2 Linear Equations

2.A Gauss Elimination Process

Let $a_1, \dots, a_m$ and b be elements of a field K. By a solutions of the linear equation

$$a x_1 + \cdots + a_m x_m = b$$

with n variables (or unknowns) $x_1, \dots, x_n$, we understand all n-tuples $x = (x_1, \dots, x_n) \in K^n$ for which the given eqn. holds. It is very easy to give all solutions of this equation. If all $a_1, \dots, a_m$ are 0, then there is a soln if and only if $b = 0$ and in this case all $x \in K^n$ are solns. On the other case at least one of $a_1, \dots, a_m$ is non-zero, say a_1, then clearly all solns are n-tuples $x = (x_1, \dots, x_n) \in K^n$ such that $x_2, \dots, x_n \in K$ are arbitrary and

$$x_1 = a_1^{-1} b - a_1^{-1} a_2 x_2 - \cdots - a_1^{-1} \cdot a_m x_n.$$

We now consider a system

$$\mathcal{E}: \begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{cases}$$

of m linear equations in n unknowns $x_1, \dots, x_n$ with coefficients $a_{ij}, b_i \in K$, $i = 1, \dots, m$; $j = 1, \dots, n$.

Solutions are the n-tuples $x = (x_1, \dots, x_n) \in K^n$ which satisfy all m equations simultaneously. $L(\mathcal{E})$.

If all $b_1, \dots, b_m$ are 0, then this system is called a homogeneous system of linear equations. In the general case the system of equations is called inhomogeneous and the system of linear equations $\mathcal{E}_0$ obtained from this system by putting all $b_i = 0$ is called the corresponding homogeneous linear system of equations.

The solutions of both these system of equations are related by the following:

2.A.1 Theorem

(1) Let x' be a (special) solution of the inhomogenous system of equations. The all solutions of this system is obtained by adding to x' a solution of the corresponding homogeneous system of linear equations, i.e.

$$L(\mathcal{E}) = x' + L(\mathcal{E}_0) = \{x' + y \mid y \in L(\mathcal{E}_0)$$

(2) The solution-set $L(\mathcal{E}_0)$ of the homogenous system of linear equations is a K-subspace of K^n.

<u>Proof</u> (1) Let $x' = (x_1', \ldots, x_n')$. Then $\sum_{j=1}^{n} a_{ij} x_j' = b_i$ for all $i = 1, \ldots, m$.

For an element $x = (x_1, \ldots, x_n) \in K^n$, we have $\sum_{j=1}^{n} a_{ij} x_j = b_i$ for all $i = 1, \ldots, m \iff \sum_{j=1}^{n} a_{ij}(x_j - x_j') = 0$ for all $i = 1, \ldots, m$, i.e. $y = x - x' \in L(\mathcal{E}_0)$.

Therefore to find all solutions of a given system of inhomogenous linear equations, it is enough to find one solution x' of this inhomogeneous system and a generating system $y^{(1)}, \dots, y^{(k)}$ for the solution-space $L(\mathcal{E}_0)$ of the corresponding system of homogeneous linear equations. Then every solution of $\mathcal{E}$ has the form

$$x' + c_1 y^{(1)} + \dots + c_k y^{(k)}$$

$c_1, \dots, c_k \in K$. The problem is to choose the no. of generators k as small as possible, for this we introduce concepts of the basis resp. dimension of a vector space in the next paragraph.

Identify an equation $a_1x_1 + \dots + a_n x_n = b$ with the coefficient tuple

$(a_1, \dots, a_n, b) \in K^n$

This allows us to use the operations

of the vectors in K^{n+1} and translate them back into equations. Therefore clearly: The solution-set of a system of linear equations will remain unchanged if we perform the following well-known <u>elementary operations</u>:

(1) Adding a multiple of an equation into other equation.

(2) Multiply ~~ing~~ an equation with $a \in K$, $a \neq 0$.

(3) Interchanging equations.

The well-known <u>Gauss-elimination</u> is a systematic application of these elementary operations to obtain all solutions algorithmically:

If atleast in one of the equations

$$a_{11}x_1 + \cdots + a_{1n}x_n = b_1$$

$$\cdots$$

$$a_{m1}x_1 + \cdots + a_{mn}x_n = b_m$$

by interchanging the equations, we may assume that $a_{11} \neq 0$ (this element a_{11} is called Pivot-element) (Pivot = angel in English)

Then multiply the first equation by a_{11}^{-1} and add it to every $i = 1, \ldots, m$ row the $(-a_{i1})$-multiple of

Then all coefficients of x_1 in each equation for $i \geq 2$ are 0.

Now, one repeat this process to the system of equations from 2^{nd} to m-th equation and eliminate the unknown x_1. If all $a_{i1} = 0$ for $i = 1, \ldots, m$, then already we can eliminate x_1. Proceeding this way we arrive at a system of equations which has the following form:

- $1 \leq i_1 < i_2 < \cdots < i_r \leq n$

$\cancel{x_{i_1} + c_{1,i_1+1} x_{i_1+1} + \cdots}$

$$x_{i_1} + c_{1,i_1+1} x_{i_1+1} + \cdots + c_{1,i_2} x_{i_2} + c_{1,i_2+1} x_{i_2+1} + \cdots + c_{1n} x_n = d_1$$

$$x_{i_2} + c_{2,i_2+1} x_{i_2+1} + \cdots + c_{2n} x_n = d_2$$

$$\cdots$$

$$x_{i_r} + \cdots + c_{rn} x_n = d_r$$

$$0 = d_{r+1}$$

$$\vdots$$

$$0 = d_m$$

This system has a solution if and only if $d_{r+1} = \cdots = d_m = 0$. Moreover, in this case the value ~~of~~ for the unknowns x_i with $i \in \{i_1, \dots, i_r\}$ can be freely chosen (in K) and the value for ~~$x_{i_1}, \dots, x_{i_r}$~~ $x_{i_r}, x_{i_{r-1}}, \dots, x_{i_2}, x_{i_1}$ are then uniquely determined from the r-th, (r-1)-th, ..., 2-nd, 1-st equations. The Calculation in this ~~last~~ step can be obtained by ~~further~~ [illegible] use of elementary operations on the coefficients of x_{i_j} in the 1-st till $(j-1)$-th equations to 0 for $j = 2, \dots, r$.

Choose Pivot-element as the biggest among a_{ij}

2.A.2 Example Find solution set of the following system by using Gauss-elimination

$$2x_1 + x_2 - 2x_3 + 3x_4 = 1$$

$$3x_1 + 2x_2 - x_3 + 2x_4 = 4$$

$$3x_1 + 3x_2 + 3x_3 - 3x_4 = b$$

over $\mathbb{Q}$.

First Pivot-element $a_{11} = 2$

$$x_1 + \frac{1}{2}x_2 - x_3 + \frac{3}{2}x_4 = 1/2$$

$$\frac{1}{2}x_2 + 2x_3 - 5/2\, x_4 = 5/2$$

$$\frac{3}{2}x_2 + 6x_3 - \frac{15}{2}x_4 = b - 3/2$$

next Pivot-element $c_{22} = 1/2$

$$x_1 + \frac{1}{2}x_2 - x_3 + 3/2\, x_4 = \frac{1}{2}$$

$$x_2 + 4x_3 - 5x_4 = 5$$

$$0 = b - 9$$

So this system has solution if and only if

$b = 9$. For the calculation of solution set in this case take the equivalent system of equations (replace R_1 by $R_1 - \frac{1}{2}R_2$)

$$x_1 - 3x_3 + 4x_4 = -2$$

$$x_2 + 4x_3 - 5x_4 = 5$$

Then the solution-set is

$$L = \left\{ (x_1, x_2, x_3, x_4) \;\middle|\; \begin{array}{l} x_1 = 3x_3 - 4x_4 - 2 \\ x_2 = -4x_3 + 5x_4 + 5 \end{array} \right\}$$

$$= \left\{ \left(3x_3 - 4x_4 - 2, -4x_3 + 5x_4 + 5, x_3, x_4 \right) \;\middle|\; x_3, x_4 \in \mathbb{Q} \right\}$$

$$= (-2, 5, 0, 0) + L(\Sigma_0), \text{ where}$$

$$L(\Sigma_0) =$$

$$= \mathbb{Q}(3, -4, 1, 0) + \mathbb{Q}(-4, 5, 0, 1).$$

The vectors $(3, -4, 1, 0), (-4, 5, 0, 1) \in \mathbb{Q}^4$ generate the K-subspace $L(\Sigma_0)$ and the sp. soln. $(-2, 5, 0, 0)$ used to obtain the representation of $L(\mathcal{E})$ given in 2.A.1 (1).