E0 219 Linear Algebra and Applications / August-December 2016

(ME, MSc. Ph. D. Programmes)

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Lectures : Monday and Wednesday ; 11:00–12:30 **Venue:** CSA, Lecture Hall (Room No. 117)

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Midterms : 1-st Midterm : Saturday, September 17, 2016; 15:00 – 17:00 **2-nd Midterm :** Sunday, October 23, 2016; 15:00 – 17:00

Final Examination : Thursday, December 08, 2016, 09:00 – -12:00

Evaluation Weightage : Assignments : 20% **Midterms (Two) :** 30% **Final Examination :** 50%

Range of Marks for Grades (Total 100 Marks)						
	Grade S	**Grade A**	**Grade B**	**Grade C**	**Grade D**	**Grade F**
Marks-Range	> 90	76 — 90	61 — 75	46 — 60	35 — 45	< 35

	Grade A$^+$	**Grade A**	**Grade B$^+$**	**Grade B**	**Grade C**	**Grade D**	**Grade F**
Marks-Range	> 90	81 — 90	71 — 80	61 — 70	51 — 60	40 — 50	< 40

9. Matrices — The Matrix of a linear map — Rank of matrices — Elementary matrices

Submit a solution of the $*$-Exercise ONLY. **Due Date :** Monday, 10-10-2016 (Before the Class)

Complete Correct solution of the Exercise 9.5 carry 20 BONUS POINTS!!!

Let K be arbitrary field and let $\mathbb{K}$ denote either the field $\mathbb{R}$ or the field $\mathbb{C}$.

9.1 Let V be a vector space of dimension n over a field K and let $f \in \operatorname{End}_K V$. Then there exists a basis $\mathfrak{w}$ of V such that the matrix $\mathfrak{M}^{\mathfrak{w}}_{\mathfrak{w}}(f)$ of f with respect to $\mathfrak{w}$ is of the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1,n-1} & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2,n-1} & a_{2n} \\ 0 & a_{32} & a_{33} & \cdots & a_{3,n-1} & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & 0 & \cdots & a_{n,n-1} & a_{nn} \end{pmatrix},$$

where the elements $a_{21}, a_{32}, \ldots, a_{n,n-1}$ below the main-diagonal are either 1 or 0.
(**Remark :** A matrix of this form, where the elements $a_{21}, a_{32}, \ldots, a_{n,n-1}$ are arbitrary is called a Hessenberg[1]– matrix. The existence of such a matrix representation is much simpler than what the applied mathematicians will make you think, when they are using *Householder type reflections*[2] for the construction (which works over $\mathbb{C}$ only), see also[3]. However, it is much simpler to construct a basis $\mathfrak{w} = (w_1, \ldots, w_n)$ of V (over arbitrary field K), see — **Proof :** To construct a basis $w_1, \ldots, w_n$ of V (over arbitrary field K), choose any $w_1 \neq 0$ in V. If $f(w_1) \in Kw_1$, say $f(w_1) = a_1 w_1$, then choose $w_2 \in V$, $w_2 \notin Kw_1$, and take $a_{11} := a_1, a_{i,1} := 0$ for $i = 2, \ldots, n$. If $f(w_1) \notin Kw_1$, put $w_2 := f(w_1)$ and $a_{11} := 0, a_{21} := 1$, $a_{i,1} = 0$ for $i = 3, \ldots, n$. Then w_1, w_2 are linearly independent, and the first column of the matrix will have the required form. Now, assume that we have chosen linearly independent vectors $w_1, \ldots, w_j$, $j < n$, such that the first $j-1$ columns of the matrix have the right form. Then proceed as follows: If $f(w_j) \in Kw_1 + \cdots Kw_j$, say $f(w_j) = a_1 w_1 + \cdots + a_j w_j$, choose a vector $w_{j+1} \in V$, $w_{j+1} \notin Kw_1 + \cdots + Kw_j$, and put $a_{ij} := a_i$ for $i = 1, \ldots, j$ and $a_{ij} := 0$ for $i = j+1, \ldots, n$. If $f(w_j) \notin Kw_1 + \cdots Kw_j$, put $w_{j+1} := f(w_j)$ and $a_{ij} := 0$ for $i = 1, \ldots j$, $a_{j+1,j} := 1$ and $a_{ij} = 0$ for $i = j+2, \ldots, n$.. Then $w_1, \ldots, w_{j+1}$ are linearly independent, and the

[1] Hessenberg matrices were first investigated by Karl Hessenberg (1904-1959), a German engineer whose dissertation investigated the computation of eigenvalues and eigenvectors of linear operators, see [Hessenberg, K. Thesis. Darmstadt, Germany: Technische Hochschule, 1942.].

[2] Householder transformation was introduced in 1958 by Alston Scott Householder (1904-1993) an American mathematician who specialized in mathematical biology and numerical analysis.

[3] [Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Reduction of a General Matrix to Hessenberg Form." § 11.5 in Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed. Cambridge, England: Cambridge University Press, pp. 476-480, 1992.]

first j columns of the matrix will have the required form. This method stops after having chosen w_n, because there are no requirements on the last column of that matrix. •)

9.2 Compute the inverse of the matrix (called the Heisenberg-matrix[4]) of the form

$$\mathfrak{B} = \begin{pmatrix} 1 & a_1 & a_2 & \cdots & a_n & c \\ 0 & 1 & 0 & \cdots & 0 & b_1 \\ 0 & 0 & 1 & \cdots & 0 & b_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & b_n \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \in \mathrm{M}_{n+2}(K).$$

(**Hint:** Let $w_0,\ldots,w_{n+1}$ be a basis of the $n+2$-dimensional vector space V over K. Then $v_0 := w_0$, $v_j := w_j + a_i w_0$, $j = 1,\ldots,n$ and $v_{n+1} := w_{n+1} + b_n w_n + \cdots + b_1 w_1 + c w_0$ is also a basis (see Exercise 3.2 (b)) of V over K. Further, the Heisenberg-matrix $\mathfrak{B} = \mathfrak{M}_{\mathfrak{v}}^{\mathfrak{w}}$ is the transition matrix of the basis $v_0,\ldots,v_{n+1}$ onto the basis $w_0,\ldots,w_{n+1}$. Therefore the inverse $\mathfrak{B}^{-1}$ is the transition matrix of the basis $w_0,\ldots,w_{n+1}$ onto the basis $v_0,\ldots,v_{n+1}$. Then the inverse $\mathfrak{B}^{-1} = \mathfrak{M}_{\mathfrak{w}}^{\mathfrak{v}}$ is the transition matrix of the basis $v_0,\ldots,v_{n+1}$ onto the basis $w_0,\ldots,w_{n+1}$. Since $w_0 = v_0$, $w_j = v_j - a_i v_0$, $j = 1,\ldots,n$ and $w_{n+1} = v_{n+1} - (b_n v_n - a_n v_0) - \cdots - b_1(v_1 - a_1 v_0) - c v_0 = v_{n+1} - b_n v_n - \cdots - b_1 v_1 + (b_n a_n + \cdots + b_1 a_1 - c) v_0$, it follows that

$$\mathfrak{B}^{-1} = \mathfrak{M}_{\mathfrak{w}}^{\mathfrak{v}} = \begin{pmatrix} 1 & -a_1 & -a_2 & \cdots & -a_n & b_n a_n + \cdots + b_1 a_1 - c \\ 0 & 1 & 0 & \cdots & 0 & -b_1 \\ 0 & 0 & 1 & \cdots & 0 & -b_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -b_n \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \in \mathrm{M}_{n+2}(K).)$$

9.3 **(a)** Let I, J be finite sets. Two matrices $\mathfrak{A}, \mathfrak{A}' \in \mathrm{M}_{I,J}(K)$ have the same rank if and only if there exist invertible matrices $\mathfrak{B} \in \mathrm{GL}_I(K)$ and $\mathfrak{C} \in \mathrm{GL}_J(K)$ such that $\mathfrak{A}' = \mathfrak{B}\mathfrak{A}\mathfrak{C}$.
(**Hint:** Let $f, g\colon K^J \to K^I$ be linear maps defined by $f(\mathbf{x}) := \mathfrak{A}\mathbf{x}$ and $f'(\mathbf{x}) := \mathfrak{A}'\mathbf{x}$, $\mathbf{x}$ is column-vector in K^J, and let $\mathfrak{A}$ respectively $\mathfrak{A}'$ be the matrices with respect to the standard bases. Let $\mathrm{Rank}\,\mathfrak{A} = \mathrm{Rank}\,\mathfrak{A}' = r$, and so $\mathrm{Rank}\, f = \mathrm{Rank}\, f' = r$. By the proof of the Rank-Theorem, there exist a basis $v_1,\ldots,v_n$ of K^J and a basis $v'_1,\ldots,v'_n$ of K^J such that $w_1 := f(v_1),\ldots,w_r := f(v_r)$ is a basis of $\mathrm{Im}\, f$ and $w'_1 := f(v'_1),\ldots,w'_r := f(v'_r)$ is a basis of $\mathrm{Im}\, f'$ and that $v_{r+1},\ldots,v_n$ and $v'_{r+1},\ldots,v'_n$ are bases of $\mathrm{Ker}\, f$ respectively $\mathrm{Ker}\, f'$. We also extend $w_1,\ldots,w_r$ and $w'_1,\ldots,w'_r$ to bases $w_1,\ldots,w_m$ respectively $w'_1,\ldots,w'_m$ of K^I. Now, we define isomorphisms $h\colon K^J \to K^J$ and $g\colon K^I \to K^I$ by $h(v_i) := v'_i$, $i = 1,\ldots,n$, and $g(w_i) := w'_i$, $i = 1,\ldots,m$. By construction, we have $g(f(v_i)) = g(w_i) = w'_i = f'(v'_i) = f'(h(v_i))$ for $i = 1,\ldots,r$ and $g(f(v_i)) = g(0) = 0 = f'(v'_i) = f'(h(v_i))$ for $i = r+1,\ldots,n$. Therefore, altogether $g \circ f = f' \circ h$, where the matrices $\mathfrak{C}' := \mathfrak{M}_{\mathfrak{e}}^{\mathfrak{e}}(h)$ and $\mathfrak{B} := \mathfrak{M}_{\mathfrak{e}}^{\mathfrak{e}}(g)$ are invertible. It follows that $\mathfrak{B}\mathfrak{A} = \mathfrak{M}_{\mathfrak{e}}^{\mathfrak{e}}(g)\,\mathfrak{M}_{\mathfrak{e}}^{\mathfrak{e}}(f) = \mathfrak{M}_{\mathfrak{e}}^{\mathfrak{e}}(g \circ f) = \mathfrak{M}_{\mathfrak{e}}^{\mathfrak{e}}(f' \circ h) = \mathfrak{M}_{\mathfrak{e}}^{\mathfrak{e}}(f')\,\mathfrak{M}_{\mathfrak{e}}^{\mathfrak{e}}(h) = \mathfrak{A}'\,\mathfrak{C}'$ and hence $\mathfrak{A}' = \mathfrak{B}\,\mathfrak{A}\,\mathfrak{C}$ mit $\mathfrak{C} := (\mathfrak{C}')^{-1}$.
For the converse the isomorphisms g and h defined above by $\mathfrak{B}$ respectively $\mathfrak{C}^{-1}$, naturally $\mathrm{Rank}\,\mathfrak{A} = \mathrm{Dim}\,\mathrm{Im}\, f = \mathrm{Dim}\,\mathrm{Im}\, g \circ f = \mathrm{Dim}\,\mathrm{Im}\, f' \circ h = \mathrm{Dim}\,\mathrm{Im}\, f' = \mathrm{Rank}\,\mathfrak{A}'$. — **Remark :** In this case we say that $\mathfrak{A}$ and $\mathfrak{A}'$ are (rank)-equivalent. The corresponding equivalence classes are precisely the set of all matrices of same rank. Therefore the rank is the only invariant of such equivalence classes. See also Supplement S9.6.)

(b) Let $m, n \in \mathbb{N}^*$, $s := \mathrm{Min}\{m, n\}$. For every r with $0 \le r \le s$, let $\mathfrak{U}_r := \sum_{i=1}^{r} \mathfrak{E}_{ii} \in \mathrm{M}_{m,n}(K)$. If $\mathfrak{A} \in \mathrm{M}_{m,n}(K)$, then $\mathfrak{A}$ (rank)-equivalent to $\mathfrak{U}_r$, where $r := \mathrm{Rank}\,\mathfrak{A}$. The matrices $\mathfrak{U}_0,\ldots,\mathfrak{U}_s$ form a full representative system in $\mathrm{M}_{m,n}(K)$ with respect to the relation of equivalence of matrices given in the part (a) above. (**Remark :** Multiplying by elementary matrices $\mathfrak{B}_{ij}(a)$, $i < j$ from right and $\mathfrak{B}_{ij}(a)$, $i > j$ from left, we can even find an invertible upper triangular matrix $\mathfrak{A}_2$ and an invertible lower triangular matrix $\mathfrak{A}_1$ such that from the matrix $\mathfrak{A}_1\mathfrak{A}\mathfrak{A}_2$ one can obtain $\mathfrak{U}_r$ by multiplying columns and rows by suitable scalars and permuting them.)

[4] These matrices were first investigated by Werner Heisenberg (1901-1976) a German theoretical physicist who made foundational contributions to quantum mechanics and is best known for asserting the uncertainty principle of quantum theory. Matrix mechanics is a formulation of quantum mechanics created by Werner Heisenberg, Max Born, and Pascual Jordan in 1925. Matrix mechanics was the first complete and correct definition of quantum mechanics. It extended the Bohr Model by interpreting the physical properties of particles as matrices that evolve in time. It is equivalent to the Schrödinger wave formulation of quantum mechanics, and is the basis of Dirac's bracket notation for the wave function.

9.4 **(a)** Suppose that the solution spaces of the system of linear equations $\mathfrak{A}\,\mathbf{x}=\mathfrak{b}$ and $\mathfrak{A}'\,\mathbf{x}=\mathfrak{b}'$ with $\mathfrak{A}\in \mathrm{M}_{m,n}(K)$, $\mathfrak{A}'\in \mathrm{M}_{m',n}(K)$ and column-vectors $\mathfrak{b}\in K^m$, $\mathfrak{b}'\in K^{m'}$, $m,m',n\in\mathbb{N}$, are non-empty affine subspaces of K^n. Show that these subspaces are parallel if and only if the block-matrix $\binom{\mathfrak{A}}{\mathfrak{A}'}\in \mathrm{M}_{m+m',n}(K)$ is of rank Max $(\mathrm{Rank}\,\mathfrak{A},\mathrm{Rank}\,\mathfrak{A}')$. (By definition, two affine spaces $x+U$ and $x'+U'$ in a K-vector space V are p a r a l l e l if either $U\subseteq U'$ or $U'\subseteq U$. With this definition, the solution spaces $L(\mathfrak{E})=x+L_0(\mathfrak{E})$ and $L(\mathfrak{E}')=x'+L_0(\mathfrak{E}')$ of two systems $\mathfrak{E}\,:\,\mathfrak{A}\,\mathbf{x}=\mathfrak{b}$ and $\mathfrak{E}'\,:\,\mathfrak{A}'\,\mathbf{x}=\mathfrak{b}'$ of liner equations which are affine spaces in K^n are parallel if either $L_0(\mathfrak{E})\subseteq L_0(\mathfrak{E}')$ or $L_0(\mathfrak{E}')\subseteq L_0(\mathfrak{E})$, where $L_0(\mathfrak{E})$ and $L_0(\mathfrak{E}')$ are the solution spaces of the homogeneous systems $\mathfrak{A}\,\mathbf{x}=0$ and $\mathfrak{A}'\,\mathbf{x}=0$ corresponding to $\mathfrak{E}$ and $\mathfrak{E}'$, respectively.)

(b) Let $r\in\mathbb{N}^*$, $s\in\mathbb{N}$ and $\mathfrak{B}\in \mathrm{M}_s(K)$. Show that for every matrix $\mathfrak{A}\in \mathrm{M}_{s,r}(K)$ and every column-vector $\mathbf{x}\in K^r$ there exists a column-vector $\mathbf{y}\in K^s$ with $\mathfrak{A}\,\mathbf{x}+\mathfrak{B}\,\mathbf{y}=0$, if and only if $\mathfrak{B}$ is invertible. Moreover, in this case, one can choose $\mathbf{y}=-\mathfrak{B}^{-1}\mathfrak{A}\,\mathbf{x}$.

****9.5** Let K be an arbitrary field and let $\mathbf{a}:=(a_1,\ldots,a_n)\in K^n$, $n\in\mathbb{N}^+$. Let $U\subseteq K^n$ be a K-subspace of the K-vector space K^n generated by the $n!$ vectors $\mathbf{a}_\sigma:=(a_{\sigma(1)},\ldots,a_{\sigma(n)})$, $\sigma\in\mathfrak{S}_n$, obtained by permuting the coordinates of $(a_1,\ldots,a_n)$. Compute the dimension $\mathrm{Dim}_K U$ of U. (**Hint :** Let $\mathfrak{A}:=\big(a_{\sigma(i)}\big)_{\substack{\sigma\in\mathfrak{S}_n,\\ 1\le i\le n}}\in \mathrm{M}_{n!\times n}(K)$ and let ${}^{\mathrm{t}}f: K^n\to K^{n!}$ be the K-linear map defined by ${}^{\mathrm{t}}f(e_i):=\mathfrak{c}_i=\sum_{\sigma\in\mathfrak{S}_n}a_{\sigma(i)}e_\sigma$, $i=1,\ldots,m$, where $e_1,\ldots,e_n\in K^n$ and $e_\sigma\in K^{n!}=K^{\mathfrak{S}_n}$, $\sigma\in\mathfrak{S}_n$ are the standard bases of K^n and $K^{n!}$, respectively and $\mathfrak{c}_i$ denote the i-th column of $\mathfrak{A}$. Then $\mathrm{Dim}_K U=\mathrm{Rank}\,\mathfrak{A}=\mathrm{Rank}\,{}^{\mathrm{t}}\mathfrak{A}=\mathrm{Rank}\,{}^{\mathrm{t}}f$. Now, compute the kernel $\mathrm{Ker}\,{}^{\mathrm{t}}f$ and use the Rank-Theorem to compute $\mathrm{Rank}\,{}^{\mathrm{t}}f$.)

$$\textbf{Ans: } \mathrm{Dim}_K U=\begin{cases}0, & \text{if } a_1=\cdots=a_n=0,\\ 1, & \text{if } a_1=\cdots=a_n\neq 0,\\ n-1, & \text{if } a_1\neq a_2 \text{ and } \sum_{i=1}^n a_i=0,\\ n, & \text{if } a_1\neq a_2 \text{ and } \sum_{i=1}^n a_i\neq 0,\end{cases}$$

9.6 **(a)** For $r,s\in\{1,\ldots,m\}$ with $m\in\mathbb{N}$, $r\neq s$ and $a,b\in K$, show that $\mathfrak{B}_{rs}(a+b)=\mathfrak{B}_{rs}(a)\,\mathfrak{B}_{rs}(b)$ in $\mathrm{M}_m(K)$, i. e., the map $(K,+)\longrightarrow \mathrm{GL}_m(K)$, $a\longmapsto\mathfrak{B}_{rs}(a)$ is an injective homomorphism from the group $(K,+)$ into the (multiplicative) group $\mathrm{GL}_m(K)$ of the invertible matrices. (**Hint :** Since $\mathfrak{E}_{rs}\mathfrak{E}_{rs}=\delta_{rs}\mathfrak{E}_{rs}=0$, $(\mathfrak{E}_n+a\mathfrak{E}_{rs})(\mathfrak{E}_n+b\mathfrak{E}_{rs})=\mathfrak{E}_n+b\mathfrak{E}_{rs}+a\mathfrak{E}_{rs}+ab\mathfrak{E}_{rs}\mathfrak{E}_{rs}=\mathfrak{E}_n+(a+b)\mathfrak{E}_{rs}$.)

(b) Show that the elementary matrices $\mathfrak{B}_{j+1,j}(a_{j+1}),\ldots,\mathfrak{B}_{m,j}(a_m)\in \mathrm{M}_m(K)$, $j\in\{1,\ldots,m\}$ and $a_{j+1},\ldots,a_m\in K$, are pairwise commutative and their product $\mathfrak{B}_{j+1,j}(a_{j+1})\cdots\mathfrak{B}_{m,j}(a_m)$ is the normalized (all diagonal entries are 1) upper triangular matrix $\mathfrak{B}_j(a_{j+1},\ldots,a_m)$ which is obtained from the identity matrix by replacing j-th column by adding the elements $a_{j+1},\ldots,a_m$ under the main-diagonal, i. e., $\mathfrak{B}_j(a_{j+1},\ldots,a_m)=\mathfrak{E}_n+\sum_{k=1}^{m-j}a_{j+k}\mathfrak{E}_{j+k,j}$. Further, show that the map $(K^{m-j},+)\longrightarrow \mathrm{GL}_m(K)$, $(a_{j+1},\ldots,a_m)\longmapsto\mathfrak{B}_j(a_{j+1},\ldots,a_m)$ is a homomorphism from the group $(K^{m-j},+)$ into the group $\mathrm{GL}_m(K)$. In particular, $\mathfrak{B}_j(a_{j+1},\ldots,a_m)^{-1}=\mathfrak{B}_j(-a_{j+1},\ldots,-a_m)$. (**Remark :** In the concrete situation it is practical for the row-operations to pre-multiply by the matrices of the type $\mathfrak{B}_j(a_{j+1},\ldots,a_m)$. Similarly for column-operations.)