# E0 219 Linear Algebra and Applications / August-December 2016

(ME, MSc. Ph. D. Programmes)



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**Lectures :** Monday and Wednesday ; 11:00–12:30 **Venue:** CSA, Lecture Hall (Room No. 117)

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**Midterms : 1-st Midterm :** Saturday, September 17, 2016; 15:00 – 17:00 **2-nd Midterm :** Saturday, October 22, 2016; 15:00 – 17:00

**Final Examination :** December ??, 2016, 09:00 – -12:00

**Evaluation Weightage : Assignments :** 20% **Midterms (Two) :** 30% **Final Examination :** 50%

| Range of Marks for Grades (Total 100 Marks) | | | | | | |
|---|---|---|---|---|---|---|
| | **Grade S** | **Grade A** | **Grade B** | **Grade C** | **Grade D** | **Grade F** |
| **Marks-Range** | > 90 | 76 — 90 | 61 — 75 | 46 — 60 | 35 — 45 | < 35 |

| | **Grade A+** | **Grade A** | **Grade B+** | **Grade B** | **Grade C** | **Grade D** | **Grade F** |
|---|---|---|---|---|---|---|---|
| **Marks-Range** | > 90 | 81 — 90 | 71 — 80 | 61 — 70 | 51 — 60 | 40 — 50 | < 40 |

## 10. Determinants

### Permutations , Determinant functions , Determinant of a linear operator , Orientations , Determinants and Volumes

Submit a solution of the $*$-**Exercise** ONLY. **Due Date :** Monday, 17-10-2016 (Before the Class)

**• Highly recommended to solve the Exercise 10.2 to win 10 BONUS POINTS!!!**
**•• Complete Correct solution of the Exercise 10.3 carry 20 BONUS POINTS!!!**
**••• Complete Correct solution of the Exercise 10.7 carry 20 BONUS POINTS!!!**

Let $K$ be arbitrary field and let $\mathbb{K}$ denote either the field $\mathbb{R}$ or the field $\mathbb{C}$.

**10.1** For the following permutations compute the number of inversions and the *signum*.[1]

**(a)** The permutation $i \mapsto n-i+1$ in $\mathfrak{S}_n$. **(b)** $\begin{pmatrix} 1 & 2 & \dots & n & n+1 & \dots & 2n \\ 1 & 3 & \dots & 2n-1 & 2 & \dots & 2n \end{pmatrix} \in \mathfrak{S}_{2n}$.

**(c)** $\begin{pmatrix} 1 & 2 & \dots & n & n+1 & \dots & 2n \\ 2 & 4 & \dots & 2n & 1 & \dots & 2n-1 \end{pmatrix} \in \mathfrak{S}_{2n}$.

**(d)** $\begin{pmatrix} 1 & \dots & n-r+1 & n-r+2 & \dots & n \\ r & \dots & n & 1 & \dots & r-1 \end{pmatrix} \in \mathfrak{S}_n,\ 1 \leq r \leq n$. (**Ans :** $(-1)^{(r-1)(n+1)}$.)

**(e)** $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \dots & 2n \\ 1 & 2n & 3 & 2(n-1) & 5 & 2(n-2) & \dots & 2 \end{pmatrix} \in \mathfrak{S}_{2n}$.

(**Hint :** It is easier to compute the sign of the permutation in (c) with the help of the canonical cycle decomposition, in this case, one can already give a decomposition in transpositions : If $n=2m$, the the given permutation is the product of the $m$ transpositions $\langle 2,2n\rangle,\dots,\langle 2m,2m+2\rangle$ and if $n=2m+1$, then it is

[1] S i g n u m is the Latin word for "mark" or "token", of course, it has become the word s i g n a t u r e or just s i g n. Another notation for the sign of a permutation is given by the more general L e v i - C i v i t a s y m b o l $\varepsilon_\sigma$, which is defined for all maps from $\sigma : \{1,\dots,n\} \to \{1,\dots,n\}$, and has value zero for non-bijective maps, in fact: $\varepsilon_\sigma = \prod_{i=1}^{n-1}\left(\frac{1}{i!}\prod_{j=i+1}^{n}(\sigma(j)-\sigma(i))\right)$. The Levi-Civita symbol, also called the a n t i s y m m e t r i c s y m b o l, or a l t e r n a t i n g s y m b o l, is a mathematical symbol used in particular in *tensor calculus*. It is named after the Italian mathematician and physicist T u l l i o L e v i - C i v i t a (1873-1941). In 1900 he and G r e g o r i o R i c c i - C u r b a s t r o (1853-1925) published the theory of tensors in "Méthodes de calcul différentiel absolu et leurs applications", which A l b e r t E i n s t e i n (1879-1955) (Einstein contributed more than any other scientist to the modern vision of physical reality. His special and general theories of relativity are still regarded as the most satisfactory model of the large-scale universe that we have) used as a resource to master the tensor calculus, a critical tool in Einstein's development of the theory of general relativity. In one of the letters, regarding Levi-Civita's work, Einstein wrote "I admire the elegance of your method of computation; it must be nice to ride through these fields upon the horse of true mathematics while the like of us have to make our way laboriously on foot". In 1933 Levi-Civita contributed to Paul Dirac's equations in quantum mechanics as well.

the product of the $m$ transpositions $\langle 2,2n\rangle,\ldots,\langle 2m,2m+4\rangle$. In any case in these transpositions only even numbers occur. Therefore the sign is equal to $(-1)^m=(-1)^{[n/2]}$. — The permutations in (b), (d), (e) are so-called shuffle-permutations and the computation of their signs, in the general case, is given in Supplement S10.2.)

**10.2 The following game is played between the Jailer $J$ of the prison and the team $P$ of $2m$ prisoners, numbered $P_1,\ldots,P_{2m}$. Player $J$ put the passports of prisoners randomly into lockers numbered $1,\ldots,2m$. Prisoners are admitted to the locker room one at a time and is allowed to open an examine exactly 50 lockers. Prisoners win if every prisoner discovers the locker containing his own passport. Team $P$ is allowed to have an initial strategy meeting and no other communication is allowed after the initial meeting. Each team member must leave the locker room exactly as he found it. (**Remarks:** It is important to realize that the solution does not involve some trick to pass information from one player to another. Random guessing by each player will succeed to find his passport with probability $1/2$ and if they act independently, they must be lucky $2m$ times in a row and the the team will with probability $(\frac{1}{2})^{2m}$ which is hopeless. Therefore team $P$ needs some help to find a clever strategy. Amazingly there is a strategy which give significant probability (over 30%) of success for team $P$. Your problem is to find this strategy! — **Hint:** Can the Remark in Supplement S10.17 (a) is helpful to implement a good strategy?)

***10.3 For $1\le i<n$, let $m_i$ be the number of *inversions*[2] $(i,j)$, $i<j\le n$, in the permutation $\sigma\in\mathfrak{S}_n$ and let $\sigma_i:=\langle i{+}m_i\,,i{+}m_i{-}1\rangle\cdots\langle i{+}1,i\rangle$. Show that $\sigma=\sigma_1\cdots\sigma_{n-1}$. (**Remarks:** (1) This proves Lecture-Notes 9.A.10 again and one can recover the permutation $\sigma$ from its inversions. More precisely : The permutation $\sigma$ is uniquely determined by the $(n-1)$-tuple $(m_1,\ldots,m_{n-1})$ with $0\le m_i\le n-i$ and every such tuple uniquely determine a permutation $\sigma\in\mathfrak{S}_n$. This encoding of the elements of $\mathfrak{S}_n$ is frequently used. — One can also examine the analogous problem with the numbers $m'_i$ of the inversions $(j,i)$, $j<i$, $i=2,\ldots,n$.

(2) *All permutations in $\mathfrak{S}_n$ have altogether $\frac{1}{2}n!\binom{n}{2}$ inversions.* For a **Proof** If $n\ge 2$, then for each permutation with the inversion tuple $(m_1,\ldots,m_{n-1})$, there is a distinct complementary permutation with the inversion tuple $((n-1)-m_1,\ldots,1-m_{n-1})$, and both together have inversions $(n-1)+\cdots+1=\binom{n}{2}$.Therefore, if $n\ge 2$, then altogether there are $\frac{1}{2}n!\binom{n}{2}$ inversions ; this also holds for $n\in\{0,1\}$. — Therefore, average number of the inversions is $\frac{1}{2}\binom{n}{2}$. — Note that, maximum number of inversions of a permutation in $\mathfrak{S}_n$ is $\binom{n}{2}$ and its is attained only by the permutation $\begin{pmatrix}1&2&\cdots&n-1&n\\ n&n-1&\cdots&2&1\end{pmatrix}\in\mathfrak{S}_n$ ; its corresponding complementary permutation is the identity permutation which has 0 inversions.)

**10.4** Let $V$ and $W$ be vector spaces over a field $K$ and let $I$ be a finite indexed set with $n$ elements.

**(a)** Suppose that in $K$ the element $n!=n!\cdot 1_K$ is non-zero, i.e. $\operatorname{Char}K=0$ or $\operatorname{Char}K>n$. Then the maps $f\mapsto\frac{1}{n!}Af$ and $f\mapsto\frac{1}{n!}Sf$ are projections of the $K$-vector space of the multi-linear maps $V^I\to W$ onto the subspace of the alternating respectively, the symmetric $I$-linear maps.

**(b)** Suppose that $\operatorname{Char}K\neq 2$. The space of the bilinear maps $V\times V\to W$ is the direct sum of the subspace of the alternating (i. e. skew-symmetric) and the subspace the symmetric bilinear maps. The corresponding projections are $\frac{1}{2}A$ resp. $\frac{1}{2}S$. (**Remark:** A bilinear map $f:V\times V\to W$ can be decomposed into its skew-symmetric part $\frac{1}{2}Af$ and its symmetric part $\frac{1}{2}Sf$.)

**10.5 (a)** (Cramer's[3] Formula) Suppose that $V$ is a $n$-dimensional vector space over a field $K$. For every determinant function $\Delta\colon V^n\to K$ and for arbitrary $x_0,\ldots,x_n\in V$, prove that

$$\sum_{i=0}^{n}(-1)^i\,\Delta(x_0,\ldots,x_{i-1}\,,x_{i+1}\,,\ldots x_n)\,x_i=0\,.$$

(**Hint:** The left-hand side is by Supplement S10.26 (with $g=\mathrm{id}_V$) is an alternating multi-linear map $V^{n+1}\to V$, and hence vanish by Corollary 9.B.6, since $\operatorname{Dim}V=n$.)

**(b)** Let $\mathfrak{A}\in\mathrm{M}_{m,n}(K)$ be an $m\times n$-matrix over a field $K$ and let $\operatorname{Det}\mathfrak{A}_{I,J}$ be a non-zero $r$-minor of $\mathfrak{A}$. Show that $\operatorname{Rank}\mathfrak{A}=r$ if and only if every $(r+1)$-minor $\operatorname{Det}\mathfrak{A}_{I',J'}=0$ with $I\subset I'$ and $J\subset J'$.

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[2] Inversions of a permutation $\sigma\in\mathfrak{S}_n$ are the pairs $(i,j)$ such that $1\le i<j\le n$ and $\sigma(i)>\sigma(j)$.

[3] Gabriel Cramer (1704-1752) was a Swiss mathematican who worked on analysis and determinants. He is best known for his formula for solving simultaneous equations.

*10.6 **(a)** (Vandermonde's determinant) For elements $a_0,\ldots,a_n \in K$, show that

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_0 & a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_0^n & a_1^n & a_2^n & \cdots & a_n^n \end{vmatrix} = \prod_{0\le i<j\le n} (a_j - a_i)\,.$$

( **Hint :** Induction on $n$. — See also Exercise 8.3. )

**(b)** Show that

$$\begin{vmatrix} 1^n & 2^n & 3^n & \cdots & (n+1)^n \\ 2^n & 3^n & 4^n & \cdots & (n+2)^n \\ 3^n & 4^n & 5^n & \cdots & (n+3)^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (n+1)^n & (n+2)^n & (n+3)^n & \cdots & (2n+1)^n \end{vmatrix} = (-1)^{\binom{n+1}{2}}\,(n!)^{n+1}\,.$$

(**Hint :** Since $(i+j-1)^n = \sum_{k=1}^{n+1}\binom{n}{k-1} i^{k-1}(j-1)^{n+1-k}$ by Binomial Theorem, the above matrix is the product of two matrices $\left(\binom{n}{k-1} i^{k-1}\right)_{1\le i,k\le n+1}$ and $\left(j^{n+1-k}\right)_{1\le k,j\le n+1}$. Now, use (a) to compute the determinants of these two matrices.)

***10.7 Let $\mathfrak{A} = (a_{ij}) \in \mathrm{M}_n(\mathbb{R})$ be $n\times n$-matrix over real numbers .

**(a)** Suppose that for each $i = 1,\ldots,n$, $|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}|$. Then $a_{11}\cdots a_{nn}\,\mathrm{Det}\,\mathfrak{A} > 0$. (**Hint:** Note that $\mathrm{Det}\,\mathfrak{A} \neq 0$ for such a matrix by Exercise 4.3 Therefore the continuous polynomial function

$$f(t) := \begin{vmatrix} a_{11} & ta_{12} & ta_{13} & \cdots & ta_{1n} \\ ta_{21} & a_{22} & ta_{23} & \cdots & ta_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ ta_{n-1,1} & ta_{n-1,2} & ta_{n-1,3} & \cdots & ta_{n-1,n} \\ ta_{n1} & ta_{n2} & ta_{n3} & \cdots & a_{nn} \end{vmatrix}$$

has no zero in the interval $[0,1]$ and so the values $f(0)$ and $f(1)$ have the same sign by the *Intermediate Value Theorem* [4]. — **Remarks:** Two important special cases are : (1) (Hadamard [5]) Let $(a_{ij}) \in \mathrm{M}_n(\mathbb{C})$. For every $i$, suppose that in the $i$-th row there is at most one element $a_{i,j(i)} \neq 0$ with $j(i) \neq i$ and for this element $|a_{i,j(i)}| < |a_{ii}$. Then the matrix $(a_{ij})$ is invertible. (2) (Minkowski [6]) Let $(a_{ij}) \in \mathrm{M}_n(\mathbb{R})$ with $a_{ij} \le 0$ for every $i \neq j$. For every $i$, suppose that $\sum_{j=1}^{n} a_{ij} > 0$. Then the matrix $(a_{ij})$ is invertible. )

**(b)** Suppose that $n \in \mathbb{N}$ is odd. Then show that there exists a real $t \in \mathbb{R}$ such that

$$\mathrm{Det}\begin{pmatrix} a_{11}+t & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}+t & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}+t \end{pmatrix} = 0\,.$$

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[4] **Intermediate Value Theorem** *Every continuous function* $f:[a,b]\to\mathbb{R}$ *on an interval* $[a,b]$ *attains every value in between* $f(a)$ *and* $f(b)$, i. e. *for every* $c\in\mathbb{R}$ *in between* $f(a)$ *and* $f(b)$ *there exists* $x_0\in[a,b]$ *such that* $f(x_0)=c$.

— For c = 0 above, the statement is also known as Bolzano's theorem. This theorem was first proved by Bernard Bolzano (1781-1848) (a mathematicain from Prague, Bohemia, Austrian Habsburg domain, now Czech Republic, who successfully freed calculus from the concept of the infinitesimal. He also gave examples of 1-1 correspondences between the elements of an infinite set and the elements of a proper subset.) in 1817. A French mathematician Augustin Louis Cauchy (1789-1857) (Cauchy pioneered the study of analysis, both real and complex, and the theory of permutation groups. He also researched in convergence and divergence of infinite series, differential equations, determinants, probability and mathematical physics) provided a proof in 1821. Both were inspired by the goal of formalizing the analysis of functions and the work of Lagrange. The insight of Bolzano and Cauchy was to define a general notion of continuity (in terms of infinitesimals in Cauchy's case, and using real inequalities in Bolzano's case), and to provide a proof based on such definitions.

[5] Jacques Hadamard (1865-1963) was a French mathematician whose most important result is the prime number theorem which he proved in 1896. This states that the number of primes $< n$ tends to infinity as fast as $n/\ln n$.

[6] Hermann Minkowski (1864-1909) was a German mathematician who developed a new view of space and time and laid the mathematical foundation of the theory of relativity.

(**Hint:** The determinant is a polynomial of odd degree $n$ in $t$ and hence it has (by intermediate value theorem, see the Footnote 4) a zero in $\mathbb{R}$.)

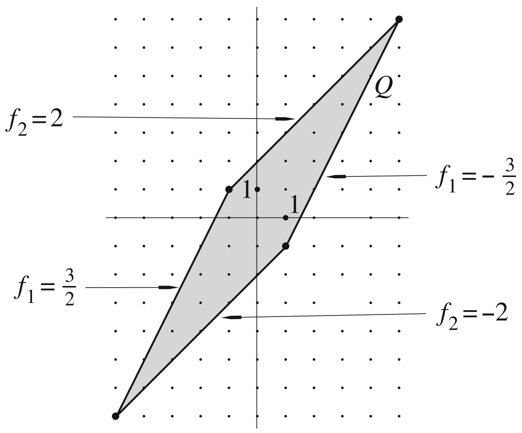
**10.8** Let $V$ be a finite dimensional $K$-vector space. Compute the determinant of the $K$-linear map $f : V \to V$ in the following cases : **(a)** $f$ is the homothecy $a\,\mathrm{id}_V$. **(b)** $f$ is a projection. **(c)** $f$ is an involution (see Supplement S8.9 (b)). **(d)** $f$ is a transvection or a dilatation (see Supplement S8.9 (c), (d)).

**10.9** **(a)** Let $V$ be an oriented $n$-dimensional $\mathbb{R}$-vector space and let $\sigma \in \mathfrak{S}_n$ be a permutation. Suppose that the orientation of $V$ is represented by the $v_1, \ldots, v_n$. Show that $v_{\sigma(1)}, \ldots, v_{\sigma(n)}$ represent the orientation of $V$ if and only if $\sigma$ is an even permutation. Further, show that the basis $v_n, \ldots, v_1$ represent the orientation of $V$ if and only if $n \equiv 0$ or $n \equiv 1$ modulo 4.

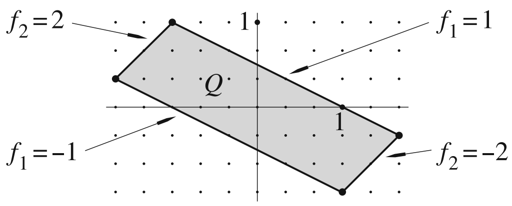
**(b)** The bases $\mathfrak{E}_{11}, \ldots, \mathfrak{E}_{1n}, \ldots, \mathfrak{E}_{m1}, \ldots, \mathfrak{E}_{mn}$ and $\mathfrak{E}_{11}, \ldots, \mathfrak{E}_{m1}, \ldots, \mathfrak{E}_{1n}, \ldots \mathfrak{E}_{mn}$ represent the same orientation of $\mathrm{M}_{m,n}(\mathbb{R})$ if and only if $\binom{m}{2}\binom{n}{2}$ is even.

**10.10** Using the Supplement S10.85 (with its notation) compute the surface area of the following parallelograms :

**(a)** $Q := \{x = (x_1, x_2) \in \mathbb{R}^2 \mid |x_1 - \tfrac{1}{2}x_2| \leq \tfrac{3}{2}, |-x_1 + x_2| \leq 2\}$.




**(b)** $Q := \{x = (x_1, x_2) \in \mathbb{R}^2 \mid |x_1 + 2x_2| \leq 1, |x_1 - x_2| \leq 2\}$ (**Ans :** 8/3 )




(**Hint :** $f_1(x_1, x_2) = -x_1 + \tfrac{1}{2}x_2$, $f_2(x_1, x_2) = -x_1 + x_2$, and hence $\mathfrak{A} = \begin{pmatrix} -1 & -1 \\ 1/2 & 1 \end{pmatrix}$ and $d = |\mathrm{Det}\,(\mathfrak{A})| = \tfrac{1}{2}$ and $\lambda^2(Q) = 2^2 \cdot \tfrac{3}{2} \cdot 2 / \tfrac{1}{2} = 24$.)