E0 219 Linear Algebra and Applications / August-December 2016

(ME, MSc. Ph. D. Programmes)

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Lectures : Monday and Wednesday ; 11:00–12:30 **Venue:** CSA, Lecture Hall (Room No. 117)

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Midterms : 1-st Midterm : Saturday, September 17, 2016; 15:00 – 17:00 **2-nd Midterm :** Saturday, October 22, 2016; 15:00 – 17:00

Final Examination : December ??, 2016, 09:00 – -12:00

Evaluation Weightage : Assignments : 20% **Midterms (Two) :** 30% **Final Examination :** 50%

Range of Marks for Grades (Total 100 Marks)

	Grade S	Grade A	Grade B	Grade C	Grade D	Grade F
Marks-Range	> 90	76 — 90	61 — 75	46 — 60	35 — 45	< 35

	Grade A$^+$	Grade A	Grade B$^+$	Grade B	Grade C	Grade D	Grade F
Marks-Range	> 90	81 — 90	71 — 80	61 — 70	51 — 60	40 — 50	< 40

7. Direct Sums and Projections ; — Dual spaces

Submit a solution of the $*$-Exercise ONLY. **Due Date :** Monday, 26-09-2016 (Before the Class)

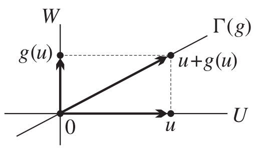
Let K be arbitrary field and let $\mathbb{K}$ denote either the field $\mathbb{R}$ or the field $\mathbb{C}$.

7.1 Let $f : U \to V$ and $g : V \to W$ be homomorphisms of K-vector spaces. If gf is an isomorphism of U onto W, then show that V is the direct sum of $\operatorname{Im} f$ and $\operatorname{Ker} g$, i. e., $V = \operatorname{Im} f \oplus \operatorname{Ker} g$.

7.2 Assume that K has at least n elements. Let $U_1, \ldots, U_n$ be subspaces (of a finite dimensional K-vector space V) of equal dimension. Then show that $U_1, \ldots, U_n$ have a common complement in V, i. e. $V = U_i \oplus W$ for every $i = 1, \ldots, n$. (**Hint :** Use the Exercise 4.5.)

7.3 Suppose that the K-vector space V is the direct sum of the subspaces U and W.

(a) For every linear map $g : U \to W$, show that the graph $\Gamma(g) := \{u + g(u) \mid u \in U\} \subseteq V$ of g is a complement of W in V.



(b) Show that the map $\operatorname{Hom}_K(U,W) \to \mathcal{C}(W,V)$ defined by $g \mapsto \Gamma(g)$ is bijective, where $\mathcal{C}(W,V)$ denote the set of all complements of W in V. Describe this bijection for $V = \mathbb{R}^2$ and $U = \mathbb{R} \times \{0\} (=$ x-axis explicitly.

(c) Suppose that $\operatorname{Dim}_K U = \operatorname{Dim}_K W = n$. Let $u_1, \ldots, u_n$ and $w_1, \ldots, w_n$ be bases of U and W, respectively. Then show that $u_1 + w_1, \ldots, u_n + w_n$ is a basis of a complement of U as well as a complement of W in V.

***7.4** Let V be a K-vector space and let $f_1, \ldots, f_n \in V^*$ be linear forms on V. Let $f : V \to K^n$ be the homomorphism defined by $f(x) := \big(f_1(x), \ldots, f_n(x)\big)$. Then show that $\operatorname{Dim}_K (Kf_1 + \cdots + Kf_n) = \operatorname{Dim}_K (\operatorname{Im} f)$. In particular, $f_1, \ldots, f_n$ are linearly independent if and only if the homomorphism f is surjective. (**Hint :** Note that $\operatorname{Im} f$ is finite dimensional and hence $\operatorname{Rank}_K f = \operatorname{Rank}_K f^* = \operatorname{Dim}_K (Kf_1 + \cdots + Kf_n)$, see also Supplement S7.33.)

7.5 A K-linear map $f : V \to W$ be a homomorphism of K-vector spaces is injective (resp. surjective, bijective) if and only if the dual map $f^* : W^* \to V^*$ is surjective (resp. injective, bijective) (**Remark:** It is not really necessary to assume that V and W are finite dimensional.)

7.6 Let $x_1, \ldots, x_n$ be all non-zero vectors in a K-vector space V over a field K with $|K| \geq n$. Then Show that there exists a hyperplane H in V such that the vectors $x_i \notin H$ for all $i = 1, \ldots, n$. (**Hint :** There exist a linear form $f_i : V \to K$ such that $f_i(x_i) = 1 \neq 0$ for each $i = 1, \ldots, n$. Therefore the subspaces $(Kx_i)^\circ$, $i = 1, \ldots, n$ are proper subspaces of the K-vector space V^* and hence $(Kx_1)^\circ \cup \cdots \cup (Kx_n)^\circ \subsetneq V^*$ by Exercise 2.2. Now, choose $f \in V^* \setminus (Kx_1)^\circ \cup \cdots \cup (Kx_n)^\circ$ and take $H := \operatorname{Ker} f$.)