

### 3.A Generating systems, linear independence, Bases

Let  $V$  be a  $K$ -vector space

3.A.1 Definition A family  $v_i, i \in I$  of elements of  $V$  is called a generating system of  $V$  if every element of  $V$  is a linear combination of  $v_i, i \in I$ , equivalently,  $V = \sum_{i \in I} K v_i$  (= the subspace generated by  $v_i, i \in I$ ).

If  $v_i, i \in I$ , is a generating system of  $V$ , then every element  $x \in V$  can be written in the form

$$x = \sum_{i \in I} a_i v_i, \quad a_i \in K, \quad a_i = 0 \text{ for all most } i \in I$$

The coefficients  $a_i$  in this representation are in general not uniquely determined by  $x$ .

If  $x = \sum_{i \in I} b_i v_i$  is another ~~decom-~~<sup>representa-</sup>tion of  $x$  as linear combination of  $v_i, i \in I$ , then

$$0 = x - x = \sum_{i \in I} (a_i - b_i) v_i = \sum_{i \in I} c_i v_i$$

as a representation of 0 with coefficients  $c_i := a_i - b_i, i \in I$ .

If  $c_i \neq 0$  for all  $i \in I$ , i.e.  $a_i \neq b_i$  for all  $i \in I$ , then this is a non-trivial representation of 0, i.e.

a representation of 0 with coefficients in  $K$  not all zero. Conversely, from every such non-trivial representation of 0, we get another representation of  $x$  as linear combination of the  $v_i, i \in I$ .

Therefore the problem of uniqueness of such a representation reduces to the question of the "linear independence" of the  $v_i, i \in I$ , in the sense of the following definition:

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3.A.2 Definition A family  $\{v_i, i \in I\}$  of elements in  $Y$  is called linear independent if the following holds:

If  $(a_i)_{i \in I} \in K^{(I)}$ , and  $\sum_{i \in I} a_i v_i = 0$

then  $a_i = 0$  for all  $i \in I$ .

The linear independence of the  $v_i, i \in I$  means: A linear combination of the  $v_i, i \in I$ , is 0 if and only if (not only almost all, but) all its coefficients are 0.

The  $v_i, i \in I$ , are linearly dependent, i.e. not linearly independent, if there exists an  $I$ -tuple  $(a_i)_{i \in I} \in K^{(I)}$  such that  $\sum_{i \in I} a_i v_i = 0$ , but at least

for one  $i_0 \in I$ ,  $a_{i_0} \neq 0$ . Then

$$v_{i_0} = \sum_{\substack{i \in I \\ i \neq i_0}} a_i v_i = \sum_{\substack{i \in I \\ i \neq i_0}} (-a_{i_0}^{-1} a_i) v_i$$

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is a linear combination of the remaining  $v_i, i \in I, i \neq i_0$ .

Therefore, a system of vectors is linearly independent if and only if no vector in this system can be represented as a linear combination of the remaining vectors.

In particular, in a linearly independent family  $v_i, i \in I, v_i \neq v_j$  for all  $i, j \in I, i \neq j$ .

A family  $v_i, i \in I$ , is linearly independent if and only if every finite subfamily  $v_j, j \in J, J \subseteq I \rightarrow$  finite is linearly independent.

By the remark before def. 3.A-2:

If the system  $v_i, i \in I$  of vectors in  $V$  is linearly independent and if  $x = \sum_{i \in I} a_i x_i \in V$  is a linear combination of the  $v_i, i \in I$ , then the

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I-tuple  $(a_i) \in K^{(I)}$  is uniquely  
determined by  $x$ .

3.A.3 Definition A family  $v_i, i \in I$ ,  
of elements of  $V$  is called a basis  
of  $V$  if  $v_i, i \in I$ , is a linearly indep  
endent generating system of  $V$ .

If  $v_i, i \in I$ , is a basis of  $V$ , then  
every vector  $x \in V$  is a linear  
combination  $x = \sum_{i \in I} a_i v_i$ ,  $(a_i) \in K^{(I)}$

of  $v_i, i \in I$  and the coefficients  $a_i$ ,  
 $i \in I$ , are uniquely determined by  $x$ .

The  $a_i, i \in I$ , are called coordinates  
or components or coefficients of  $x$   
with respect to the basis  $v_i, i \in I$ ,  
and the functions: for  $i \in I$ ,

$$N_i^* : V \longrightarrow K, \quad x \mapsto N_i^*(x) := a_i$$

are called the coordinate-functions  
on  $V$  with resp. to the basis  $v_i, i \in I$ .

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Therefore by definition: for all  $x \in V$ ,

$$x = \sum_{i \in I} a_i v_i = \sum_{i \in I} v_i^*(x) v_i$$

From the computation-rules for the linear combinations, <sup>we have</sup> the following computation-rules for the coordinate functions  $v_i^*$ ,  $i \in I$ :

For all  $x, y \in V$  and all  $a \in K$

$$v_i^*(x+y) = v_i^*(x) + v_i^*(y) \text{ and}$$

$$v_i^*(ax) = a \cdot v_i^*(x).$$

From the earlier definitions we immediately have:

3.A.4 Theorem Let  $V$  be a vector space over  $K$  and let  $v_i, i \in I$  be a family of elements in  $V$ . Let  $f: K^{(I)} \rightarrow V$  be the map defined by  $(a_i)_{i \in I} \mapsto \sum_{i \in I} a_i v_i$ . Then

(1)  $v_i, i \in I$ , is a generating system of  $V \iff f$  is surjective.

(2)  $v_i, i \in I$ , is linearly independent  $\iff f$  is injective.

(3)  $v_i, i \in I$ , is a basis of  $V \iff f$  is bijective.

### 3.A.5 Example (Standard bases)

Let  $K$  be a field,  $I$  an indexed set,  
 $V = K^{(I)} =$  the  $I$ -tuples  $(a_i)_{i \in I}$  with elements in  $K$  with almost all  $a_i = 0$ .

For  $i \in I$ ,  $e_i := (\delta_{ji})_{j \in I} : I \rightarrow K$   
 $i \mapsto 1$   
 $j \mapsto 0, j \neq i$

Then  $e_i \in K^{(I)}, i \in I$  and they form a basis of  $K^{(I)}$  - called the standard basis of  $K^{(I)}$ .

$$(a_i)_{i \in I} = \sum_{i \in I} a_i e_i \quad \text{for } (a_i)_{i \in I} \in K^{(I)}$$

The coordinate functions  $e_i^*$  are the canonical projections (A.1.  $\mapsto a_i, i \in I$ )

In particular, for  $n \in \mathbb{N}^*$ , 3A/8  
 $I = \{1, 2, \dots, n\}$ ,  $K^I = K^{(I)} = \underbrace{K \times \dots \times K}_{n \text{ times}}$   
 $= K^n$

$e_1 = (1, 0, \dots, 0)$ ,  $e_2 = (0, 1, 0, \dots, 0)$ ,  $\dots$ ,  
 $e_n = (0, \dots, 0, 1)$  is a standard basis  
of  $K^n$  with coordinate functions:

$$e_i^* : K^n \rightarrow K, (a_1, \dots, a_n) \mapsto a_i, i=1, \dots, n.$$

The standard basis of  $K = K^1$  is 1.

3.A.6 Example If  $v_i, i \in I$ , is  
a linearly independent family of  
elements in a vector space  $V$  over  $K$ ,  
then  $v_i, i \in I$ , is a basis of the  
 $K$ -subspace  $W = \sum_{i \in I} K v_i$  of  $V$  genera-  
ted by the family  $v_i, i \in I$ .

3.A.7 Example (Polynomial functions)  
Let  $D \subseteq K$  be a subset with at-



least  $n$  elements,  $n \in \mathbb{N}$ . Then the power functions:  $D \rightarrow K$ ,

$$1 = t^0, t^1, \dots, t^{n-1}$$

which are elements of  $K^D :=$  the vector space of all  $K$ -valued functions on  $D$ , are linearly independent over

$K$ : For, if  $a_0 + a_1 t + \dots + a_{n-1} t^{n-1} = 0$ ,

$a_0, a_1, \dots, a_{n-1} \in K$ , i.e. the function

$$p: D \rightarrow K, t \mapsto \sum_{i=0}^{n-1} a_i t^i = 0 \text{ on } D \text{ and}$$

hence  $p$  has at least  $n$  zeros, but  $\deg p \leq n-1$ . Therefore  $p(t) = 0$ , i.e.

$$a_0 = a_1 = \dots = a_{n-1} = 0.$$

In particular, the power-functions  $t^m, m=0, \dots, n-1$ , is a basis of

the space  $K[t]_n$  of the polynomial functions of degree  $< n$  on  $D$ .

If  $D$  has infinitely many elements, then the power-functions  $t^m, m \in \mathbb{N}$ , are

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linearly independent and hence  
form a basis of the space  $K[t]$   
( $\subseteq K^D$ ) of all polynomial functions  
on  $D$ .

(\*) 3.A.8 Example (Rational functions)

Let  $D \subseteq \mathbb{C}$  be an infinite subset.  
By the theorem on division with  
remainder and the theorem on the  
partial fractions (11.B.1 and 11.B.2  
or 10.A.2 of Vol 1) and Example 3.B.8  
the rational functions

$$t^n, n \in \mathbb{N}, \text{ and } \frac{1}{(t-a)^m}, m \in \mathbb{N}^*, a \in \mathbb{C} \setminus D.$$

together form a basis of the  $\mathbb{C}$ -  
vector space of the rational func-  
tions on  $D$ , see also Example 3.B.8.

- A corresponding assertion also holds  
for an arbitrary "algebraically closed"  
field  $K$ , instead of  $\mathbb{C}$ .

3.A.9 Example The  $\mathbb{R}$ -valued functions (on  $\mathbb{R}$ )  $f_a: \mathbb{R} \rightarrow \mathbb{R}, t \mapsto |t-a|$ ,  $a \in \mathbb{R}$ , are linearly independent elements in the  $\mathbb{R}$ -vector space  $\mathbb{R}^{\mathbb{R}}$  of all  $\mathbb{R}$ -valued functions on  $\mathbb{R}$ . (See Exercise 17a) (See 12.A, ver 1)

Simple proof: Let  $n > 0$  and

$$\sum_{i=1}^n c_i f_{a_i} = 0, \quad c_1, \dots, c_n \in \mathbb{R}, \quad \text{c.e.} \quad \sum_{i=1}^n c_i |t-a_i| = 0$$

for all  $t \in \mathbb{R}$ . To prove  $c_1 = \dots = c_n = 0$ .

We may assume  $a_1 < a_2 < \dots < a_n$ .

Then  $c_1 |t-a_1| = -\sum_{i=2}^n c_i |t-a_i|$  for

all  $t \in \mathbb{R}$ . On the interval  $(-\infty, a_2)$

LHS is  $c_1 |t-a_1| = c_1 (t-a_1)$  is a polynomial function of degree  $\leq 1$ .

The RHS on  $(-\infty, a_2)$  is always  $-ve$ .

So the only possibility is  $c_1 = 0$ . Continuing this way conclude that  $c_2 = \dots = c_n = 0$ .

\* 3.A.10 Example Let  $D \subseteq \mathbb{C}$

and for  $\alpha \in \mathbb{C}$ , let  $e^{\alpha t}$  be the function  $\mathbb{D} \rightarrow \mathbb{C}$   
 $t \mapsto e^{\alpha t}$

$e^{\alpha t} \in \mathbb{C}^{\mathbb{D}}$ . Suppose that  $D$  has a limit point in  $\mathbb{C}$ . Then the family  $e^{\alpha t}, \alpha \in \mathbb{C}$  is linearly independent in the  $\mathbb{C}$ -vector space  $\mathbb{C}^{\mathbb{D}}$  of  $\mathbb{C}$ -valued functions on  $D$ .

Prf For this one need to use identity theorem:

If  $\sum_{\nu=1}^n a_{\nu} e^{\alpha_{\nu} z} = 0$  for a set  $z$  of

Complex numbers which has a limit pt.

then  $\sum_{\nu=1}^n a_{\nu} e^{\alpha_{\nu} z} = 0$  for all  $z \in \mathbb{C}$  (by

the identity thm) and similarly

$\sum_{\nu=1}^n a_{\nu} \alpha_{\nu} e^{\alpha_{\nu} z} = 0$  and hence

$\sum_{\nu=2}^n a_{\nu} (\alpha_1 - \alpha_{\nu}) e^{\alpha_{\nu} z} = 0$ . Now, use induction on  $n$

Similarly, the functions  $t^\alpha$ ,  $\alpha \in \mathbb{C}$ , defined on every subset  $D \subseteq \mathbb{C} \setminus \mathbb{R}_-$  which has a limit pt in  $\mathbb{C} \setminus \mathbb{R}_-$ , are linearly independent in the  $\mathbb{C}$ -vector-space  $\mathbb{C}^D$ .

Proof. If  $\sum_{\nu=1}^n a_\nu t^{\alpha_\nu} = 0$  for a set  $z$  of complex numbers in  $\mathbb{C} \setminus \mathbb{R}_-$  which has a limit pt in  $\mathbb{C} \setminus \mathbb{R}_-$ . Then prove (as in the ex. above) that  $a_1 = \dots = a_n = 0$ .

### \* 3.A.12 Example (Quasi-polynomials)

The functions  $t^n e^{\alpha t}$ ,  $n \in \mathbb{N}$ ,  $\alpha \in \mathbb{C}$  are linearly independent in the space  $\mathbb{C}^D$  of  $\mathbb{C}$ -valued functions on a subset  $D \subseteq \mathbb{C}$  which has a limit point in  $\mathbb{C}$ . The subspace generated by these functions is called the space of quasi-polynomials. The quasi-polynomials are the solutions of the linear

differential equations with constant coefficients.  $P(D)y = 0$ ,  $P \in \mathbb{C}[X] \setminus \{0\}$ ,  
(See Vol. 1. 19.D. and also Theorem 11.E.3)

3.A.12 Example (IR-bases of

$\mathbb{C}$ -vector spaces) Let  $V$  be a  $\mathbb{C}$ -complex vector space. Then  $V$  is also a  $\mathbb{R}$ -vector space in a natural way. If  $v_j, j \in J$  is a  $\mathbb{C}$ -basis of  $V$ , then  $v_j, iv_j, j \in J, i = \sqrt{-1}$  is a  $\mathbb{R}$ -basis of  $V$ .

Proof These vectors is a  $\mathbb{R}$ -generating system of  $V$ : for  $x \in V$ , there are complex numbers  $z_j, j \in J$  with  $x = \sum_{j \in J} z_j v_j$ .

$$\begin{aligned} \text{Then } x &= \sum_{j \in J} (\operatorname{Re} z_j + i \operatorname{Im} z_j) v_j \\ &= \sum_{j \in J} (\operatorname{Re} z_j) v_j + \sum_{j \in J} (\operatorname{Im} z_j) i v_j \end{aligned}$$

is a representation of  $x$  as a linear

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Combination ~~of~~ in the  $v_j, iv_j, j \in J$   
with coefficients  $\operatorname{Re} z_j, \operatorname{Im} z_j \in \mathbb{R}$ .

Further, if  $0 = \sum_{j \in J} a_j v_j + \sum_{j \in J} b_j (iv_j)$

with real coefficients  $a_j, b_j \in \mathbb{R}$ , then

$0 = \sum_{j \in J} (a_j + ib_j) v_j$ . Now, since  $v_j,$

$j \in J$ , are linearly independent over  $\mathbb{C}$ ,

it follows that  $a_j + ib_j = 0$  for all  $j \in J$

i.e.  $a_j = 0$  and  $b_j = 0$  for all  $j \in J$ .

As a simple example, we note that  
from the  $\mathbb{C}$ -basis  $1$  of  $\mathbb{C}$ , we obtain

the  $\mathbb{R}$ -basis  $1, i$  of  $\mathbb{C}$ . More generally,

from the  $\mathbb{C}$ -standard basis  $e_1, \dots, e_n$

$\in \mathbb{C}^n$ , we get the  $\mathbb{R}$ -basis

$(1, 0, \dots, 0), (i, 0, \dots, 0), \dots, (0, \dots, 0, 1), (0, \dots, 0, i)$

of  $\mathbb{C}^n$  with  $2n$  elements.

3.A.13 Definition Let  $v_i, i \in I$ , be a family of vectors in a  $K$ -vector space  $V$ .

(1)  $v_i, i \in I$ , is called maximal linearly independent in  $V$  if  $v_i, i \in I$ , is linearly independent and the family  $v_i, i \in I \cup \{i_0\}$  with  $i_0 \notin I$  is linearly dependent for every  $v_{i_0} \in V$ .

(2)  $v_i, i \in I$ , is called a minimal generating system for  $V$  if  $v_i, i \in I$ , is a generating system for  $V$  and if  $v_i, i \in I \setminus \{i_0\}$  is not a generating system for every  $i_0 \in I$ .

3.A.14 Theorem Let  $v_i, i \in I$ , be a family of vectors in a  $K$ -vector space  $V$ . Then the following statements are equivalent:

(1)  $v_i, i \in I$ , is a basis of  $V$ .

(2)  $v_i, i \in I$ , is a maximal linearly independent in  $V$ .



(3)  $v_i, i \in I$ , is a minimal generating system for  $V$ .

Proof (1)  $\Rightarrow$  (2): Extend  $v_i, i \in I$ , by another element  $v_{i_0} \in V, i_0 \notin I$ . Since  $v_i, i \in I$ , is a basis of  $V$ ,  $v_{i_0}$  is a linear combination of  $v_i, i \in I$ , i.e.  $v_{i_0} = \sum_{i \in I} a_i v_i, (a_i) \in K^{(I)}$ . But then  $v_i, i \in I \cup \{i_0\}$  is not linearly independent

(2)  $\Rightarrow$  (1): <sup>only</sup> To show that  $v_i, i \in I$ , is a generating system for  $V$ . Let  $x \in V$ . Then the family  $v_i, i \in I, x$ , is linearly independent. Therefore there is a linear combination  $0 = \sum_{i \in I} a_i v_i + ax$  with  $(a_i) \in K^{(I)}, a \in K$  and not all coefficients are zero. If  $a = 0$ , then we get a contradiction to the linear independence of  $v_i, i \in I$ . Therefore  $a \neq 0$  and  $x = \sum_{i \in I} (-a^{-1} a_i) v_i$  is a linear combination of the  $v_i, i \in I$ .

(1)  $\Rightarrow$  (3): Let  $v_0 \in I$ . If  $v_i, i \in I \setminus \{v_0\}$  is also a generating system for  $V$ , then  $v_{v_0}$  is a linear combination of  $v_i, i \in I \setminus \{v_0\}$ , i.e.  $v_{v_0} = \sum_{i \in I \setminus \{v_0\}} a_i v_i$  and hence  $v_i, i \in I$  is linearly dependent.

(3)  $\Rightarrow$  (1): If  $v_i, i \in I$  is linearly dependent, then there exists  $v_{v_0}, v_0 \in I$ , which is a linear combination of  $v_i, i \in I \setminus \{v_0\}$ . Therefore  $v_i, i \in I \setminus \{v_0\}$  is a generating system for  $V$  and so  $v_i, i \in I$  is not a minimal generating system for  $V$ .

3.A.15 Theorem Let  $v_i, i \in I$ , be a countable generating system for a  $K$ -vector space  $V$ . Then this system contains a basis of  $V$ . In particular, every vector space with a countable generating system also has a countable basis.

Proof We may assume that  $I \subseteq \mathbb{N}$ .

$$\text{Let } J := \{i \in I \mid v_i \notin \sum_{\substack{j \in I \\ j < i}} K v_j\}.$$

Then  $v_j, j \in J$ , is a basis of  $V$ .

These vectors generate  $V$ : Suppose on the contrary that the subspace generated by  $v_j, j \in J$  does not contain all the vectors  $v_i, i \in I$ . Then choose a minimal element  $i_0 \in I$  with  $v_{i_0} \notin \sum_{j \in J} K v_j$  (exists since  $I \subseteq \mathbb{N}$ )

Then  $i_0 \notin J$  but this is a contradiction to the definition of  $J$ .

The  $v_j, j \in J$ , are linearly independent:

If  $\sum_{j \in J} a_j v_j = 0$  with at least one  $a_{j_0} \neq 0, j_0 \in J$ , ~~let~~ then let  $j_0$  be maximal

with this property (there are only finitely many non-zero, so maximal exists!). Then  $v_{j_0} \in \sum_{j < j_0} K v_j$  and hence  $j_0 \notin J$  by def. of  $J$ , a contradiction.

Corollary Let  $v_1, \dots, v_n$  be a finite generating system for  $V$ . Then  $V$  has a basis. 3A/20

Theorem 3.A.15 can easily be generalised to a basis-extension theorem:

Supplement

3.A.16 Theorem Let  $v_i, i \in I$ , be a generating system of the  $K$ -vector space  $V$ . Suppose that the subfamily  $v_r, r \in I_0$  with  $I_0 \subseteq I$  is linearly independent and  $I \setminus I_0$  is countable. Then there exists a subset  $J \subseteq I \setminus I_0$  such that  $v_s, s \in I_0 \cup J$ , is a basis of  $V$ .

Proof We may assume  $I \setminus I_0 \subseteq \mathbb{N}$ .

Let ~~Choose~~  $J := \{i \in I \setminus I_0 \mid v_i \notin \sum_{r \in I_0} K v_r + \}$

$$\left. \begin{array}{l} \sum_{\substack{j \in I \setminus I_0 \\ j < i}} K v_j \end{array} \right\}$$

3.A.17 Remark (Bases for arbitrary vector spaces -- Well-ordering theorem). Zorn's lemma)

If  $x_0, x_1, x_2, \dots$  is a sequence of vectors in the  $K$ -vector space  $V$  which generate  $V$ , then by the method of proof of 3.A.15, we obtain a basis of  $V$ .

For this one remove all "unnecessary"  $x_i$  from this sequence, namely, all the  $x_i$  who already belong to the subspace generated by the earlier vectors  $x_0, \dots, x_{i-1}$ .

As in 3.A.15 one can <sup>always construct</sup> (obtain) a basis from an arbitrary generating system  $v_i, i \in I$ , of  $V$  if the index set  $I$  is so-called Well-ordered, i.e. a total

order with the following property:

Every non-empty subset of  $I$  has a smallest element. It is well-known

Well-ordering Theorem which says that every set has such a well-ordering.

Cantor considered even more natural assertion. A first proof was given by

by Zermelo with the help of the following well-known Axiom of choice: If  $A_j, j \in J$ , is a non-empty (i.e.  $J \neq \emptyset$ ) family of non-empty subsets, then the cross-product  $\prod_{j \in J} A_j$  is also non-empty.

(equivalently, there is a map (choice map)  $f: J \rightarrow \bigcup_{j \in J} A_j$  such that  $f(j) \in A_j$  for all  $j \in J$ )

Existence of choice function:

Therefore from the well-ordering theorem we immediately have:

3.A.18 Theorem (Hamel) Every vector space has a basis.

Hamel proved only the existence of bases for the  $\mathbb{Q}$ -vector space  $\mathbb{R}$ ; such bases are called Hamel's bases

Hamel basis is not known explicitly.

More generally, 3.A.16 also holds without the hypothesis that  $I \cup J_0$  is countable. We shall cite this as the general basis-supplement theorem if we state it without countability, or the hypothesis

The proof of 3.A.18 and the just mentioned generalisation of theorem 3.A.16 can be proved by using a well-order on the index set  $I$  to the generating system  $v_i, i \in I$ ,

A natural proof is obtained by using <sup>Well-known</sup> Zorn's Lemma

3.A.19 Zorn's lemma Every inductively ordered set  $M$  has (at least) a maximal element.

An ordered set  $(M, \leq)$  is called inductively ordered if every chain in  $M$ , (totally ordered subset of  $M$ )  $N \subseteq M$  has an upper bound in  $M$ , i.e. an element  $x_S \in M$  with  $x \leq x_S$  for all  $x \in N$ . In particular,  $M$  is then non-empty since  $M$  contains an upper bound for the empty chain  $N := \emptyset \subseteq M$ .

The proof of Zorn's lemma essentially uses the Axiom of choice. For this proof we refer to SS, vol. 1, Anhang 1.A

We now prove once again 3.A.18 as an application of Zorn's lemma

For this let  $v_i, i \in I$ , be a generating system of the  $K$ -vector space  $V$  and



let  $\mathcal{K}$  be the set of those subsets  $J \subseteq I$  for which the <sup>sub-</sup>family  $v_j, j \in J$  is linearly independent, i.e.

$$\mathcal{K} := \left\{ J \subseteq I \mid v_j, j \in J \text{ is linearly independent} \right\}$$

Then  $\mathcal{K}$  is inductively ordered with resp. to the natural inclusion. For, if

$\mathcal{L} \subseteq \mathcal{K}$  is a totally ordered subset of  $\mathcal{K}$ , then  $J_{\mathcal{L}} := \bigcup_{J \in \mathcal{L}} J$  belongs to

$\mathcal{K}$  (for  $\mathcal{L} = \emptyset$ ,  $J_{\mathcal{L}} = \emptyset \in \mathcal{K}$ ), since if  $j_1, \dots, j_n$  are finitely many distinct elements in  $J_{\mathcal{L}}$ ,  $j_\nu \in J_\nu$  for  $\nu = 1, \dots, n$ , then  $j_1, \dots, j_n \in J_{j_0}$ , where  $J_{j_0}$  is a biggest element of  $\{J_1, \dots, J_n\}$ , which exists, since  $\mathcal{L}$  is totally ordered. In particular,  $v_{j_1}, \dots, v_{j_n}$  are linearly independent. Therefore we can apply Zorn's lemma and hence there exists a

maximal element  $J_0 \in \mathcal{J}$ . Therefore  
 clearly  $\{v_j, j \in J_0\}$  is a maximal  
 linearly independent system in  $V$  <sup>contained in  $\{v_j, j \in I\}$</sup>   
 and hence by 3.A.14 is a basis of  $V$   
 (Exercise!). Exactly similar way  
 one can prove the basis supplement  
 theorem 3.A.16 without hypothesis  
 that  $I \setminus I_0$  is countable.

(1) It follows from 3.A.15 that:

Exercise 7 of 3A: If a vector space  $V$  has a finite (resp. countable) generating system, then every generating system of  $V$  contains a finite (resp. countable) generating system.

Proof (a) Suppose  $y_1, \dots, y_m \in V$  is a finite generating system for  $V$  and  $x_i, i \in I$ , be an arbitrary generating system of  $V$ . Then  $V = Ky_1 + \dots + Ky_m =$

$\sum_{i \in I} Kx_i$  and for every  $j = 1, \dots, m$ ,

$$y_j = \sum_{i \in E(j)} a_{ij} x_i, \text{ where } E(j) \text{ is a}$$

finite subset of  $I$  and  $a_{ij} \in K$  for

all  $i \in E(j)$ . Then  $E := \bigcup_{j=1}^m E(j)$

is a finite subset of  $I$  and the

subspace generated by  $x_i, i \in E$  contains all  $y_1, \dots, y_m$  and hence is equal to  $V$ .

(b) As an extension of (a) above we note the following important theorem:

Let  $Y \subseteq V$  be an infinite generating system for  $V$ . Then an arbitrary system  $x_i, i \in I$ , of  $V$  contains a generating system  $x_j, j \in J$  with  $|J| \leq |Y|$ .

Proof for every  $y \in Y$ , there exists a finite subset  $E(y) \subseteq I$  with  $y \in \sum_{i \in E(y)} \lambda x_i$ . Then  $x_j, j \in J := \bigcup_{y \in Y} E(y)$

is a generating system for  $V$ , and  $|J| \leq |Y|$  (this need  $Y$  is infinite),  
 But, if  $Y$  is countable, then  $J$  is a countable union of countable (finite) sets and hence  $J$  is countable and in particular,  $|J| \leq |Y|$ . This proves the assertion that if there is (infinite) countable generating system for  $V$ , then every generating system of  $V$  contains a countable generating system.

(Exercise 8)

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(2) Let  $V$  be a  $K$ -vector space which has a countable infinite basis. Then every basis of  $V$  is also countable infinite.

Proof Let  $v_n, n \in \mathbb{N}$  be a countable infinite basis of  $V$  and let  $x_i, i \in I$  be an arbitrary basis of  $V$ . Then the generating system  $x_i, i \in I$ , must contain a countable generating system for  $V$ . But since  $x_i, i \in I$ , is linearly independent, every proper subfamily  $x_j, j \in J \subsetneq I$  cannot be generating system for  $V$ . This proves that  $x_i, i \in I$ , itself must be countable, i.e.  $I$  is countable.

(Exercise 9)

(3) Let  $V$  be a vector space over  $K$ . Suppose that there is an uncountable linearly independent system  $x_i, i \in I_0$ , in  $V$ . Then every system of generators of  $V$  is also uncountable.

Proof Suppose that there

is a countable system  $v_n, n \in \mathbb{N}$  of generators for  $V$ . Let  $I := I_0 \cup \mathbb{N}$  and let  $v_i := x_i$  for  $i \in I_0$ . Then  $v_i, i \in I$  is also a generating system for  $V$ . Now, since  $x_i, i \in I_0$ , is linearly independent and  $I \setminus I_0 = \mathbb{N}$  is countable, by 3.A.16 there exists a subset  $J \subseteq I \setminus I_0 = \mathbb{N}$  such that  $v_i, i \in I_0 \cup J$  is a basis of  $V$ .

This proves that the family  $x_i, i \in I_0$ , is extended to a basis of  $V$  and this basis of  $V$  is uncountable. But by 3.A.15  $V$  also has a countable basis, since there is a countable generating system  $v_n, n \in \mathbb{N}$  for  $V$ . This is a contradiction to (2) above.

(4) (Exercise 10) Let  $K$  be a field which has only countably many elements

Then every  $K$ -vector space  $V$  which has a countable system of generators has countably many elements, i.e.  $V$  is countable set.

Proof By 3.A.15  $V$  has a countable basis  $v_m, m \in \mathbb{N}$ . Then the map  $K^{(\mathbb{N})} \rightarrow V$  is bijective. Therefore it is enough to show that  $K^{(\mathbb{N})}$  is countable, if  $K$  is countable.

For, this first note that  $M_f(K) \cong M_f(\mathbb{N})$  is countable and the map

$K^{(\mathbb{N})} \rightarrow M_f(K), (a_m)_{m \in \mathbb{N}} \mapsto \{a_m\}$

is injective. Therefore  $K^{(\mathbb{N})}$  is a subset of a countable set  $M_f(K)$  and hence countable.

(5) (Exercise 11) Let  $v_i, i \in I$ , be a generating system of the  $K$ -vector space  $V$ . Show that every maximal linearly independent subsystem of  $v_i, i \in I$ , is a basis of  $V$ .

Proof Let  $J \subseteq I$  be such that  $v_j, j \in J$ , is maximal linearly independent subsystem of  $v_i, i \in I$ . Then for every  $i \in I \setminus J$ , the family  $v_j, j \in J, v_i$ , is not linearly independent, i.e. there is a linear combination 
$$\sum_{j \in J} a_j v_j + b v_i = 0$$

$a_j = 0$  for almost all  $j \in J$ ,  $a_j, b, v_i \in K$ .  
 and not all  $a_j, b$  are zero.  
 Then  $b \neq 0$ , since  $v_j, j \in J$ , is linearly independent and hence  $b$

$$v_i = -b^{-1} \left( \sum_{j \in J} a_j v_j \right) \in \sum_{j \in J} K v_j$$

This proves that  $v_i \in \sum_{j \in I} K v_j$  for all  $i \in I$ .



and hence  $v_j, j \in J$  is a linearly independent generating system for  $V$ , i.e. it is a basis of  $V$ .

(6) Remark Using Zorn's lemma and the assertion (5) one can prove the existence of basis in arbitrary  $K$ -vector spaces. This proves the basis-supplement theorem for arbitrary vector spaces:

Let  $v_i, i \in I$ , be a generating system for  $V$  and let

$$\mathcal{F} := \{ J \mid J \subseteq I, v_j, j \in J \text{ linearly independent} \}$$

Note that  $\mathcal{F} \neq \emptyset$ , since  $\{i\} \in \mathcal{F}$  for  $i \in I$  with  $v_i \neq 0$ .  $\mathcal{F}$  is ordered by the natural inclusion. To show that  $\mathcal{F}$  is inductively ordered, let  $\mathcal{C} \subseteq \mathcal{F}$  be a chain (totally ordered) subset. Then  $\tilde{J} := \bigcup_{J \in \mathcal{C}} J$  is an upper

bound for  $\mathcal{C}$  and  $\tilde{\mathcal{J}} \in \mathcal{F}$ . For, if  $v_j, j \in \tilde{\mathcal{J}}$  is not linearly independent, then there are finitely many  $j_1, \dots, j_m \in \tilde{\mathcal{J}}$  such that  $(v_{j_1}, \dots, v_{j_m})$  are not linearly independent. But  $j_1, \dots, j_m \in \mathcal{J}$  for some  $\mathcal{J} \in \mathcal{C}$ , since  $\mathcal{C}$  is a chain.

Then  $\{v_{j_1}, \dots, v_{j_m}\} \subseteq \{v_j \mid j \in \mathcal{J}\}$  which is linearly independent, a contradiction.

This proves that  $\tilde{\mathcal{J}} \in \mathcal{F}$  and  $\tilde{\mathcal{J}}$  is clearly an upper bound for  $\mathcal{C}$  in  $\mathcal{F}$ .

Now we can apply Zorn's lemma to  $(\mathcal{F}, \subseteq)$  to conclude that  $\mathcal{F}$  has maximal element  $\mathcal{J} \in \mathcal{I}$ . This means that the <sup>sub-</sup>family  $v_j, j \in \mathcal{J}$ , is maximal linearly independent family of  $v_a, a \in \mathcal{I}$ . By (5)  $v_j, j \in \mathcal{J}$ , is a basis of  $V$ .

(7) Let  $V$  be a  $K$ -vector space and let  $v_i, i \in I$ , be a generating system for  $V$ . Suppose that the subsystem  $v_j, j \in J; J \subseteq I$ , is linearly ~~base~~ independent.

Then there exists a subset  $L$  with  $J \subseteq L \subseteq I$  such that  $v_\ell, \ell \in L$ , is a basis of  $V$ .

Proof Similar to that of (6).

Apply Zorn's lemma to the ordered set with resp. to natural inclusion

$$\mathcal{F} := \{ L \mid J \subseteq L \subseteq I, v_\ell, \ell \in L, \text{ is linearly independent} \}.$$

Then  $J \in \mathcal{F}$  and check that  $\mathcal{F}$  is inductively ordered and hence by Zorn's lemma  $\mathcal{F}$  has a maximal element  $L$ . By (5)  $v_\ell, \ell \in L$  is a basis of  $V$ .

Therefore: Every linearly independent system in  $V$  can be extended to a basis of  $V$ .