# 9.E The Determinant of a Linear operator

In this subsection, let $V$ be a finite dimensional vector space over a field.

Let $f : V \longrightarrow V$ be a $K$-linear operator on $V$. For every determinant function $\Delta : V^I \longrightarrow K$,

$$\Delta' : V^I \longrightarrow K, \quad (x_j)_{j\in I} \longmapsto \Delta\big((f(x_j))_{j\in I}\big)$$

is also a determinant function. Therefore, if $\Delta \neq 0$, then $\Delta' = \lambda \Delta$ with $\lambda \in K$, since the space of all determinant functions $V^I \to K$ is of dimension 1 (see 9.C.5). This scalar $\lambda$ is apparently independent of the choice of the determinant function $\Delta \neq 0$ (and independent of the choice of the index set $I$ with $|I| = \mathrm{Dim}_K V$)

**9.E.1 Definition** Let $f : V \longrightarrow V$ be a $K$-linear operator on the finite dimensional $K$-vector space and $\Delta : V^I \longrightarrow K$ be a non-zero determinant function. Then the unique scalar $\operatorname{Det} f \in K$, defined by the equation

$$\Delta\big((f(x_j))_{j\in I}\big) = \operatorname{Det} f \cdot \Delta\big((x_j)_{j\in I}\big)$$

$(x_j)_{j\in I} \in V^I$, is called the determinant of $f$.

In particular, let $\underline{v} = (v_i)_{i\in I}$ be a $K$-basis of $V$ and $\Delta_{\underline{v}} : V^I \longrightarrow$ be the determinant function with $\Delta_{\underline{v}}\big((v_i)_{i\in I}\big) = 1$.

If $f(v_j) = \sum_{i\in I} a_{ij} v_i$, $j \in I$, i.e. if $(a_{ij})_{i,j\in I} = \mathfrak{M}^{\underline{v}}_{\underline{v}}(f)$ is the matrix of $f$ w.r.to the basis $\underline{v}$, then by 9.C.3, we have:

$$\Delta_{\underline{v}}\left((f(v_j))_{j\in I}\right) = \mathrm{Det}\,(a_{ij})_{i,j\in I}$$
$$= \mathrm{Det}\,\mathcal{M}^{\underline{v}}_{\underline{v}}(f).$$

On the other hand by definition:

$$\Delta_{\underline{v}}\left((f(v_j))_{j\in I}\right) = (\mathrm{Det}\,f)\cdot\Delta_{\underline{v}}\left((v_j)_{j\in I}\right)$$
$$= \mathrm{Det}\,f$$

Therefore, we have proved:

**9.E.2 Theorem** Let $\underline{v}$ be a $K$-basis of the finite dimensional $K$-vector space $V$. For every $K$-linear operator $f: V \longrightarrow V$ on $V$, we have:

$$\mathrm{Det}\,f = \mathrm{Det}\,\mathcal{M}^{\underline{v}}_{\underline{v}}(f).$$

Theorem 9.E.2 says that the following ~~th~~ diagram is commutative:

$$\begin{tikzcd}
\operatorname{End}_K V \arrow[rr, "\mathcal{M}^{v}_{v}"] \arrow[dr, "\mathrm{Det}"'] & & M_I(K) \arrow[dl, "\mathrm{Det}"] \\
& K &
\end{tikzcd}$$

**9.E.3 Product formula** For two $K$-linear operators $f, g : V \to V$ on the finite dimensional $K$-vector space $V$, we have:

$$\mathrm{Det}(f \circ g) = (\mathrm{Det} f)(\mathrm{Det} g)$$

**Proof** Let $\Delta : V^I \longrightarrow K$ be a non-zero determinant function on $V$. Then for $(x_j)_{j \in I} \in V^I$,

$$\Delta(f(g(x_j))) = (\mathrm{Det} f)\, \Delta(g(x_j)) = (\mathrm{Det} f)(\mathrm{Det} g)\, \Delta((x_j)_{j \in I}).$$

The product formula 9.E.3 also follows from the product formula 9.D.5 for the matrices, since

$$\mathfrak{M}_{\underline{v}}^{\underline{v}}(f\circ g) = \mathfrak{M}_{\underline{v}}^{\underline{v}}(f)\cdot\mathfrak{M}_{\underline{v}}^{\underline{v}}(g)$$

for arbitrary basis $\underline{v}$ of $V$.

Conversely, 9.E.3 supply a new proof for 9.D.5 (which is of course based on the same idea).

If one define the determinant $\operatorname{Det} f$ of a linear operator $f: V \longrightarrow V$ directly by the formula (from 9.E.2)

$$\operatorname{Det} f = \operatorname{Det} \mathfrak{M}_{\underline{v}}^{\underline{v}}(f),$$

then we need to check that $\operatorname{Det} f$ is independent of the choice of the basis $\underline{v} = (v_i)_{i\in I}$ of $V$.

For another basis $\underline{v}' = (v_i')_{i\in I}$ of $V$, by 8.A.15

$$\mathfrak{M}_{\underline{v}'}^{\underline{v}'}(f) = \mathfrak{B}\cdot\mathfrak{M}_{\underline{v}}^{\underline{v}}(f)\,\mathfrak{B}^{-1},$$

9E16

where $\mathcal{B} = \mathfrak{M}^{\underline{v}}_{\underline{v}'}(\mathrm{id}_V) \in GL_I(V)$ is the transition matrix from $\underline{v}$ to $\underline{v}'$. Now, use the product-formula for the determinant of matrices, to get the required equality:

$$\operatorname{Det} \mathfrak{M}^{\underline{v}'}_{\underline{v}'}(f) = \operatorname{Det} \mathfrak{M}^{\underline{v}}_{\underline{v}}(f).$$

As in the case of matrices, the determinant function induces a group homomorphism

$$\operatorname{Det} : \operatorname{End}_K V \longrightarrow K$$

$$\operatorname{Det} : \operatorname{Aut}_K \longrightarrow K^{*}.$$

Analogous to 9.C.10, we have:

**9.E.4 Theorem** An operator $f : V \to V$ on the finite dimensional $K$-vector space $V$ is bijective (i.e. an Automorphism) if and only if its determinant is non-zero.

From the formula 9.D.4 for block-matrices, the folloning often used formula:

<u>9.E.5 Theorem</u> Let $V$ be a finite dimensional $K$-vector space and $f: V \longrightarrow V$ a $K$-linear operator on $V$. Further, let $U \subseteq V$ be a $f$-invariant subspace, i.e. $f(U) \subseteq U$ and $\overline{f}: \overline{V} \longrightarrow \overline{V}$ be the operator induced by $f$ on the residue-class space $\overline{V} := V/U$, $\overline{f}(\overline{x}) = \overline{f(x)}$, $x \in V$. Then

$$\mathrm{Det}\, f = \mathrm{Det}\,(f|U) \cdot \mathrm{Det}\, \overline{f}$$

<u>Proof</u> Let $u_1, \dots, u_r$ be a basis of $U$ and $\underline{v} = \{u_1, \dots, u_r, w_1, \dots, w_s\}$ be a basis of $V$. Since $f(U) \subseteq U$, the matrix $\mathfrak{M}^{\underline{v}}_{\underline{v}}(f)$ of $f$ w.r.to the basis $\underline{v}$ is a block-matrix:

9E|8

$$\mathfrak{B} := \begin{pmatrix} a_{11} & \cdots & a_{1r} & b_{11} & \cdots & b_{1s} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{r1} & \cdots & a_{rr} & b_{r1} & \cdots & b_{rs} \\ 0 & \cdots & 0 & c_{11} & \cdots & c_{1s} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & c_{s1} & \cdots & c_{ss} \end{pmatrix}$$

where the blocks

$$\mathfrak{A} := \begin{pmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & \ddots & \vdots \\ a_{r1} & \cdots & a_{rr} \end{pmatrix}$$ resp.

$$\mathfrak{C} = \begin{pmatrix} c_{11} & \cdots & c_{1s} \\ \vdots & \ddots & \vdots \\ c_{s1} & \cdots & c_{ss} \end{pmatrix}$$

is the matrix $\mathfrak{M}^{\underline{u}}_{\underline{u}}(f|U)$ of $f|U : U \to U$ w.r.to the basis $\underline{u} = \{u_1, \cdots, u_r\}$ resp. the matrix $\mathfrak{M}^{\overline{\underline{w}}}_{\overline{\underline{w}}}(\overline{f})$ of $\overline{f} : \overline{V} \longrightarrow \overline{V}$ w.r.to the basis $\overline{\underline{w}} = \{\overline{w}_1, \cdots, \overline{w}_s\}$ of $\overline{V}$. Therefore, by 9.D.4, we have:

$$\begin{aligned} \operatorname{Det} f = \operatorname{Det} \mathfrak{B} &= \operatorname{Det} \mathfrak{A} \cdot \operatorname{Det} \mathfrak{C} \\ &= \operatorname{Det}(f|U) \cdot \operatorname{Det} \overline{f}. \end{aligned}$$

If $f : E \longrightarrow E$ be an affine map of the finite dimensional affine space $E$ over the $K$-vector space $V$, then the determinant of the linear part $\underbrace{f_0)\ \text{of}\ f}_{\in \operatorname{End}_K V)}$ is also called the *determinant of $f$*.