9.D Computational Rules for Determinants

The determinant of a $n \times n$ matrix $(a_{ij}) \in M_n(K)$ is also denoted by

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

Therefore by definition:

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = \sum_{\sigma \in \mathfrak{S}_n} (\operatorname{Sign} \sigma)\, a_{1,\sigma 1} \cdots a_{n,\sigma n}$$

$$= \sum_{\sigma \in \mathfrak{S}_n} (\operatorname{Sign} \sigma)\, a_{\sigma 1,1} \cdots a_{\sigma n,n}$$

For $n = 1, 2, 3$, we note explicitly:

$$|a_{11}| = a_{11}, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

The last formula can be remembered by the following scheme which is also known as Sarruss-Rule:

$$
\begin{array}{ccccc}
+ & + & + & - & - \quad - \\
a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\
a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\
a_{31} & a_{32} & a_{33} & a_{31} & a_{32}
\end{array}
$$

We repeat 9.C.7:

9.D.1 Theorem Let $\mathfrak{A} \in M_I(K)$.

Then: $\quad \mathrm{Det}\,\mathfrak{A} = \mathrm{Det}\,{}^t\mathfrak{A}$.

The determinant function

$$\mathrm{Det}: M_I(K) \longrightarrow K$$

is an alternating multilinear function of the columns of the matrix.

Because of 9.D.1 it is also of the rows of the matrix. In particular, one can completely determine the behaviour of the determinant of a matrix by applying elementary operations.

$\mathfrak{A}' = \mathfrak{A}(R_i \mapsto R_i + aR_j \mid i, j \in I, i \neq j, a \in K)$

For a square matrix $\mathfrak{A} \in M_I(K)$:

$= \mathfrak{A}(C_i \mapsto C_i + aC_j \mid i, j \in I, i \neq j, a \in K)$

(1) Let $\mathfrak{A}'$ be the matrix obtained from the matrix $\mathfrak{A}$ by adding a-th multiple of one column (resp. row) to another column (resp. row). Then $\operatorname{Det} \mathfrak{A}' = \operatorname{Det} \mathfrak{A}$.

$= \mathfrak{A}(R_i \mapsto aR_i \mid i \in I, a \in K^\times)$

(2) Let $\mathfrak{A}'$ be the matrix obtained from the matrix $\mathfrak{A}$ by multiplying a column (resp. row) by a non-zero scalar $a \in K$. Then $\operatorname{Det} \mathfrak{A}' = \operatorname{Det} \mathfrak{A}$.

$= (C_{\sigma(i)} \mid \sigma \in \mathfrak{S}(I))$ (resp. ...)

(3) Let $\mathfrak{A}_\sigma$ be the matrix obtained from the matrix $\mathfrak{A}$ by permuting the columns (resp. rows) of $\mathfrak{A}$ according to the permutation $\sigma \in \mathfrak{S}(I)$. Then

$$\operatorname{Det} \mathfrak{A}_\sigma = (\operatorname{Sign} \sigma) \operatorname{Det} \mathfrak{A}$$

__Proof__ (1) Let $x_j, j \in I$ be columns of $\mathfrak{A}$ and $\mathfrak{A}' = (x_1, \cdots x_j, \cdots x_i + ax_j, \cdots, x_n)$

Then $\operatorname{Det} \mathfrak{A}' = \Delta_{\underline{e}}(x_1, \cdots, x_j, \cdots, x_i + ax_j, \cdots x_n)$

$= \Delta_{\underline{e}}(x_1, \cdots, x_j, \cdots, x_i, \cdots, x_n) + a \underbrace{\Delta_{\underline{e}}(\cdots x_j, \cdots, x_j \cdots)}_{=0}$

$= \operatorname{Det} \mathfrak{A}$.

All other rules follow similarly by using properties of alternating multilinear maps.

A $n \times n$ arbitrary matrix $\mathfrak{A} \in M_n(K)$ can be transformed to an upper (resp. lower) triangular matrix by using elementary operations see for example 8.B.4. Since each of elementary operations change in the determinant exactly described under for the explicit calculation of the determinant it is important to know the determinant of triangular matrices. This is very simple to calculate:

9.D.2 Theorem Let $\mathfrak{A} = (a_{ij}) \in M_n(K)$ be an upper triangular matrix, i.e. $a_{ij} = 0$ for $n \geq i > j \geq 1$. Then:

$$\operatorname{Det} \mathfrak{A} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix}$$

$$= a_{11} a_{22} \cdots a_{nn}$$

~~Proof~~ It is $\operatorname{Det} \mathfrak{A} = \sum_{\sigma \in \mathfrak{S}_n} (\operatorname{Sign} \sigma) a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}$

For every $\sigma \in \mathfrak{S}_n$, $\sigma \neq \mathrm{id}$, there exists $i_0 \in \{1, \dots, n\}$ with $i_0 > \sigma(i_0)$ and so (?) the corresponding summand is zero.

It remains $\operatorname{Det} \mathfrak{A} = a_{11} \cdots a_{nn}$.

An analogous to 9.D.2 assertion, naturally also holds for lower triangular matrix.

9.D.3 Example

$$\begin{vmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{vmatrix} \overset{C_3 \mapsto C_3 - 2C_1}{=} \begin{vmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{vmatrix}$$

$R_3 \mapsto R_3 + R_2$

$$= \begin{vmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{vmatrix} = 1 \cdot (-1) \cdot (-1) = 1.$$

9D16

A generalisation of 9.D.2 is the following theorem:

<u>9.D.4 Theorem</u> Let $\mathfrak{a} \in M_r(K)$, $\mathfrak{b} \in M_{r,s}(K)$ and $\mathfrak{c} \in M_s(K)$ be matrices over the field K. Then the ~~square~~ $(r+s)\times(r+s)$-block matrix

$$\begin{pmatrix} \mathfrak{a} & \mathfrak{b} \\ 0 & \mathfrak{c} \end{pmatrix} \in M_{r+s}(K)$$

has the determinant $\operatorname{Det}\mathfrak{a} \cdot \operatorname{Det}\mathfrak{c}$.

<u>Proof</u> By elementary operations we can transform the matrix $\mathfrak{a}$ resp. $\mathfrak{c}$ into upper triangular matrices:

$$\mathfrak{a}' = \begin{pmatrix} a'_{11} & a'_{12} & \cdots & a'_{1r} \\ 0 & a'_{22} & \cdots & a'_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a'_{rr} \end{pmatrix} \text{ resp. } \mathfrak{c}' := \begin{pmatrix} c'_{11} & c'_{12} & \cdots & c'_{1s} \\ 0 & c'_{22} & \cdots & c'_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c'_{ss} \end{pmatrix}$$

Then $\operatorname{Det}\mathfrak{a} = a \operatorname{Det}\mathfrak{a}' = a\, a'_{11} \cdots a'_{rr}$ and $\operatorname{Det}\mathfrak{c} = c \operatorname{Det}\mathfrak{c}' = c\, c'_{11} \cdots c'_{ss}$, where the factors a (resp. c) are the modifications (changes) in the determinants described by the operations.

Now, we conduct the analogous operations on the matrix:

$\begin{pmatrix} \mathfrak{A} & \mathfrak{B} \\ 0 & \mathfrak{C} \end{pmatrix}$ and obtain an upper triangular matrix $\begin{pmatrix} \mathfrak{A}' & \mathfrak{B}' \\ 0 & \mathfrak{C}' \end{pmatrix}$. It follows

$$\operatorname{Det}\begin{pmatrix} \mathfrak{A} & \mathfrak{B} \\ 0 & \mathfrak{C} \end{pmatrix} = ac \operatorname{Det}\begin{pmatrix} \mathfrak{A}' & \mathfrak{B}' \\ 0 & \mathfrak{C}' \end{pmatrix} =$$

$$ac\, a'_{11} \cdots a'_{rr}\, c'_{11} \cdots c'_{ss} = \operatorname{Det}\mathfrak{A} \cdot \operatorname{Det}\mathfrak{C}.$$

Finally, we prove the fundamental product-formula:

<u>9.D.5</u> <u>Product Formula</u> For square matrices $\mathfrak{A}, \mathfrak{B} \in M_I(K)$

$$\operatorname{Det}(\mathfrak{A}\mathfrak{B}) = (\operatorname{Det}\mathfrak{A})(\operatorname{Det}\mathfrak{B})$$

<u>Proof</u> Let x_j, $j \in I$, be the columns of $\mathfrak{B}$. Then $\mathfrak{A}x_j$, $j \in I$, are the columns of $\mathfrak{A}\mathfrak{B}$. The map

$$f : (K^I)^I \longrightarrow K, \quad \mathfrak{B} \longmapsto \overset{\displaystyle \operatorname{Det}(\mathfrak{A}\mathfrak{B})}{\overset{\shortparallel}{\Delta_{\underline{e}}\left((\mathfrak{A}x_j)_{j \in I}\right)}}$$

9D18

is an alternating multilinear map i.e. a determinant function on K^I. From 9.C.4 it follows that

$$\mathrm{Det}(\mathfrak{a}\mathfrak{b}) = f(\mathfrak{b}) = f\left((x_j)_{j\in I}\right) =$$

$$\Delta_{\underline{e}}\left((x_j)_{j\in I}\right)\cdot f\left((e_j)_{j\in I}\right)$$

$$= \Delta_{\underline{e}}\left((x_j)_{j\in I}\right)\cdot f(\mathfrak{E}_I)$$

$$= (\mathrm{Det}\,\mathfrak{b})\cdot(\mathrm{Det}\,\mathfrak{a}) \qquad \blacksquare$$

If $\mathfrak{a}\in M_I(K)$ is invertible, i.e. if $\mathfrak{a}\in GL_I(K)$, then $\mathfrak{a}\cdot\mathfrak{a}^{-1} = \mathfrak{E}_I$ and hence $1 = \mathrm{Det}\,\mathfrak{E}_I = \mathrm{Det}(\mathfrak{a}\mathfrak{a}^{-1}) = (\mathrm{Det}\,\mathfrak{a})(\mathrm{Det}\,\mathfrak{a}^{-1})$. Once again (as from 9.C.10 and also from 9.D.2) it follows that the determinant of an invertible matrix is non-zero. Moreover,

$$\mathrm{Det}\,\mathfrak{a}^{-1} = (\mathrm{Det}\,\mathfrak{a})^{-1}$$

The determinant map

$$\mathrm{Det}: GL_I(K) \longrightarrow K^\times$$

is a group homomorphism (by 9.D.5),

from the general linear group $GL_I(K)$ into the multiplicative group $(K^\times, \cdot)$ of K, and if $I \neq \phi$, then it is surjective. Its Kernel Ker Det is the subgroup

$$SL_I(K) := \{\mathfrak{A} \in GL_I(K) \mid \mathrm{Det}\,\mathfrak{A} = 1\}$$

of the matrices $\mathfrak{A} \in GL_I(K)$ whose determinant is 1. It is called the special linear group (corresponding to the index set I). In particular, ($I = \mathbb{N}_n^*$) we denote it by

$$SL_n(K)$$

for $n \in \mathbb{N}$, the group of the square $n \times n$-matrices over K whose determinant is 1. By the isomorphism theorem 6.A.11, for $n \geq 1$, we have

$$GL_n(K)/SL_n(K) \cong K^\times$$

9.D.6 Example By 8.C.8 every matrix $\mathfrak{A} \in GL_n(K)$ is a product of diagonal matrix $\mathrm{Diag}(d, 1, \cdots, 1)$ and elementary matrices $B_{ij}(a) = E_n + aE_{ij}$, $i \neq j$, $a \in K$. Since the elementary matrices have determinant 1, by the product theorem 9.D.5 the determinant of $\mathfrak{A}$ is equal to the determinant of the diagonal matrix, i.e. equal to the coefficient d. If $\mathfrak{A} \in SL_n(K)$, then necessarily $d = 1$ and $\mathfrak{A}$ is the product of elementary matrices, in other words:

9.D.7 Theorem Let K be a field. Then the special linear group $SL_n(K)$, $n \in \mathbb{N}$, is generated by the elementary matrices $B_{ij}(a)$, $i \neq j$, $a \in K$.

(important)

Theorem 9.D.7 has the following group theoretic ~~Corollary~~: Consequence:

9.D.8 Theorem Let K be a field and $n \in \mathbb{N}$, $n \geq 2$. Then the special linear group $SL_n(K)$ is the commutator group of the general linear group $GL_n(K)$. In the case $|K|=2$, we assume $n=2$.

Proof Since the residue class group $GL_n(K)/SL_n(K) \cong K^\times$ is commutative, the commutator group $N := [GL_n(K), GL_n(K)]$ of $GL_n(K)$ is contained in $SL_n(K)$.

By 9.D.7 it is enough to prove that the residue class of every elementary matrix $B_{ij}(a)$, $i \neq j$, $a \in K$, $a \neq 0$, in $GL_n(K)/N$ is trivial.

Since

$$
\begin{aligned}
B_{ij}(a) &= \sigma B_{ij}(1) \sigma^{-1} \\
&= \sigma B_{ij}(1) \sigma^{-1} B_{ij}^{-1}(1)\, B_{ij}(1),
\end{aligned}
$$

where $\sigma = \mathrm{Diag}(1, \dots, 1, a, 1, \dots, 1)$ is the diagonal matrix with a at the i-th position on the diagonal, all $B_{ij}(a)$ have the same residue class as that of $B_{ij}(1)$

Now, if $|K|>2$ and if $a \notin \{0,1\}$, then $B_{ij}(1) = B_{ij}(1-a)\,B_{ij}(a)$.

It follows that the residue class of $B_{ij}(1)$ is equal to its square and hence it is trivial.

Now, assume that $n \geq 3$. Then for every pair, i,j, there exists $r \in \{1,\dots,n\} \setminus \{i,j\}$. From the equations

$$B_{ij}(a) = B_{rj}(a) \cdot B_{ir}(-1)\,B_{rj}(a)^{-1}\,B_{ir}(-1)^{-1}$$

(which can be checked easily), it follows that every elementary matrix $B_{ij}(a)$ is itself a commutator of elementary matrices. Therefore, moreover, $SL_n(K)$, for $n \geq 3$, is its own commutator subgroup, i.e. $SL_n(K)$, $n \geq 3$, is a perfect group.

9.D.9 Corollary Let K be a field and $n \in \mathbb{N}$, $n \geq 2$; if $n=2$, then assume that $|K| > 2$. Then ~~for every~~ group homomorphism

$$\varphi : GL_n(K) \longrightarrow H$$

from $GL_n(K)$ in an abelian group H there exists a unique group homomorphism

$$\psi : K^* \longrightarrow H$$

such that $\varphi(\mathfrak{r}) = \psi(\mathrm{Det}\,\mathfrak{r})$ for all $\mathfrak{r} \in GL_n(K)$. It is

$$\psi(a) = \varphi(\mathrm{Diag}(a,1,\dots,1)),\ a \in K^*.$$

Proof By 9.D.8 φ is trivial on $SL_n(K)$ and hence the assertion follows from the Theorem 6.A.10 on induced homomorphisms. ∎

In 9.D.8 and 9.D.9 the exceptional case $K = \mathbb{Z}_2$ and $n=2$, $GL_2(K) = SL_2(K) = GL_2(\mathbb{Z}_2) \cong \mathfrak{S}_3$ permuttion group) and its commutator group $[\mathfrak{S}_3 : \mathfrak{S}_3] = \mathfrak{A}_3$ has index 2.

Let K be a field, $n \in \mathbb{N}^*$ and $\varphi : \mathrm{GL}_n(K) \longrightarrow K^\times$ be a group homomorphism, i.e. a character of $\mathrm{GL}_n(K)$ with values in K. Then it follows from 9.D.9 together with 5.A.17 that:

(1) If φ is a rational function, then there exists $m \in \mathbb{Z}$ with
$$\varphi(\mathfrak{a}) = (\mathrm{Det}\,\mathfrak{a})^m$$
for all $\mathfrak{a} \in \mathrm{GL}_n(K)$. In particular, if φ is a polynomial function, then $m \in \mathbb{N}$.

(2) If $K = \mathbb{R}$ and if φ is continuous, then there exists a real-number $\beta \in \mathbb{R}$ with $\varphi(\mathfrak{a}) = |\mathrm{Det}\,\mathfrak{a}|^\beta$ for all $\mathfrak{a} \in \mathrm{GL}_n(\mathbb{R})$ or with $\varphi(\mathfrak{a}) = |\mathrm{Det}\,\mathfrak{a}|^\beta\,\mathrm{Sign}(\mathrm{Det}\,\mathfrak{a})$ for all $\mathfrak{a} \in \mathrm{GL}_n(\mathbb{R})$.

(3) If $K = \mathbb{C}$ and if φ is continuous,

then there exists a complex-number $\alpha \in \mathbb{C}$ and an integer $m \in \mathbb{Z}$ with $\varphi(\mathfrak{a}) = |\operatorname{Det} \mathfrak{a}|^{\alpha} (\operatorname{Det} \mathfrak{a})^{m}$ for all $\mathfrak{a} \in \mathrm{GL}_n(\mathbb{C})$

In the cases (2) and (3), consider $\mathrm{GL}_n(\mathbb{K}) \subseteq \mathbb{K}^{n^2}$ with the canonical topology induced from the usual topology of $\mathbb{K}^{n^2}$.

Further, in the case (1) the exception mentioned in 9.D.9 is not significant, since for the field $\mathbb{Z}_2$ with 2 elements, the group $\mathbb{Z}_2^{\times}$ is trivial.

9.D.10 Example (Special Projective Linear Group) The center of the group $\mathrm{GL}_n(K)$, K field, $n \geq 1$, is

$$Z(\mathrm{GL}_n(K)) = \{ a E_n \mid a \in K^{\times} \} = K^{\times} E_n \cong K^{\times}.$$

The residue class group

$$\mathrm{GL}_n(K) / Z(\mathrm{GL}_n(K))$$

is isomorphic to the group $\mathrm{PGL}_n(K)$

of projective collineations of the $(n-1)$-dimensional projective space $\mathbb{P}^{n-1}(K)$, see Subsection 7.B, which is called the <u>projective linear group over K</u>.

The center of the group $SL_n(K)$, $n \geq 1$, is the group * $E_n(K) := \{a \in K^{*} \mid a^n = 1\}$ is the group of the n-th roots of unity in K.

* $Z(SL_n(K)) = \{a E_n \mid a \in E_n(K)\}$, where

The corresponding residue-class group

$$PSL_n(K) = SL_n(K) / Z(SL_n(K))$$

is called the <u>special linear group over K</u>. The embedding

$SL_n(K) \longrightarrow GL_n(K)$ induces an embedding $PSL_n(K) \longrightarrow PGL_n(K)$ and hence via this embedding $PSL_n(K)$ can be identified as subgroup of $PGL_n(K)$. Moreover, it is normal in $PGL_n(K)$. The quotient group

$$PGL_n(K) / PSL_n(K) \simeq K^\times / {}^nK^\times ,$$

where ${}^nK^\times$ is the subgroup $\{a^n \mid a \in K^\times\}$ of the n-th powers in $K^\times$. In particular, $PSL_n(K) = PGL_n(K)$ if and only if ${}^nK^\times = K^\times$, i.e. every element in K is a n-th power (in K).

We state (without proof) very important theorem:

<u>Theorem</u> Let K be a field and $n \in \mathbb{N}$, $n \geq 2$ (if $n=2$, then assume $|K| \geq 4$). Then the special projective linear group $PSL_n(K)$ is simple (and non-abelian)

For a finite field K, we note in particular, the order of this finite non-abelian simple group:

Let $\mathbb{K}_q$ be a finite field with q elements. Then (see Example 10.A.28)

$$|PSL_n(\mathbb{K}_q)| = \frac{q^{\binom{n}{2}}(q^n-1)(q^{n-1}-1)\cdots(q^2-1)}{\gcd(q-1,n)}$$

Note that proof of this formula use the fact that $K^\times$ is cyclic (see 6.A.26) of order $q-1$ and hence the subgroup $E_n(\mathbb{K}_q) = \{a\in\mathbb{K}_q^\times \mid a^n=1\}$ has the order $\gcd(q-1,n)$.

The groups $PSL_2(\mathbb{K}_2) = PGL_2(\mathbb{K}_2) \cong \mathfrak{S}_3$ and $PSL_2(\mathbb{K}_3) \cong \mathfrak{A}_4$ are not simple.

(Note that $PGL_2(\mathbb{K}_3) \cong \mathfrak{S}_4$).

The simple groups $PSL_2(\mathbb{K}_5)$ and $PSL_2(\mathbb{K}_4) = PGL_2(\mathbb{K}_4)$ both have order 60 and hence isomorphic to $\mathfrak{A}_5$, see Example 9.A.15. The smallest non-abelian simple group with more than 60 elements is (up to isomorphism) is the group $PSL_2(\mathbb{K}_7) \cong PSL_3(\mathbb{K}_2) = PGL_3(\mathbb{K}_2) = GL_3(\mathbb{K}_2)$ with 168 elements which is often denoted by G_{168}.

The group $PGL_2(\mathbb{K}_5)$ is of order 120 and by Example 7.B.4 operates transitively (even 3-fold transitively) on the projective line $\mathbb{P}^1(\mathbb{K}_5) = \overline{\mathbb{K}_5} = \{0,1,2,3,4,\infty\}$. It is an interesting subgroup (of index 6) of the (full-) permutation group $\mathfrak{S}(\overline{\mathbb{K}_5}) = \mathfrak{S}_6$.

The canonical operation of $\mathfrak{S}(\overline{\mathbb{K}_5})$ on the set $\mathfrak{S}(\overline{\mathbb{K}_5})/PGL_2(\mathbb{K}_5)$ of left-cosets induces a group homomorphism

$$\Phi : \mathfrak{S}(\overline{\mathbb{K}_5}) \xrightarrow{\approx} \mathfrak{S}\left(\mathfrak{S}(\overline{\mathbb{K}_5})/PGL_2(\mathbb{K}_5)\right)$$

which is trivially injective and hence bijective. If $\tau \in \mathfrak{S}(\overline{\mathbb{K}_5})$ is a transposition, then $\Phi(\tau) : \sigma PGL_2(\overline{\mathbb{K}_5}) \longmapsto \tau\sigma PGL_2(\overline{\mathbb{K}_5})$

has clearly no fixed point (see again Example 7.B.4) and is in particular, cannot be a transposition.

If $\Psi : \overline{\mathbb{K}_5} \longrightarrow \mathfrak{S}(\mathbb{K}_5)/\mathrm{PGL}_2(\mathbb{K}_5)$ is an arbitrary bijection, then the group automorphism

$$\mathfrak{S}(\overline{\mathbb{K}_5}) \longrightarrow \mathfrak{S}(\overline{\mathbb{K}_5}), \ \sigma \longmapsto \Psi^{-1}\Phi(\sigma)\Psi$$

of $\mathfrak{S}(\overline{\mathbb{K}_5})$ is not an inner automorphism of $\mathfrak{S}(\overline{\mathbb{K}_5})$, since for a transposition τ as above, the image $\Psi^{-1}\Phi(\tau)\Psi$ is not a transposition.

On the other hand, for $n \neq 6$, all automorphisms of $\mathfrak{S}_n$ are inner automorphisms.

The group $\mathrm{Inn}\,\mathfrak{S}_6$ ($\cong \mathfrak{S}_6/\mathrm{Z}(\mathfrak{S}_6) = \mathfrak{S}_6$, see Example 9.A.12) of inner automorphisms of $\mathfrak{S}_6$ has the index 2 in the group $\mathrm{Aut}\,\mathfrak{S}_6$ of all automorphisms of $\mathfrak{S}_6$. Therefore $|\mathrm{Aut}\,\mathfrak{S}_6| = 1440$.

Moreover, $PGL_2(\mathbb{K}_5) \cong \mathfrak{S}_5$. For a <u>proof</u> Consider the dihedral group D_4 which is isomorphic to the subgroup $D \subseteq G := PGL_2(\mathbb{K}_5)$ of the transformations $z \longmapsto az$ and $z \longmapsto bz^{-1}$, $a, b \in \mathbb{K}_5^\times$, and within the subgroup $V := \{\pm z, \pm z^{-1}\}$ which is isomorphic to the Klein's 4-group $D_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$ (for the description of elements of G, see Example 7.B.4). Moreover, V is a subgroup of $PSL_2(\mathbb{K}_5) \cong \mathfrak{A}_5$ (Note that D Contains another subgroup $\tilde{V} := \{\pm z, \pm 2z^{-1}\}$ which is isomorphic to D_2, but <u>not</u> Contained in $PSL_2(\mathbb{K}_5)$.)

The normaliser $N_G(V)$ of V in G is $\underbrace{\text{Contained in } D \text{ and is}}$ $\underbrace{\text{isomorphic to}}$ $\mathfrak{A}_4$ (a subgroup of $PSL_2(\mathbb{K}_5)$ which is) (since $\mathfrak{V}_4$ is normal in $\mathfrak{A}_4$), its order is therefore a multiple of 24. But, since

V is <u>not</u> normal in G, $|N_G(V)| = 24$

and the natural transitive not necessarily faithful operation of G on $G/N_G(V)$ induces an isomorphism

$$G \cong \mathfrak{S}(G/N_G(V)) \cong \mathfrak{S}_5.$$

(moreover, $N_G(\tilde{V}) = D$)

The inverse matrix of an invertible matrix can be explicitly given ~~by~~ with the help of determinants. For preparation we introduce of notation:

Let $\mathfrak{A} = (a_{ij}) \in M_n(K)$ be a $n \times n$ matrix. The determinants of the $(n-1) \times (n-1)$-matrix

$$\mathfrak{A}(1, \ldots, i-1, i+1, \ldots n \mid 1, \ldots, j-1, j+1, \ldots, n)$$

obtained from $\mathfrak{A}$ by deleting i-th row and j-th column is denoted by A_{ij}. By 9.A.4 (with $r = 1$ and $s = n-1$)

$$A_{ij} = \begin{vmatrix} 1 & a_{i1} & \cdots & a_{i,j-1} & a_{i,j+1} & a_{in} \\ 0 & a_{11} & & a_{1,j-1} & a_{1,j+1} & a_{1n} \\ \vdots & \vdots & & & & \\ 0 & a_{i-1,1} & & a_{i-1,j-1} & a_{i-1,j+1} & a_{i-1,n} \\ 0 & a_{i+1,1} & & a_{i+1,j-1} & a_{i+1,j+1} & a_{i+1,n} \\ \vdots & \vdots & & \vdots & & \\ 0 & a_{n1} & & a_{n,j-1} & a_{n,j+1} & a_{nn} \end{vmatrix}$$

By $(i-1)$-interchanges of rows and $(j-1)$ interchanges of columns, we get:

$$A_{ij} = (-1)^{i+j} \begin{vmatrix} a_{11} & \cdots & a_{1,j-1}, & 0, & a_{1,j+1} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{i,j-1} & 1 & a_{i,j+1} & \cdots & a_{in} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & 0 & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix}$$

$$= (-1)^{i+j} \Delta_{\underline{e}}(x_1, \dots, x_{j-1}, e_i, x_{j+1}, \dots, x_n)$$

where $x_1, \dots, x_n$ are the columns of $\mathfrak{A}$ and $\underline{e} = (e_1, \dots, e_n)$ is the standard basis of K^n.

9.D.11 Expansion Theorem for Determinants

Let $\mathfrak{A} = (a_{ij}) \in M_n(K)$. Then for all $i, j \in \{1, 2, \dots, n\}$, we have:

$$\operatorname{Det} \mathfrak{A} = \sum_{k=1}^{n} (-1)^{k+j} a_{kj} A_{kj}$$

$$= \sum_{k=1}^{n} (-1)^{i+k} a_{ik} A_{ik}$$

The first equality is known as the

<u>expansion of Det $\mathfrak{A}$ by the j-th column</u> and the second equality is known as the <u>expansion of Det $\mathfrak{A}$ by the i-th row</u>.

<u>Proof of 9.D.11</u> It is enough to prove the first equality. The second equality follows by applying the first equality to the transpose matrix ${}^t\mathfrak{A}$.

Using the above notation, we have:

$$
\begin{aligned}
\mathrm{Det}\,\mathfrak{A} &= \Delta_{\underline{e}}(x_1, \dots, x_j, \dots, x_n) \\
&= \Delta_{\underline{e}}\Big(x_1, \dots, \sum_{k=1}^{n} a_{kj}\cdot e_k, \dots, x_n\Big) \\
&= \sum_{k=1}^{n} a_{kj}\,\Delta_{\underline{e}}(x_1, \dots, x_{j-1}, e_k, x_{j+1}, \dots x_n) \\
&= \sum_{k=1}^{n} (-1)^{k+j} a_{kj} A_{kj} \qquad \blacksquare
\end{aligned}
$$

Further, for $\mathfrak{A} = (a_{ij}) \in M_n(K)$, the matrix

$$\mathrm{Adj}\,\mathfrak{A} := (\tilde{a}_{ij}) \text{ with } \tilde{a}_{ij} := (-1)^{i+j} A_{ji}$$

is called the <u>adjoint matrix of</u> $\underline{\mathfrak{a}}$ or <u>adjoint to</u> $\mathfrak{a}$.

Note that in the formation of the adjoint matrix the interchange in the indices i and j:
The (i,j)-th entry of $\mathrm{Adj}\,\mathfrak{a}$ is obtained by deleting j-th row and i-th column of $\mathfrak{a}$. The sign $(-1)^{i+j}$ is obtained from the checker:

$$\begin{pmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The element $(-1)^{i+j}A_{ij} = \tilde{a}_{ji}$ is also called the (i,j)-th <u>cofactor of</u> $\mathfrak{a}$

9.D.12 Example

$$\mathrm{Adj}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}.$$ Further,

$$\operatorname{Adj}\begin{pmatrix}1 & 0 & 2\\ 2 & -1 & 3\\ 4 & 1 & 8\end{pmatrix} = \begin{bmatrix} \begin{vmatrix}-1 & 3\\ 1 & 8\end{vmatrix} & -\begin{vmatrix}0 & 2\\ 1 & 8\end{vmatrix} & \begin{vmatrix}0 & 2\\ -1 & 3\end{vmatrix} \\ -\begin{vmatrix}2 & 3\\ 4 & 8\end{vmatrix} & \begin{vmatrix}1 & 2\\ 4 & 8\end{vmatrix} & -\begin{vmatrix}1 & 2\\ 2 & 3\end{vmatrix} \\ \begin{vmatrix}2 & -1\\ 4 & 1\end{vmatrix} & -\begin{vmatrix}1 & 0\\ 4 & 1\end{vmatrix} & \begin{vmatrix}1 & 0\\ 2 & -1\end{vmatrix}\end{bmatrix}$$

$$= \begin{pmatrix}-11 & 2 & 2\\ -4 & 0 & 1\\ 6 & -1 & -1\end{pmatrix}$$

From the Expansion Theorem 9.D.11, it follows directly:

<u>9.D.13 Theorem</u> Let $\mathfrak{A} \in M_n(K)$.

Then:

$$(\operatorname{Adj}\mathfrak{A})\cdot\mathfrak{A} = \mathfrak{A}\cdot(\operatorname{Adj}\mathfrak{A})$$
$$= (\operatorname{Det}\mathfrak{A})\,\mathcal{E}_n$$

If $\mathfrak{A}$ is invertible, then

$$\mathfrak{A}^{-1} = \frac{1}{\operatorname{Det}\mathfrak{A}}\cdot\operatorname{Adj}\mathfrak{A}.$$

<u>Proof</u> Let $\mathfrak{A} = (a_{ij})$ and

$\operatorname{Adj} \mathfrak{A} = (\tilde{a}_{ij})$ with $\tilde{a}_{ij} = (-1)^{i+j} A_{ji}$.

Then $(\operatorname{Adj} \mathfrak{A}) \cdot \mathfrak{A} = (b_{ij})$ with

$$b_{ij} = \sum_{k=1}^{n} \tilde{a}_{ik} \cdot a_{kj} = \sum_{k=1}^{n} (-1)^{k+i} a_{kj} A_{ki}$$

$$= \delta_{ij} \cdot \operatorname{Det} \mathfrak{A}$$

(Note by 9.D.11 : $\operatorname{Det} \mathfrak{A} = \sum_{k=1}^{n} (-1)^{k+j} a_{kj} \cdot A_{kj}$)

In the case $i = j$ the last equality is the Expansion of the $\operatorname{Det} \mathfrak{A}$ by the j-th column by 9.D.11. For $i \neq j$, again by 9.D.11 :

$$\sum_{k=1}^{n} (-1)^{k+i} a_{kj} A_{ki}$$

is the determinant of the matrix obtained from $\mathfrak{A}$ by replacing i-th column by the j-th column and hence this matrix has two equal columns thus its determinant is 0.

The equation $\mathfrak{A} \cdot \operatorname{Adj} \mathfrak{A} = (\operatorname{Det} \mathfrak{A}) E_n$ is proved analogously ~~with~~ by using the expansion theorem for rows.

It also follows formally from the first equality for the transpose ${}^t\mathfrak{A}$ of $\mathfrak{A}$ and the equality

$$\operatorname{Adj} {}^t\mathfrak{A} = {}^t(\operatorname{Adj} \mathfrak{A})$$

(by taking transpose).

The last assertion in 9.D.13 follows from the fact that the determinant of an invertible matrix is $\neq 0$ (see Cor. to 9.D.5).

9.D.14 Corollary (Cramer's Rule)

Let

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ \cdots \quad \cdots \quad &\cdots \\ a_{m1}x_1 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

be a system of m linear equations in n unknowns with coefficients in the field K. Suppose that the coefficient matrix $\mathfrak{A} = (a_{ij}) \in GL_n(K)$. Then

$$x_j = \frac{\begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & b_1 & a_{1,j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & b_n & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix}}{\operatorname{Det} \mathfrak{A}}$$

$j = 1, \ldots, n$

is the unique solution of the given system.

<u>Proof</u> Let $\mathfrak{b} = (b_i) \in K^n$. The system $\mathfrak{A} \cdot x = \mathfrak{b}$ for $x \in K^n = (x_i)_{i \in I}$

is equivalent with the system

$$\underline{x} = \mathfrak{A}^{-1}\mathfrak{b} = \frac{1}{\operatorname{Det}\mathfrak{A}} (\operatorname{Adj} \mathfrak{A}) \mathfrak{b}$$

Equating the j-th Component:

$$x_j = \frac{1}{\operatorname{Det}\mathfrak{A}} \sum_{k=1}^{n} (-1)^{j+k} A_{kj} \cdot b_k .$$

Now the result follows from the Expansion Theorem 9.D.11 ∎

9.D.15 Example.

The determinant of the coefficient matrix of the system of linear equations:

$$\begin{aligned} x_1 \quad\quad + 2x_3 &= 1 \\ 2x_1 - x_2 + 3x_3 &= 0 \\ 4x_1 + x_2 + 8x_3 &= -1 \end{aligned}$$

is by Example 9.D.3 is equal to 1.

Therefore by the Cramer's Rule 9.D.14 the system has the unique solution:

$$x_1 = \begin{vmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \\ -1 & 1 & 8 \end{vmatrix} = -13$$

$$x_2 = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 0 & 3 \\ 4 & -1 & 8 \end{vmatrix} = -5$$

$$x_3 = \begin{vmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ 4 & 1 & -1 \end{vmatrix} = 7$$

The inverse of the coefficient matrix (by 9.D.13) and the computation ~~[illegible]~~ in Example 9.D.12 is:

$$\begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix}^{-1} = \begin{pmatrix} -11 & 2 & 2 \\ 4 & 0 & 1 \\ 6 & -1 & -1 \end{pmatrix},$$ see also Example 8.B.7.

With these calculations, note that the unique solution of the system in the form:

$$\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}-11&2&2\\-4&0&1\\6&-1&-1\end{pmatrix}\begin{pmatrix}1\\0\\-1\end{pmatrix} = \begin{pmatrix}-13\\-5\\7\end{pmatrix}$$

We remark also that the Cramer's Rule and the formulae in 9.D.13 in general, have more theoretical interest. For finding concretely the solutions of systems of equations (resp. finding explicitly inverses of matrices) We recommend the Gauss-elimination process (resp. and the methods deduced from it ~~Gauss elimination~~, see also the subsections 8.B and 8.C.