9.F Orientations

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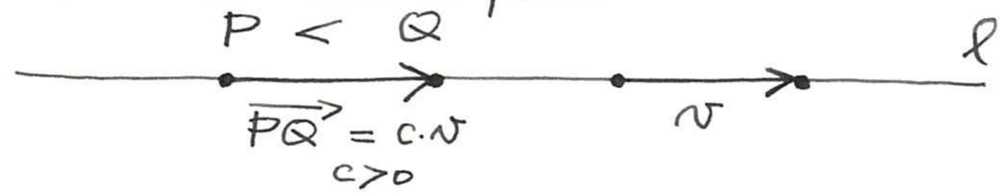
In this subsection, we denote the base scalar field $K = \mathbb{R}$ with its natural order $\leq$.

The real affine line ℓ has two natural orders which are defined by using the natural order $\leq$ on the scalar field $\mathbb{R}$. For this, let $v \in V$ be a basis of the corresponding 1-dimensional $\mathbb{R}$-vector space $V = V(\ell)$ of translations and define



With the affine isomorphism

$$\mathbb{R} \longrightarrow \ell, \quad r \longmapsto rv + O,$$

where O is a fixed point on the line ℓ, one can also transfer the natural order on $\mathbb{R}$ to ℓ. This order depends on the choice of the basis v. Namely, if $v' = av$ is another basis, then the order defined by using v' is same as the order defined by using v if and only if $a > 0$. If $a < 0$, then the order defined by using v' on ℓ is the opposite order (in the case $\leq$, it is denoted by $\geq$). Both these orders are called the <u>orientations</u> of the line ℓ. The line ℓ together one of these order is called an <u>oriented line</u>.

Now we want to define orientations on an arbitrary finite dimensional real affine space E over the $\mathbb{R}$-vector space $V = V(E)$. The $\mathbb{R}$-vector space $\mathrm{Alt}_{\mathbb{R}}(n; V, \mathbb{R})$ of the determinant functions on V, $n = \mathrm{Dim}_{\mathbb{R}} V$, associated to V, is 1-dimensional, see 9.C.5.

9.F.1 Definition An orientation on the n-dimensional $\mathbb{R}$-vector space V is an orientation on the $\mathbb{R}$-vector space $\mathrm{Alt}(n, V, \mathbb{R})$ of the determinant functions $V^n \xrightarrow{\mathbb{R}} \mathbb{R}$.

An n-dimensional $\mathbb{R}$-vector space has exactly two orientations and these are distinguished by non-zero determinant functions $\Delta : V^n \to \mathbb{R}$, where two such functions Δ and Δ' define the same orientations on V if and only if they differ by a positive (scalar) factor, i.e. $\Delta' = a\Delta$, $a \in \mathbb{R}$, $a > 0$.

If $\underline{v} = (v_1, \cdots, v_n)$ is a basis of the oriented $\mathbb{R}$-vector space V, then we say that $\underline{v}$ represents the given orientation of V if the determinant function $\Delta_{\underline{v}}$ (with $\Delta_{\underline{v}}(v_1, \cdots, v_n) = 1$ see 9.C.3) defines the given orientation on V. Let $\underline{v}' = (v_1', \cdots, v_n')$ be another basis of V with the transition matrix $\mathfrak{M}^{\underline{v}}_{\underline{v}'}(\mathrm{id}_V) = \mathfrak{B} \in GL_n(\mathbb{R})$, then $\Delta_{\underline{v}'} = (\mathrm{Det}\,\mathfrak{B})^{-1} \cdot \Delta_{\underline{v}}$ by 9.C.8. It follows that

9.F.2 Theorem Two bases $\underline{v} = (v_1, \cdots, v_n)$ and $\underline{v}' = (v_1', \cdots, v_n')$ of the $\mathbb{R}$-vector space V represents the same orientation on V if and only if the transition matrix $\mathfrak{B} = \mathfrak{M}^{\underline{v}}_{\underline{v}'}(\mathrm{id})$ has a positive determinant.

By orientation of a finite dimensional real affine space E, we simply mean an orientation of the corresponding $\mathbb{R}$-vector space $V = V(E)$ of the translations of E.
If $(P_0, \cdots, P_n)$ is an affine basis of E, i.e. a non-degenerate n-simplex $n = \mathrm{Dim}_K E$, then by definition, $(P_0, \cdots, P_n)$ represents the given orientation on E if the basis $\overrightarrow{P_0P_1}, \cdots, \overrightarrow{P_0P_n}$ represents the corresponding orientation on V.

9.F.3 Examples (1) The space of the determinant functions of a 0-dimensional $\mathbb{R}$-vector space is $\mathbb{R}$. This has the natural orientation of $\mathbb{R}$ (defined by the basis 1). The opposite orientation is defined by the basis -1. Accordingly, one can denote the orientation of a point $\{P_0\}$

Which is a 0-dimensional affine space by the sign +1 or −1. The 0-simplex (P_0) represents the orientation 1. The orientation −1 is not represented by an affine basis.

(2) The <u>standard orientation</u> on $\mathbb{R}^n$ is by definition, represented by the standard basis $e_1, \cdots, e_n$ of $\mathbb{R}^n$. On an arbitrary finite dimensional $\mathbb{R}$-vector space of positive dimension none of the two orientations is distinguished. On the $\mathbb{R}$-vector space $\mathbb{R}^I$ with $|I| \geq 2$, there is no canonical orientation. However, if $I = \{i_1, \cdots, i_n\}$ is totally ordered with $i_1 < i_2 < \cdots < i_n$, then the orientation represented by the basis $e_{i_1}, \cdots, e_{i_n}$ is a distinguished <u>standard orientation</u> of $\mathbb{R}^I$.

(3) Let V be a finite dimensional $\mathbb{R}$-vector space. The orientations on V and on the dual space $V^* = \operatorname{Hom}_K(V, K)$ Corresponds to each other in a natural way. If $\underline{v} = (v_1, \cdots, v_n)$ is a basis of V and $\underline{v}^* = (v_1^*, \cdots, v_n^*)$ is the dual basis of V^* w.r.to $\underline{v}$, then the orientations represented by $\underline{v}$ and $\underline{v}^*$ Corresponds to each other. This correspondence is independent of the choice of the basis $\underline{v}$ (resp. $\underline{v}^*$) **<u>Proof</u>** Let $\underline{v}$ and $\underline{v}'$ be bases of V and $\mathcal{B} = \mathfrak{M}^{\underline{v}}_{\underline{v}'}(\mathrm{id})$ is the transition matrix from $\underline{v}$ to $\underline{v}'$.

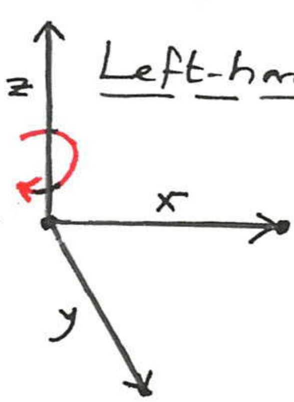
Then the contragradient matrix ${}^t\mathcal{B}^{-1} = M^{\underline{v}^*}_{\underline{v}'^*}(\mathrm{id}_{V^*})$ is the transition matrix from $\underline{v}^*$ to $\underline{v}'^*$ and the determinants $\mathrm{Det}\,\mathcal{B}$ and $\mathrm{Det}\,{}^t\mathcal{B}^{-1}$ have the same sign.

By the way with this remark the previously defined ambiguity on the concept of the orientation on 1-dimensional space is repaired. Namely, note that the space of the determinant functions on an 1-dimensional vector space and its dual space are same, i.e. $\mathrm{Alt}_{\mathbb{R}}(1; V, \mathbb{R}) = V^*$.

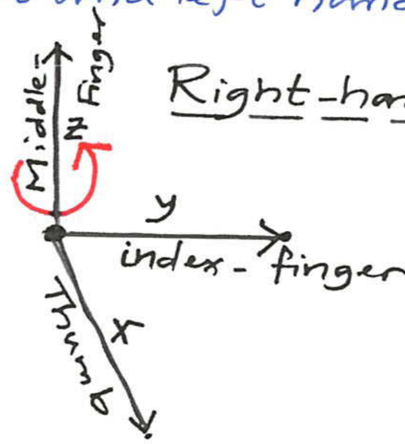
9.F.4 Example (Orientations of the Universe)

An orientation of the universe (= the 3-dimensional physical real (affine) space) is given by a sequence of three linearly independent translations v_1, v_2, v_3 of the universe.
For the standard orientation for these translations one can choose the directions of the thumb, the index finger and the middle finger of the right-hand (Right-hand Rule). --- Note that this is different for right and left hand!

Left-hand Rule



Right-hand Rule



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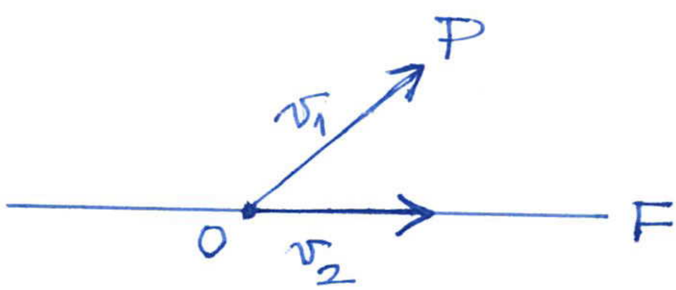
The above choice of a specific orientation is related to human phisiology: asymmetry of the right and left sides.

The question of whether or not there exist purely physical processes which would distinguish the orientation of the universe, i.e. "non-invariant relative to mirror reflection" was answered affirmatively in 1956, to everyone's surprise, by an experiment of Lee-Yang-Wu which established the non-conservation of parity in weak interactions.

9.F.5 Example (Orientation of Hyperplanes)
Let F be an hyperplane in finite dimensional real affine space E (of positive dimension). Then F defines two open-half spaces in E, namely:
$\{P \in E \mid f(P) < 0\}$ and $\{P \in E \mid f(P) > 0\}$,
where $f : E \longrightarrow K$ is an affine map with $F = f^{-1}(0)$.



The orientation of E corresponds (in a natural way) with the orientation of F if one of the half-space is specified. This is done as follows:
Let $F = U + O$ and $v_1 \in H$, where H is the specified half space, i.e.

$$v_1 = \overrightarrow{OP}, \quad O \in F \text{ and } P \in H$$

Let $v_2, \cdots, v_n$ be a basis of the subspace U corresponding to F. Then $v_1, \cdots, v_n$ (resp. $v_2, \cdots, v_n$) represents the orientation of E (resp. F). This correspondence is independent of the choice of the vector v_1 and the basis $v_2, \cdots, v_n$ of U.
Proof If v_1' is another vector in the half-

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Space H and if $v'_2, \dots, v'_n$ is another basis of U, then:

$$v'_1 = a v_1 + b_2 v_2 + \cdots + b_n v_n, \quad a, b_2, \dots, b_n \in \mathbb{R}, \; a>0.$$

If $\tau = \mathcal{M}^{v_2, \dots, v_n}_{v'_2, \dots, v'_n}(id_U)$ is the transition matrix from the basis $v_2, \dots, v_n$ to the basis $v'_2, \dots, v'_n$ of U, then

$$\mathcal{D} = \begin{pmatrix} a & 0 & \cdots & 0 \\ b_2 & & & \\ & & \tau & \\ b_n & & & \end{pmatrix} = \mathcal{M}^{v_1, \dots, v_n}_{v'_1, \dots, v'_n}(id_V)$$

is the transition matrix from the basis $v_1, \dots, v_n$ to the basis $v'_1, \dots, v'_n$ of V. Since $\mathrm{Det}\,\mathcal{D} = a \cdot \mathrm{Det}\,\tau$, the bases $v_1, \dots, v_n$ and $v'_1, \dots, v'_n$ represents the same orientation of E if and only if the bases $v_2, \dots, v_n$ and $v'_2, \dots, v'_n$ of F represents the same orientation of F.