{Rad}(\mathcal{I}(A)) := \{\mathfrak{a} \in \mathcal{I}(A) \mid \mathfrak{a} = \sqrt{\mathfrak{a}}\}$. The closure operator $\Psi\Phi = \mathrm{I} \circ \mathrm{V} : \mathcal{I}(A) \longrightarrow \mathcal{I}(A)$ maps $\mathfrak{a} \longmapsto \sqrt{\mathfrak{a}}$.

(c) It is not easy to describe the set of closed points $Y_0 = \mathrm{Img}\, \mathrm{V}$ of $Y = \mathfrak{P}(\mathrm{Spec}\, A))$ and the closure operator $\Phi\Psi = \mathrm{V} \circ \mathrm{I} : \mathfrak{P}(\mathrm{Spec}\, A) \longrightarrow \mathfrak{P}(\mathrm{Spec}\, A),\; W \longmapsto \mathrm{V}(\mathrm{I}(W)) = \mathrm{V}(\cap_{\mathfrak{p} \in W}\, \mathfrak{p})$. However, there is something nice to describe : $Y_0 =$ is precisely the closed subsets for the Zariski topology on $\mathrm{Spec}\, A$ and $\Phi\Psi(W) = \mathrm{V}(\mathrm{I}(W)) = \overline{W} :=$ the closure of the subset $W \subseteq \mathrm{Spec}\, A$ with respect to the Zariski topology of $\mathrm{Spec}\, A$.

(d) The maps $\mathrm{V} : \mathrm{Rad}(\mathcal{I}(A)) \longrightarrow \{\mathrm{V}(\mathfrak{a}) \mid \mathfrak{a} \in \mathrm{Rad}(\mathcal{I}(A))\} =$ Zariski-closed subsets in $\mathrm{Spec}\, A$, $\mathfrak{a} \longmapsto \mathrm{V}(\mathfrak{a})$ and $\mathrm{I} : \{\mathrm{V}(\mathfrak{a}) \mid \mathfrak{a} \in \mathrm{Rad}(\mathcal{I}(A))\} =$ Zariski-closed subsets in $\mathrm{Spec}\, A \longrightarrow \mathrm{Rad}(\mathcal{I}(A))$, $\mathrm{V}(\mathfrak{a}) \longmapsto \mathfrak{a}$, are antitone bijective which are inverses of each other. (Immediate from (a), (b), (c) and Supplement S6.6 (d).)

S6.11 Let A be a commutative ring and $X=(\mathfrak{I}(A),\subseteq)$, $\mathfrak{J}\subseteq \operatorname{Spec} A$ and $Y=(\mathfrak{P}(\mathfrak{J}),\subseteq)$. (For example in the above Supplement S6.9 we have considered $\mathfrak{J}=\operatorname{Spec} A$).

The maps $\Phi:\mathfrak{I}(A)\longrightarrow \mathfrak{P}(\mathfrak{J})$, $\mathfrak{a}\longmapsto \mathrm{V}(\mathfrak{a}):=\{\mathfrak{b}\in\mathfrak{J}\mid \mathfrak{a}\subseteq\mathfrak{b}\}$ and $\Psi:\mathfrak{P}(\operatorname{Spec} A)\longrightarrow \mathfrak{I}(A)$, $W\longmapsto \cap_{\mathfrak{b}\in W}\mathfrak{b}$ form an antitone Galois connection.

Various particular, choices for subset $\mathfrak{J}\subseteq\operatorname{Spec} A$ have been considered and the most important among them is the subset $\operatorname{Spm} A$ of all maximal ideals in A. In this case, the set X_0 of closed points of X consists of all ideals which can be written as the intersection of a family of maximal ideals. Such ideals are clearly radical ideals, but not all radical ideals are obtained in this way. — For a commutative ring A, the following statements are equivalent : (i) Every radical ideal $\mathfrak{a}$ is the intersection of the maximal ideals containing $\mathfrak{a}$. (ii) Every prime ideal $\mathfrak{p}$ is the intersection of the maximal ideals containing it. A commutative ring A is called a Jacobson ring if these equivalent conditions are satisfied. For example, $\mathbb{Z}$ is a Jacobson ring. The polynomial ring $K[X_1,\ldots,X_n]$ in n indeterminates $X_1,\ldots,X_n$ over a field K is a Jacobson ring. More generally, finite type algebra over a field is a Jacobson ring. This follows from Zariski-Lemma (HNS3): *If A is a K-algebra of finite type over a field K, then for every maximal ideal $\mathfrak{m}\in\operatorname{Spm} A$, $A/\mathfrak{m}$ is a finite field extension. In particular, $A/\mathfrak{m}\,|\,K$ is an algebraic field extension.*

S6.12 (Classical Algebraic Geometry — Hilbert's Nullstellensaz) Let $L|K$ be a field extension (mostly L algebraically closed, for example, $(K,L)=(\mathbb{Q},\overline{\mathbb{Q}})$, $(\mathbb{Q},\mathbb{C})$, $(\mathbb{R},\mathbb{C})$, or $K=L$ is an arbitrary algebraically closed field). For $n\in\mathbb{N}$, let $\mathbb{A}^n_L:=L^n$ be the affine space over L and let $\mathrm{P}_n:=K[X_1,\ldots,X_n]$ be the polynomial ring in n indeterminates $X_1,\ldots,X_n$ over K.

(a) The maps $\Phi:=\mathrm{V}_L:\mathfrak{I}(\mathrm{P}_n)\longrightarrow\mathfrak{P}(\mathbb{A}^n_L)$, $\mathfrak{a}\longmapsto \mathrm{V}_L(\mathfrak{a}):=\{a\in\mathbb{A}^n_L\mid f(a)=0 \text{ for every } f\in\mathfrak{a}\}$ and

$$\Psi:=\mathrm{I}_K:\mathfrak{P}(\mathbb{A}^n_L)\longrightarrow\mathfrak{I}(\mathrm{P}_n),\quad W\longmapsto \mathrm{I}_K(W):=\{f\in\mathrm{P}_n\mid f(a)=0 \text{ for every } a\in W\}$$

form an antitone Galois connection. (For $\mathfrak{a}\in\mathfrak{I}(\mathrm{P}_n)$ and $W\in\mathfrak{P}(\mathbb{A}^n_L)$, $\mathfrak{a}\subseteq\mathrm{I}_K(W)=\Psi(W)\iff W\subseteq\mathrm{V}_L(\mathfrak{a})=\Phi(\mathfrak{a})$.)

(b) The set of closed points (see Supplement S6.6 (d)) $Y_0=\operatorname{Img}\mathrm{V}_L$ in $Y=\mathfrak{I}(\mathrm{P}_n)$ is precisely the set of K-algebraic subsets $\{\mathrm{V}_L(\mathfrak{a})\mid\mathfrak{a}\in\operatorname{Rad}(\mathfrak{I}(\mathrm{P}_n))\}$ in $\mathbb{A}^n_L$. The closure operator $\Phi\Psi=\mathrm{V}_L\mathrm{I}_K:\mathfrak{P}(\mathbb{A}^n_L)\to\mathfrak{P}(\mathbb{A}^n_L)$ maps every K-algebraic subset $\mathrm{V}_L(\mathfrak{a})$ to itself, see Supplement S6.6 (c).

(c) The map I_K is injective on the set of K-algebraic subsets in $\mathbb{A}^n_L$. Generally the set of closed points $X_0=\operatorname{Img}\mathrm{I}_K$ is rather difficult to describe. However, if L is algebraically closed this can be described by using the famous geometric version of Hilbert's Nullstellensatz (HNS2): *Let $L|K$ be a field extenstion with L algebraically closed and $\mathfrak{a}\in\mathfrak{I}(K[X_1,\ldots,X_n])$ be an ideal. Then* $\mathrm{I}_K(\mathrm{V}_L(\mathfrak{a}))=\sqrt{\mathfrak{a}}$. One proves easily that HNS2 and HNS3 (Zariski's Lemma) are equivalent.

(d) If L is algebraically closed, then the maps

$\mathrm{V}_L:\operatorname{Rad}(\mathfrak{I}(\mathrm{P}_n))\longrightarrow\{\mathrm{V}_L(\mathfrak{a})\mid\mathfrak{a}\in\operatorname{Rad}(\mathfrak{I}(\mathrm{P}_n))\}=K$-algebraic subsets in $\mathbb{A}^n_L$, $\mathfrak{a}\longmapsto\mathrm{V}_L(\mathfrak{a})$ and

$\mathrm{I}_K:\{\mathrm{V}_L(\mathfrak{a})\mid\mathfrak{a}\in\operatorname{Rad}(\mathfrak{I}(\mathrm{P}_n))\}=K$-algebraic subsets in $\mathbb{A}^n_L\longrightarrow\operatorname{Rad}(\mathfrak{I}(\mathrm{P}_n))$, $\mathrm{V}_L(\mathfrak{a})\longmapsto\mathfrak{a}$,

are antitone bijective which are inverses of each other. (Immediate from (a), (b), (c) and Supplement S6.6 (d).)

(e) In the special case of $K=L$ algebraically closed field, the Galois connection between $X=\mathfrak{I}(\mathrm{P}_n)$ and $\mathfrak{P}(\operatorname{Spm}\mathrm{P}_n)$ has the following description :

(i) (HNS4) *The canonical injective map $\mathbb{A}^n_K\longrightarrow\operatorname{Spm}\mathrm{P}_n$, $a=(a_1,\ldots,a_n)\longmapsto\mathfrak{m}_a:=\langle X_1-a_1,\ldots,X_n-a_n\rangle$, is bijective. Moreover, for every $\mathfrak{a}\in\mathfrak{I}(\mathrm{P}_n)$, $a\in\mathrm{V}_K(\mathfrak{a})$ if and only if $\mathfrak{a}\subseteq\mathfrak{m}_a$.* (By the way HNS4 is also equivalent to HNS2, Proof!).

(ii) The closure operator $\mathfrak{I}(\mathrm{P}_n)\longrightarrow\mathfrak{I}(\mathrm{P}_n)$ maps $\mathfrak{a}$ to its radical ideal $\sqrt{\mathfrak{a}}$.

(iii) The closure operator $\mathfrak{P}(\operatorname{Spm}\mathrm{P}_n)\longrightarrow\mathfrak{P}(\operatorname{Spm}\mathrm{P}_n)$ coincides with the topological closure with respect to the Zariski topology on $\mathbb{A}^n_K\xrightarrow{\sim}\operatorname{Spm}\mathrm{P}_n$, see (i).

S6.13 (Algebraic topology) Let X be a path connected topological space and let $\pi_1(X)$ be the first fundamental group of X. Then there is an antitone Galois connection between subgroups $\mathcal{S}(\pi_1(X))$ of $\pi_1(X)$ and path-connected covering spaces of X.

S6.14 (Galois Connections Decorticated) To relations between sets one can associate antitone Galois connection naturally as follows :

(a) Let X, Y be sets and $R\subseteq X\times Y$ be a relation between X and Y. For $A\in\mathfrak{P}(X)$ and $y\in Y$, write ARy if $(x,y)\in R$ for every $x\in A$. Similarly, $B\in\mathfrak{P}(Y)$ and $x\in X$, (dually) write xRB if $(x,y)\in R$ for every $y\in B$. Further, for $A\in\mathfrak{P}(X)$ and $B\in\mathfrak{P}(Y)$, write ARB if $(x,y)\in R$ for all $x\in A$ and all $y\in B$.

The maps $\Phi_R:\mathfrak{P}(X)\longrightarrow\mathfrak{P}(Y)$, $\Phi_R(A):=\{y\in Y\mid ARy\}$ and $\Psi_R:\mathfrak{P}(Y)\longrightarrow\mathfrak{P}(X))$, $\Psi_R(B):=\{x\in Y\mid xRB\}$ are antitone and $\mathcal{G}_R:=(\mathfrak{P}(X),\mathfrak{P}(Y),\Phi_R,\Psi_R)$ is a antitone Galois connection. Indeed, For all $A\in\mathfrak{P}(X)$ and all $B\in\mathfrak{P}(Y)$, we have $A\subseteq\Psi_R(B)\iff ARB\iff B\subseteq\Phi_R(A)$.

(b) Let X, Y be sets and $\mathcal{G} = (\mathfrak{P}(X), \mathfrak{P}(Y), \Phi, \Psi)$ be any antitone Galois connection. Let $R \subseteq X \times Y$ be the relation defined by $(x,y) \in R$ if $y \in \Psi(\{x\})$. Then $\mathcal{G} = \mathcal{G}_R$. (**Hint:** Since $(\mathfrak{P}(X), \subseteq)$ and $(\mathfrak{P}(X), \subseteq)$ are complete lattices, we can apply Supplement 6.4 (b). For $A \in \mathfrak{P}(X)$, $A = \cup_{x\in A}\{x\} = \vee_{x\in A}\{x\}$, we have $\Phi(A) = \cap_{x\in A}\Phi(\{x\}) = \cap_{x\in A}\{y \in Y \mid xRy\} = \{y \in Y \mid ARy\} = \Phi_R(A)$. Moreover, since $\mathcal{G}$ is a galois connection, we have $\{x\} \subseteq \Psi(\{y\}) \iff \{y\} \subseteq \Phi(\{x\}) \iff xRy$. Therefore, for every $B \in \mathfrak{P}(Y)$, $B = \cup_{y\in B}\{x\} = \vee_{y\in B}\{x\}$, we have $\Psi(B) = \cap_{y\in B}\Psi(\{y\}) = \cap_{y\in B}\{x \in X \mid xRy\} = \{x \in X \mid xRB\} = \Psi_R(B)$.)

(c) Every antitone Galois connection $\mathcal{G} = (X, Y, \Phi, \Psi)$ with X and Y complete lattices can be extended to an antitone Galois connection between $\mathfrak{P}(X)$ and $\mathfrak{P}(Y)$. In particular, every antitone Galois connection between complete lattices is the antitone Galois connection induced by a relation between sets. (— Recall that: Every ordered sets can be embedded into a power set lattice, see for example, Supplement S6.4. — **Hint:** For $A \subseteq X$, put $\Phi(A) = \wedge\{\Phi(\{x\})\}_{x\in A}$ and for $Y \subseteq Y$, put $\Psi(B) = \wedge\{\Psi(\{y\})\}_{y\in B}$. For the supplement use part (c).)

S6.15 Most examples of the Galois connections given above arise from a relation. For example:

(a) For the Galois connection in Galois Theory, see Supplement S6.9, take $X = \mathfrak{P}(L)$ and $Y = \mathfrak{P}(\mathrm{Gal}(L|K))$. In this case, the Galois connection is induced by the relation $\sigma x = x$ on $L \times \mathrm{Gal}(L|K)$.

(b) For the Galois connection in Commutative algebra/Modern Algebraic Geometry, see Supplement S6.10, take $X = \mathfrak{P}(A)$ and $Y = \mathfrak{P}(\mathrm{Spec}\,A)$. In this case, the Galois connection is induced by the relation $x \in \mathfrak{p}$ on $A \times \mathrm{Spec}\,A$.