8.B Rank of Matrices

Let $f: V \longrightarrow W$ be a K-linear map of finite dimensional K-vector spaces with the matrix $\mathfrak{A} = \mathfrak{M}_{\underline{w}}^{\underline{v}}(f)$ with resp. to the given bases $\underline{v} = (v_j)_{j\in J}$ of V and $\underline{w} = (w_i)_{i\in I}$ of W. Then the rank of f $\mathrm{Rank}\, f = \mathrm{Dim}_K \mathrm{Im} f$ is the dimension of the K-subspace $\sum_{j\in J} K f(v_j)$ of W generated by $f(v_j)$, $j\in J$. The inverse image of this subspace under the isomorphism $K^I \longrightarrow W$ $e_i \longmapsto w_i$, $i\in I$, defined by the basis $\underline{w} = (w_i)_{i\in I}$, is a K-subspace of K^I generated by the columns of the matrix $\mathfrak{A}$. Its dimension is nothing but the $\mathrm{Rank}\, f$. We therefore define:

8.B.1 Definition Let $\mathfrak{A} \in M_{I,J}(K)$ be a matrix over the field K with I, J finite. The dimension of the K-subspace generated by the columns of $\mathfrak{A}$ is called the

(column) rank of $\mathfrak{A}$ and is denoted by

$$\operatorname{Rank}_K \mathfrak{A} \quad \text{or} \quad \operatorname{Rank} \mathfrak{A}$$

Therefore the rank of $\mathfrak{A}$ is the maximal number of linearly independent column vectors of $\mathfrak{A}$. In particular, $\operatorname{Rank} \mathfrak{A} \leq |J|$. As a K-subspace of K^I the columns of $\mathfrak{A}$ generate the column-subspace (of $\mathfrak{A}$) of dimension $\leq \operatorname{Dim}_K K^I = |I|$.

Altogether, for a matrix $\mathfrak{A} \in M_{I,J}(K)$,

$$\operatorname{Rank} \mathfrak{A} \leq \operatorname{Min}(|I|, |J|).$$

If equality holds, then we say that $\mathfrak{A}$ has <u>maximal rank</u>.

By the remark before the definition 8.B.1, we have:

<u>8.B.2 Theorem</u> The rank of a K-homomorphism $f : V \longrightarrow W$ of finite dimensional K-vector spaces is equal to the rank of the matrix of f w.r.to any bases $\underline{v}$ of V and $\underline{w}$ of W, i.e.,

$\operatorname{Rank} f = \operatorname{Rank} \mathfrak{M}_{w}^{v}(f)$.

The rank of the matrix $\mathfrak{A} \in M_{I,J}(K)$ defined above is used the columns of $\mathfrak{A}$ and hence it is also called the column-rank of $\mathfrak{A}$. Analogously, one can define the row-rank of $\mathfrak{A}$ as the dimension of the K-subspace of K^J generated by the rows of $\mathfrak{A}$. Clearly, it is the column-rank of the transpose ${}^t\mathfrak{A}$ of $\mathfrak{A}$.

We have the following basic theorem:

8.B.3 Theorem Let I, J be finite sets and $\mathfrak{A} \in M_{I,J}(K)$ be a $I \times J$-matrix. Then:

$$\operatorname{Rank} \mathfrak{A} = \operatorname{Rank} {}^t\mathfrak{A}.$$

Proof Let $\mathfrak{A} = (a_{ij})_{i \in I,\, j \in J}$.

We give two proofs.

First Proof: Let $R_i := (a_{ij})_{j \in J} \in K^J$, $i \in I$, be the rows and $C_j := (a_{ij})_{i \in I} \in K^I$, $j \in J$, be the columns of $\mathfrak{A}$. Further, let

$v_1, \dots, v_r \in K^I$ with $v_k = (b_{ik})_{i \in I}$, $k = 1, \dots, r$, be a basis of the column-space $\sum_{j \in J} K c_j$ of K^I generated by the columns c_j, $j \in J$, of $\mathfrak{A}$. Then there exist $c_{kj} \in K$ with

$$(a_{ij})_{i \in I} = c_j = c_{1j} v_1 + \dots + c_{rj} v_r$$
$$= (c_{1j} b_{i1} + \dots + c_{rj} b_{ir})_{i \in I}.$$

For the vectors $w_k := (c_{kj})_{j \in J} \in K^J$, $k = 1, \dots, r$, we have:

$$R_i = (a_{ij})_{j \in J} = (c_{1j} b_{i1} + \dots + c_{rj} b_{ir})_{j \in J}$$
$$= b_{i1} w_1 + \dots + b_{ir} w_r \text{ for all}$$

$i \in I$. Therefore the row-space $\sum_{i \in I} K R_i \subseteq K w_1 + \dots + K w_r$ and the

$$\text{row-rank of } \mathfrak{A} = \operatorname{Dim}_K \sum_{i \in I} K R_i \leq r =$$

the column-rank of $\mathfrak{A}$. By replacing $\mathfrak{A}$ by its transpose ${}^t\mathfrak{A}$, the reverse inequality follows. Altogether, we have the equality:

$$\underset{\text{Rank } {}^t\mathfrak{A}}{\overset{\shortparallel}{\text{row-rank of } \mathfrak{A}}} = \underset{\text{Rank } \mathfrak{A}}{\overset{\shortparallel}{\text{column-rank of } \mathfrak{A}}}.$$

Second-Proof: Let $f : V \longrightarrow W$ be the K-linear map defined by the matrix $\mathfrak{A} = (a_{ij}) \in M_{I,J}(K)$, i.e.

$$f(v_j) := \sum_{i \in I} a_{ij} w_i \,,\; j \in J, \text{ where } \underline{v} =$$

$(v_j)_{j \in J}$ is a basis of V and $\underline{w} = (w_i)_{i \in I}$ is a basis of W. Then $\mathfrak{A} = \mathfrak{M}^{\underline{v}}_{\underline{w}}(f)$

and $\operatorname{Rank} f = \operatorname{Rank} \mathfrak{A}$.

Let $f^* : W^* \longrightarrow V^*$ be the dual map of f, $\underline{v}^* = (v_j^*)_{j \in J}$ and $\underline{w}^* = (w_i^*)_{i \in I}$ be the dual bases of V and W w.r.to bases $\underline{v}$ and $\underline{w}$, respectively. Then

$$\mathfrak{M}^{\underline{w}^*}_{\underline{v}^*}(f^*) = {}^t\mathfrak{M}^{\underline{v}}_{\underline{w}}(f) = {}^t\mathfrak{A}$$ by Theorem 8.A.20 and hence by Theorem 5.G.19

$$\operatorname{Rank} \mathfrak{A} = \operatorname{Rank} f = \operatorname{Rank} f^* = \operatorname{Rank} {}^t\mathfrak{A}.$$

8.B.4 Example (Calculation of rank)

The rank of matrices can be, relatively easier to determine by using so-called

<u>elementary operations.</u>

Let $\mathfrak{A} \in M_{I,J}(K)$ be a matrix with I, J finite sets. From $\mathfrak{A}$ we obtain a matrix $\mathfrak{A}' \in M_{I,J}(K)$ such that the row-space of $\mathfrak{A}$ is equal to the row-space of $\mathfrak{A}'$. In particular, $\operatorname{Rank} \mathfrak{A} = \operatorname{Rank} \mathfrak{A}'$. Therefore, we consider the following elementary (row-) operations on $\mathfrak{A}$ to obtain a matrix $\mathfrak{A}'$ which has the same rank as $\mathfrak{A}$:

(1) Addition of the a-times a row to another row, $a \in K$.

$$\left(R_i \longmapsto R_i + aR_j,\ \begin{matrix} i,j \in J, \\ i \neq j, \end{matrix}\ a \in K\right)$$

(2) Multiplying a row with a scalar $a \in K, a \neq 0$.

$$\left(R_i \longmapsto aR_i,\ i \in J,\ a \in K,\ a \neq 0\right)$$

(3) Interchanging two rows.

$$\left(R_i \longleftrightarrow R_j,\ i, j \in J\right)$$

Analogous, elementary operations on columns are also defined. These operations don't alter the column-rank and hence the row-rank of a matrix.

In the description of the Gauss-elimination in Subsection 2A we have already demonstrated that using <u>only</u> elementary row-operations a $m \times n$ matrix can be transformed to a matrix whose rank is seen immediately. If one also allows permutations of columns, then one can reach to a matrix of the form:

$$\begin{pmatrix} c_{11} & * & \cdots & * & * & \cdots & * \\ 0 & c_{22} & \cdots & * & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_{rr} & * & \cdots & * \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}$$

where $c_{ii} \neq 0$ for $i = 1, \ldots, r$. Then naturally, rank of this matrix and hence the rank of $\mathfrak{A}$ is r.

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Since c_{ii} are not necessarily 1, one can use elementary row operations of type (2) to make them all 1.

8.B.5 Example On the matrix

$$\mathfrak{A} = \begin{pmatrix} 0 & 1 & 3 & -2 \\ 2 & 1 & -4 & 3 \\ 2 & 3 & 2 & -1 \end{pmatrix}$$

the following sequence of elementary operations transforms $\mathfrak{A}$ to the matrix $\mathfrak{A}' = \begin{pmatrix} 2 & 1 & -4 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$:

$$\begin{pmatrix} 0 & 1 & 3 & -2 \\ 2 & 1 & -4 & 3 \\ 2 & 3 & 2 & -1 \end{pmatrix} \xmapsto{R_1 \leftrightarrow R_2} \begin{pmatrix} 2 & 1 & -4 & 3 \\ 0 & 1 & 3 & -2 \\ 2 & 3 & 2 & -1 \end{pmatrix}$$

$$\xmapsto{R_3 \mapsto R_3 - R_1} \begin{pmatrix} 2 & 1 & -4 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 2 & 6 & -4 \end{pmatrix}$$

$$\xmapsto{R_3 \mapsto R_3 - 2R_2} \begin{pmatrix} 2 & 1 & -4 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \mathfrak{A}'$$

Therefore $\operatorname{Rank} \mathfrak{A} = \operatorname{Rank} \mathfrak{A}' = 2$.

Sometimes it is beneficial to combine row and column operations.

In general, it is difficult to compute the rank of a real or complex matrix using computer programs, since as a consequence of unavoidable rounding-errors, computer is forced to recognize small-values as 0 which are $\neq 0$.

Using elementary matrix operations, one can easily compute the inverse of an invertible matrices. For this, we first prove:

8.B.6 Lemma Let I be a finite set. For an $I \times I$ matrix $\mathfrak{A} \in M_I(K)$, the following statements are equivalent:

(i) $\mathfrak{A}$ is invertible, i.e. $\mathfrak{A} \in GL_I(K)$.

(ii) The columns of $\mathfrak{A}$ are linearly independent, i.e. a basis of K^I.

(ii') The rows of $\mathfrak{A}$ are linearly independent, i.e. a basis of K^I.

(iii) $\mathfrak{A}$ has maximal rank, i.e. $\operatorname{Rank} \mathfrak{A} = |I|$.

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<u>Proof</u> Let $f_{\mathfrak{A}} : K^I \longrightarrow K^I$ be the K-linear map such that the matrix $\mathfrak{M}_{\underline{e}}^{\underline{e}}(f_{\mathfrak{A}}) = \mathfrak{A}$, where $\underline{e} = (e_i)_{i \in I} \in K^I$ is the standard basis of K^I (f is defined by $f(e_j) = \sum_{i \in I} a_{ij} e_i$, $j \in I$, where $\mathfrak{A} = (a_{ij})$.). Then by 8.A.10 $\mathfrak{A}$ is invertible if and only if $f_{\mathfrak{A}}$ is bijective. Since $\operatorname{Im} f_{\mathfrak{A}}$ is generated by the columns of $\mathfrak{A}$, $\operatorname{Im} f_{\mathfrak{A}} = K^I$ if and only if the columns of $\mathfrak{A}$ forms a basis of K^I. This proves the equivalence (i) $\Longleftrightarrow$ (ii). Remaining implications follow from 8.B.3.

Now let $\mathfrak{A} = (a_{ij}) \in M_n(K)$ be a square $n \times n$ matrix. The $n \times n$ matrix $\mathfrak{X} = (x_{jk}) \in M_n(K)$ is the inverse of $\mathfrak{A}$ if and only if $\mathfrak{A} \cdot \mathfrak{X} = \mathfrak{E}_n$, (see 8.A.11), i.e. for every $k = 1, \ldots, n$, the elements x_{jk} of the k-th column of $\mathfrak{X}$ is a solution of the system of only; in this case, the matrix-equation $\mathfrak{X}\mathfrak{A} = \mathfrak{E}_n$ is solved for the matrix $\mathfrak{X}$. In any case, it is not allowed to use both row and column operations for the computation of the inverse of a matrix, see also next Subsection.

8.B.7 Example We want to compute the inverse of the matrix

$$\mathfrak{A} = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix} \in M_3(\mathbb{Q}).$$

For this note the following elementary row-operations

$$\left(\begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix} \middle| \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) \xmapsto[R_3 \mapsto R_3 - 4R_1]{R_2 \mapsto R_2 - 2R_1} \left(\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix} \middle| \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix}\right)$$

$$\xmapsto{R_2 \mapsto -R_2} \left(\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \middle| \begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ -4 & 0 & 1 \end{pmatrix}\right) \xmapsto{R_3 \mapsto R_3 - R_2} \left(\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \middle| \begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ -6 & 1 & 1 \end{pmatrix}\right)$$

$$\xmapsto[R_2 \mapsto R_2 + R_3]{R_1 \mapsto R_1 + 2R_3} \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \middle| \begin{pmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ -6 & 1 & 1 \end{pmatrix}\right) \xmapsto{R_3 \mapsto -R_3} \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \begin{pmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{pmatrix}\right)$$

Therefore, the matrix $\mathfrak{A}$ is invertible and it's inverse is:

$$\begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix}^{-1} = \mathfrak{A}^{-1} = \begin{pmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{pmatrix}.$$

<u>8.B.8 Example</u> (<u>System of linear equations</u>) The system of linear equations

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\ \cdots \qquad \cdots \qquad \cdots \quad &\quad \cdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

with the coefficient matrix $\mathfrak{A} = (a_{ij}) \in M_{m,n}(K)$ and the column vectors

$$\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \text{ and } \underline{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

can be represented by the matrix-equation $\mathfrak{A} \cdot \underline{x} = \underline{b}$. The rank of this system of equations, in the sense of Example 5.E.5, is equal to the rank of the matrix $\mathfrak{A}$ and the equations:

$$a_{i1}x_{1k} + \cdots + a_{in}x_{nk} = \delta_{ik},$$

$i = 1, \dots, n$.

The coefficient matrix of this system of equations is the given matrix $\mathfrak{A} = (a_{ij})$ and the vectors on the right-hand side are the standard basis vectors $e_1, \dots, e_n \in K^n$. To find the solution space, we apply Gauss-elimination to transform the coefficient matrix $\mathfrak{A}$ by using elementary row operations to the matrix $\mathfrak{A}'$ of the form: $\mathfrak{A}' = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ 0 & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_{nn} \end{pmatrix}$

At the same time, we apply the same elementary row-operations on the unit-matrix E_n on the right-hand side, and get the matrix equation:

$$\mathfrak{A}'\mathfrak{X} = \mathfrak{B}'$$

Since Rank $\mathfrak{A}$ = Rank $\mathfrak{A}'$, $\mathfrak{A}$ is invertible if and only if all entries $c_{11}, \dots, c_{nn}$ on the main diagonal of $\mathfrak{A}'$ are $\neq 0$. Further, using elementary row-operations on ~~the~~ matrix $\mathfrak{A}'$ (and the matrix $\mathfrak{B}'$) transform the matrix $\mathfrak{A}'$ to the unit-matrix $\mathfrak{E}_n$, and get the matrix-equation:

$$\mathfrak{X} = \mathfrak{E}_n \mathfrak{X} = \mathfrak{B}.$$

The columns of the matrix $\mathfrak{B}$ are the solutions of the original system of linear equations. The matrix $\mathfrak{B}$ obtained from $\mathfrak{E}_n$ by elementary row-operations, is the inverse $\mathfrak{A}^{-1}$ of $\mathfrak{A}$.

<u>Remark</u> In the above method of finding the inverse of a matrix, as in the Gauss-elimination Column-pivot search should applied. Moreover, <u>note that</u> only elementary <u>row</u> operations are allowed.

This method also works, analogously, if one uses elementary <u>Column</u> operations assertion 5.E.6 can be reformulated as:

The solution space of $\mathfrak{A}\underline{x} = \underline{b}$ is an affine subspace of K^n of the dimension $n - \operatorname{Rank}\mathfrak{A}$ if it is non-empty.

The system $\mathfrak{A}\underline{x} = \underline{b}$ has a solution if and only if $\underline{b}$ belongs to the subspace (of K^m) generated by the columns of $\mathfrak{A}$.

Therefore: The system of equations $\mathfrak{A}\underline{x} = \underline{b}$ has a solution $\underline{x} \in K^n$ if and only if $\operatorname{Rank}\mathfrak{A} = \operatorname{Rank}(\mathfrak{A}, \underline{b})$, where the (augmented) $m\times(n+1)$-matrix $(\mathfrak{A}, \underline{b})$ is obtained by adding the vector $\underline{b}$ as the $(n+1)$-th column to the matrix $\mathfrak{A}$.

If $m = n$, then the system of equations $\mathfrak{A}\underline{x} = \underline{b}$, for a fixed $\mathfrak{A} \in M_n(K)$ has a solution for every choice of $\underline{b} \in K^n$ if and only if $\mathfrak{A}$ is invertible. The unique solution for the given $\underline{b} \in K^n$ is $\underline{x} = \mathfrak{A}^{-1}\underline{b}$.