E0 219 Linear Algebra and Applications / August-December 2016

(ME, MSc. Ph. D. Programmes)

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Lectures : Monday and Wednesday ; 11:00–12:30 **Venue:** CSA, Lecture Hall (Room No. 117)

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Midterms : 1-st Midterm : Saturday, September 17, 2016; 15:00 – 17:00 **2-nd Midterm :** Saturday, October 22, 2016; 15:00 – 17:00

Final Examination : December ??, 2016, 09:00 – -12:00

Evaluation Weightage : Assignments : 20% **Midterms (Two) :** 30% **Final Examination :** 50%

Range of Marks for Grades (Total 100 Marks)						
	Grade S	**Grade A**	**Grade B**	**Grade C**	**Grade D**	**Grade F**
Marks-Range	> 90	76 — 90	61 — 75	46 — 60	35 — 45	< 35

	Grade A$^+$	**Grade A**	**Grade B$^+$**	**Grade B**	**Grade C**	**Grade D**	**Grade F**
Marks-Range	> 90	81 — 90	71 — 80	61 — 70	51 — 60	40 — 50	< 40

Supplement 3

Generating systems, Linear independence, Bases

To understand and appreciate the Supplements which are marked with the symbol † one may possibly require more mathematical maturity than one may have! These are steps towards applications to various other branches of mathematics, especially to analysis, number theory and Affine and Projective Geometry.

Participants may ignore these Supplements — altogether or in the first reading!!

S3.1 Let $x_1,\ldots,x_n, x$ be elements of a vector space over a field K. Then

(a) The family $x_1,\ldots,x_n\,, x_1+\cdots+x_n$ is linearly dependent over K, but every n of these vectors are linearly independent over K.

(b) Show that $x_1,\ldots,x_n,x$ are linearly independent over K if and only if $x_1,\ldots,x_n$ are linearly independent over K and $x \notin Kx_1+\cdots+Kx_n$.

(c) Show that $x_1,\ldots,x_n$ is a generating system of V if and only if $x_1,\ldots,x_n\,,x$ is a generating system of V and $x \in Kx_1+\cdots+Kx_n$.

S3.2 Let V be a vector space over a field K.

(a) If V has a finite (respectively, a countable) generating system, then every generating system of V has a finite (respectively, a countable) generating system.

(b) If V has a countable infinite basis, then every basis of V is countable infinite.

(c) If there is an uncountable linearly independent system in V, then no generating system of V is countable.

(d) If K is countable and if V has a countable generating system, then V is countable. In particular, every Hamel-basis of $\mathbb{R}$ over $\mathbb{Q}$ is uncountable.

(e) If $v_i\,, i\in I$, is a generating system for V, then every maximal linearly independent subsystem of $v_i\,, i \in I$, is a basis of V. (**Remark :** Using this assertion and the Zorn's Lemma one can easily prove that : Every vector space has a basis and deduce the general Supplementary Basis Theorem.)

S3.3 Let $a_n,\ n\in\mathbb{N}^*$, be a sequence of elements in K. Show that :

(a) For every $m\in\mathbb{N}$, the polynomials $1\,, X-a_1,\ldots,(X-a_1)\cdots(X-a_{m-1})$ form a K-basis of the K-vector space $K[X]_m$ of polynomials of degrees $< m$. (**Hint :** Use Exercise 3.2 (b).)

(b) The polynomials $(X-a_1)\cdots(X-a_n)\,, n\in\mathbb{N}$, form a K-basis of $K[X]$. (**Hint :** Use part (a).)

S3.4 Let $\lambda_1,\ldots,\lambda_n$ be pairwise distinct elements in a field K. Then :

(a) The vectors $x_1 := (1, \lambda_1, \lambda_1^2, \ldots, \lambda_1^{n-1}), \ldots, x_n := (1, \lambda_n, \lambda_n^2, \ldots, \lambda_n^{n-1}) \in K^n$ are linearly independent over K (and hence is a basis of K^n, See Lecture-Notes Theorem 3.B.7). (**Hint :** Induction on n. Assume the result for $n-1$ and that $a_1x_1 + \cdots + a_nx_n = 0$. Then we have the equations: $a_1\lambda_n x_1' + \cdots + a_n\lambda_n x_n' = 0$ and $a_1\lambda_1 x_1' + \cdots + a_n\lambda_n x_n' = 0$, and so $a_1(\lambda_n - \lambda_1)x_1' + \cdots + a_{n-1}(\lambda_n - \lambda_{n-1})x_{n-1}' = 0$, where $x_i' := (1, \lambda_i, \ldots, \lambda_i^{n-2})$, $i = 1, \ldots, n$.)

(b) The vectors $y_1 := (1, 1, \ldots, 1), y_2 := (\lambda_1, \lambda_2, \ldots, \lambda_n), \ldots, y_n := (\lambda_1^{n-1}, \ldots, \lambda_n^{n-1}) \in K^n$ are linearly independent over K (and hence is a K-basis of K^n). (**Hint :** Note that a representation $b_1y_1 + \cdots + b_ny_n = 0$ with $b_1, \ldots, b_n \in K$ is equivalent with the system of equations $b_1 + b_2\lambda_i + \cdots + b_n\lambda_i^{n-1} = 0$, $i = 1, \ldots, n$. Therefore the vectors x_i, $i = 1, \ldots, n$, are solutions of the homogeneous system of linear equations $b_1z_1 + \cdots + b_nz_n$. Since $x_1, \ldots, x_n$ is a generating system of K^n, the solution space of this equation is K^n which is possible only in the case $b_1 = \cdots = b_n = 0$. **Another Argument :** The equations $b_1 + b_2\lambda_i + \cdots + b_n\lambda_i^{n-1} = 0$, $i = 1, \ldots, n$, mean that the polynomial $b_1 + b_2X + \cdots + b_nX^{n-1} \in K[X]$ of degree $< n$ has n pairwise distinct zeros $\lambda_1, \ldots, \lambda_n \in K$ and hence $b_1 = \cdots = b_n = 0$.)

S3.5 The $\mathbb{R}$-valued functions $f_a : \mathbb{R} \to \mathbb{R}$, $t \mapsto |t-a|$, $a \in \mathbb{R}$, are linearly independent in the $\mathbb{R}$-vector space $\mathbb{R}^{\mathbb{R}}$ of all real-valued functions on $\mathbb{R}$. (**Hint :** Let $n > 0$ and $\sum_{i=1}^n c_i\,|t - a_i| = 0$ with $a_1 < a_2 < \cdots < a_n$ and coefficients $c_1, \ldots, c_n \in \mathbb{R}$. Then the function $c_1|t-a_1| = -\sum_{i=2}^n c_i\,|t-a_i|$ is a polynomial function of degree ≤ 2 on the interval $]-\infty, a_2)$. But, this is possible of if $c_1 = 0$ (why?). Now, apply induction to conclude $c_2 = \cdots = c_n = 0$.)

S3.6 Let K be a field, $D \subseteq K$ and $n \in \mathbb{N}$. Then the power-functions $t^\nu|D$, $\nu = 0, \ldots, n-1$, are linearly independent over K in the K-vector space K^D of all K-valued functions on D if and only if D has at least n elements. (**Hint :** Use Identity Theorem, Supplement S2.6 (d).). Moreover, the power functions $t^\nu|D$, $\nu \in \mathbb{N}$, are linearly independent over K if and only if D is infinite.

S3.7 Let x_i, $i \in I$, be a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$. Then the corresponding coordinate functions $x_i^* : \mathbb{R} \to \mathbb{Q} \subseteq \mathbb{R}$, $i \in I$, are functions $f : \mathbb{R} \to \mathbb{R}$ with $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$ which are not of the form $x \mapsto ax$ for some $a \in \mathbb{R}$. (**Remark :** For the solution of the problem (formulated by Cauchy, see Math. Ann **60**, 459-462 (1905)) of the existence of such functions, Hamel proved the existence of $\mathbb{Q}$-bases of $\mathbb{R}$.)

S3.8 Determine which of the following systems of functions are linearly independent over $\mathbb{R}$ in the $\mathbb{R}$-vector space $\mathbb{R}^{\mathbb{R}}$ of all functions : **(a)** $1, \sin t, \cos t$. **(b)** $\sin t, \cos t, \sin(\alpha + t)$ ($\alpha \in \mathbb{R}$ fixed). **(c)** $t, |t|, \operatorname{Sign} t$. **(d)** $e^t, \sin t, \cos t$.

S3.9 The functions $t \mapsto \cos kt$, $k \in \mathbb{N}$, and $t \mapsto \sin \ell t$, $\ell \in \mathbb{N}^*$, altogether are linearly independent in the $\mathbb{R}$-vectors space of real-valued functions on the closed interval $[0, 2\pi]$. (**Hint :** From a representation $\sum_{k=0}^m a_k \cos kt + \sum_{\ell=1}^n b_\ell \sin \ell t = 0$, $a_k, b_\ell \in \mathbb{R}$, to show that all a_k and all b_ℓ are 0, use

$$\int_0^{2\pi} \cos kt\, \cos jt\, dt = \begin{cases} 2\pi & k = j = 0, \\ \pi, & k = j \neq 0, \\ 0 & \text{otherwise} \end{cases} \quad \int_0^{2\pi} \sin \ell t \cos jt = 0 \text{ and } \int_0^{2\pi} \sin \ell t\, \sin jt\, dt = \begin{cases} \pi, & k = j \neq 0, \\ 0 & \text{otherwise.} \end{cases})$$

S3.10 The functions $t \mapsto e^{\lambda t}$, $\lambda \in \mathbb{C}$, are linearly independent over $\mathbb{C}$ in the $\mathbb{C}$-vector space $\mathbb{C}^{[a,b]}$ of all complex-valued functions on the closed interval $[a,b] \subseteq \mathbb{R}$, $a < b$. (**Hint :** Suppose that $\sum_{j=1}^n c_j e^{\lambda_j t} = 0$, $c_j \in \mathbb{C}$, be a representation of the 0 function with pairwise distinct $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$. Then differentiating i-times, we get $\sum_{j=1}^n c_j \lambda_j^i e^{\lambda_j t} = 0$ and in particular, $\sum_{j=1}^n c_j \lambda_j^i e^{\lambda_j t_0} = 0$, $i \in \mathbb{N}$ and for an arbitrary $t_0 \in [a,b]$. Now, by Exercise 3.4 (b), $c_1 e^{\lambda_1 t_0} = \cdots = c_n e^{\lambda_n t_0} = 0$, and hence $c_1 = \cdots = c_n = 0$ because $e^{\lambda_j t_0} \neq 0$, $j = 1, \ldots, n$.)

S3.11 Let K be a field. Let $f_i \in K^X$, $i \in I$, and $g_j \in K^Y$, $j \in J$, be linearly independent K-valued functions on the sets X resp. Y. Then the functions $f_i \otimes g_j$: $(x,y) \longmapsto f_i(x)\,g_j(y)$, $(i,j) \in I \times J$, are linearly independent in $K^{X \times Y}$.

S3.12 Let $K \subseteq L$ be a field extension and let b_i, $i \in I$, be a K-basis of L. If V is a L-vector space with L-basis y_j, $j \in J$, then b_iy_j, $(i,j) \in I \times J$, is a K-basis of V.

S3.13 (R a t i o n a l f u n c t i o n s) Let K be a field. The quotient of two polynomials over K are called a r a t i o n a l f u n c t i o n (i n o n e v a r i a b l e X o v e r K). Therefore a rational function is of the form F/G with $F, G \in K[X]$, $G \neq 0$ (We may even assume that $\gcd(F,G) = 1$, see Supplement S2.5). The addition, multiplication and scalar multiplication of K of polynomials can be canonically extended to the set $K(X) := \{F/G \mid F, G \in K[X], G \neq 0 \text{ and } \gcd(F,G) = 1\}$ of rational functions over K. With these operations, the set $K(X)$ is a commutative ring and a vector space over K, moreover, it is a K-algebra and $K[X]$ is a K-subalgebra of $K(X)$. Further, $K(X)$ is a field (in fact, the quotient field of the integral domain $K[X]$) and is called the r a t i o n a l f u n c t i o n f i e l d (i n o n e v a r i a b l e X o v e r K).

(a) Every rational function F/G in one indeterminate X over K can also be represented as $F/G = Q + R/G$, where Q and R are polynomials over K with $\deg R < \deg G$. (**Hint :** Use the Division with Remainder for polynomials.)

(b) (P a r t i a l f r a c t i o n d e c o m p o s i t i o n) Let F and G be polynomials over K with $\deg F < \deg G$ and $G = (X-\alpha_1)^{n_1} \cdots (X-\alpha_r)^{n_r}$, $\alpha_i \neq \alpha_j$ for $i \neq j$, $n_i \in \mathbb{N}^*$. Then there exists a unique representation

$$\begin{aligned}\frac{F}{G} = &\ \frac{\alpha_{11}}{(X-\alpha_1)} + \frac{\alpha_{12}}{(X-\alpha_1)^2} + \cdots + \frac{\alpha_{1n_1}}{(X-\alpha_1)^{n_1}} \\ &+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &+ \frac{\alpha_{r1}}{(X-\alpha_r)} + \frac{\alpha_{r2}}{(X-\alpha_r)^2} + \cdots + \frac{\alpha_{rn_r}}{(X-\alpha_r)^{n_r}}\end{aligned}$$

with $\alpha_{ik} \in K$, $i = 1, \ldots, r$; $k = 1, \ldots, n_i$. (**Remark :** If $K = \mathbb{C}$, then by Supplement S2.6 (g) Theorem (FTA), every rational function F/G, $F, G \in \mathbb{C}[X]$, $G \neq 0$, $\deg F < \deg G$ has the partial fraction decomposition over $\mathbb{C}$. — **Hint :** By Induction on $n := \deg G = n_1 + \cdots + n_r$. The case $n \leq 1$ is trivial. To determine $\alpha := \alpha_{rn_r}$, consider a representation

$$\frac{F}{G} = \frac{\alpha}{(X-\alpha_r)^{n_r}} + \frac{\widetilde{F}}{\widetilde{G}}$$

with $\widetilde{G} := (X-\alpha_1)^{n_1}(X-\alpha_2)^{n_2} \cdots (X-\alpha_r)^{n_r-1}$, $\deg \widetilde{F} < n-1$. This representation is equivalent with

$$F = \alpha(x-\alpha_1)^{n_1} \cdots (X-\alpha_{r-1})^{n_{r-1}} + (X-\alpha_r)\widetilde{F}\,.$$

Substituting $X = \alpha_r$, we get

$$\alpha = \frac{F(\alpha_r)}{(\alpha_r-\alpha_1)^{n_1} \cdots (\alpha_r-\alpha_{r-1})^{n_{r-1}}}\,.$$

For this choice of α, obviously, α_r is a zero of the polynomial $F - \alpha\,(X-\alpha_1)^{n_1} \cdots (X-\alpha_{r-1})^{n_{r-1}}$ and hence it is of the form $(X-\alpha_r)\widetilde{F}$ with $\deg \widetilde{F} < n-1$. Now, applying the induction hypothesis to the rational function $\widetilde{F}/\widetilde{G}$, the existence of the $\alpha_{ik} \in K$, is immediate and since α and $\widetilde{F}$ are uniquely determined, their uniqueness also follows. — The above proof is constructive and the coefficients $\alpha_{rn_r}, \ldots, \alpha_{r1}, \ldots, \alpha_{1n_1}, \ldots, \alpha_{11}$ are successively determined. In particular, for a simple zero α_i, i. e., $n_i = 1$, we get the representation

$$\alpha_{i1} = \frac{F(\alpha_i)}{(\alpha_i-\alpha_1)^{n_1} \cdots (\alpha_i-\alpha_{i-1})^{n_{i-1}}(\alpha_i-\alpha_{i+1})^{n_{i+1}} \cdots (\alpha_i-\alpha_r)^{n_r}} = \frac{F(\alpha_i)}{G'(\alpha_i)}$$

(where G' is the derivative of G is used in the denominator). Therefore, in the case $n_1 = \cdots = n_r = 1$, the partial fraction decomposition is

$$\frac{F}{G} = \sum_{i=1}^{r} \frac{F(\alpha_i)}{G'(\alpha_i)} \cdot \frac{1}{(X-\alpha_i)}\,.$$

In the case of non-simple zeros the coefficients α_{ik} are described by using higher derivatives of G, see???.)

(c) (P a r t i a l f r a c t i o n d e c o m p o s i t i o n o v e r $\mathbb{R}$) Let F and $G \in \mathbb{R}[X]$ be real polynomials with $\deg F < \deg G$ and we choose (by dividing by the leading coefficient of G) the monic representation

$$G = (X-\alpha_1)^{n_1} \cdots (X-\alpha_s)^{n_s} q_1^{m_1} \cdots q_t^{m_t},\ \alpha_i \neq \alpha_j \text{ for } i \neq j,\ n_i \in \mathbb{N}^* \text{ and } q_j = (X-\beta_j)(X-\overline{\beta}_j)$$

with a non-real (complex) zero $\beta_j \in \mathbb{C}$ (see Supplement S2.6 (h) Theorem (Real-Version of FTA)). In the complex partial fraction decomposition of F/G (see the part (b)), the coefficients of

$$\frac{1}{(X-\beta_j)^k} \quad \text{and} \quad \frac{1}{(X-\overline{\beta}_j)^k}$$

are conjugates of each other (this follows from the uniqueness in part (b) by conjugation). Altogether, the summands of the type

$$\frac{\beta}{(X-\beta_j)^k}+\frac{\overline{\beta}}{(X-\overline{\beta}_j)^k}=\frac{\beta(X-\overline{\beta}_j)^k+\overline{\beta}(X-\beta_j)^k}{q_j^k}=\frac{p}{q_j^k}$$

occur in the partial fraction decomposition of F/G with numerator $p\in\mathbb{R}[X]$. On can write these summands in the form

$$p=p_0+p_1q_j+\cdots+p_{k-1}q_j^{k-1}$$

with real polynomials p_i of degrees ≤ 1, see[1], and hence

$$\frac{p}{q_j^k}=\frac{p_{k-1}}{q_j}+\cdots+\frac{p_0}{q_j^k}\,.$$

Altogether : There exists a representation

$$\begin{aligned}\frac{f}{g}=&\frac{\alpha_{11}}{(x-\alpha_1)}+\cdots+\frac{\alpha_{1n_1}}{(x-\alpha_1)^{n_1}}+\cdots+\frac{\alpha_{s1}}{(x-\alpha_s)}+\cdots+\frac{\alpha_{sn_s}}{(x-\alpha_s)^{n_s}}\\&+\frac{p_{11}}{q_1}+\cdots+\frac{p_{1m_1}}{q_1^{m_1}}+\cdots+\frac{p_{t1}}{q_t}+\cdots+\frac{p_{tm_t}}{q_t^{m_t}}\end{aligned}$$

with $\alpha_{ik}\in\mathbb{R}$ and real linear polynomials p_{jl}. Moreover, the α_{ik} and p_{jl} are uniquely determined.

S3.14 **(a)** Let $D\subseteq\mathbb{C}$ be an *infinite* subset. Then the rational functions

$$t^n,\ n\in\mathbb{N},\quad \frac{1}{(t-a)^m},\ a\in\mathbb{C}\smallsetminus D,\ m\in\mathbb{N}^*,$$

together form a $\mathbb{C}$-basis of the $\mathbb{C}$-vector space of the complex rational functions defined on D. The corresponding assertion also holds for every algebraically closed field K instead of $\mathbb{C}$, see Supplement S2.6 (g). (**Hint :** See Supplement S2.13.)

(b) Let $D\subseteq\mathbb{R}$ be an *infinite* subset and $Q\subseteq\mathbb{R}[X]$ be the set of all monic polynomials of degree 2 witout any real zeros. Then the rational functions

$$t^n,\ n\in\mathbb{N},\quad \frac{1}{(t-a)^m},\ m\in\mathbb{N}^*,\ a\in\mathbb{R}\setminus D,\quad \text{and}\quad \frac{t^r}{q^\ell},\ r\in\{0,1\},\ \ell\in\mathbb{N}^*,\ q\in Q,$$

together form a $\mathbb{R}$-basis of the $\mathbb{R}$-vector space of real rational functions defined on D. (**Hint :** See Supplement S2.13.)

†**S3.15** **(a)** The vector space of all sequences $K^{\mathbb{N}}$ has no countable generating system over K. (**Hint :** Consider the cases K countable and uncountable separately to show that $K^{\mathbb{N}}$ is never countable and use Supplements S3.2 (c), (d) and Exercise 3.4 (b).)

(b) Let I be an infinite set. Then the K-vector space K^I of K-valued functions on I has no countable generating system over K.

(c) The K-subspace of $K^{\mathbb{N}}$ generated by the characteristic functions e_A, $A\subseteq\mathbb{N}$ has no countable generating system. (**Hint :** *If $\mathfrak{K}$ is a totally ordered subset of $\mathfrak{P}(\mathbb{N})\setminus\{\emptyset\}$, then the family e_A, $A\in\mathfrak{K}$ is linearly independent.* Now, use the fact that there are uncountable totally ordered subsets in the ordered set $(\mathfrak{P}(\mathbb{N}),\subseteq)$.)

†**S3.16** **(a)** Let $I\subseteq\mathbb{R}$ be an interval which contain more than one point. Then none of the $\mathbb{K}$-vector space $\mathrm{C}^{\alpha}_{\mathbb{K}}(I)$, $\alpha\in\mathbb{N}\cup\{\infty,\omega\}$, has a countable generating system.

[1] Let $G\in K[X]$ be a polynomial of degree $n\geq 1$. For every polynomial $F\in K[X]$, $F\neq 0$, there exist uniquely determined polynomials $p_0,\ldots,p_r\in K[X]$ with

$$F=p_0+p_1G+\cdots+p_rG^r,\quad p_r\neq\ \text{ and }\ \deg p_i<n\,.$$

(This expansion corresponds to the g-adic expansion of natural numbers.)

(b) The $\mathbb{K}$-vector space of all convergent power series $\sum_{n=0}^{\infty} a_n x^n$ with coefficients $a_n \in \mathbb{K}$, $n \in \mathbb{N}$, has no countable generating system over $\mathbb{K}$.

†**S3.17** **(a)** The functions $e^{\alpha}z$, $\alpha \in \mathbb{C}$, are linearly independent over $\mathbb{C}$ in the $\mathbb{C}$-vector space $\mathbb{C}^D$ of all $\mathbb{C}$-valued functions on D for every subset $D \subseteq \mathbb{C}$ which has a limit point in $\mathbb{C}$. (Suppose that there exist complex numbers $a_1, \ldots, a_n \in \mathbb{C}$ and pairwise distinct complex numbers $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ such that $\sum_{\nu=1}^{n} a_\nu e^{\alpha_\nu z} = 0$ for all $z \in D$. Then, by the *Identity Theorem for analytic functions*[2] $\sum_{\nu=1}^{n} a_\nu e^{\alpha_\nu z} = 0$ for all $z \in \mathbb{C}$ and hence (by differentiating) $\sum_{\nu=1}^{n} a_\nu \alpha_\nu e^{\alpha_\nu z} = 0$ for all $z \in D$. It follows that $\sum_{\nu=2}^{n} a_\nu (\alpha_1 - \alpha_\nu) e^{\alpha_\nu z} = 0$ for all $z \in D$. Now, use induction on n.)

(b) The functions z^{α}, $\alpha \in \mathbb{C}$, are linearly independent over $\mathbb{C}$ in the $\mathbb{C}$-vector space $\mathbb{C}^D$ of all $\mathbb{C}$-valued functions on D for every subset $D \subseteq \mathbb{C} \setminus \mathbb{R}_-$ which has a limit point in $\mathbb{C} \smallsetminus \mathbb{R}_-$. (Recall that: The exponential function exp is injective on the strip $E := \{z \in \mathbb{C} \mid -\pi < \operatorname{Im} z < \pi\}$ and its image is the complex plane without the negative real axis $\mathbb{C} \smallsetminus \mathbb{R}_- := \{z \in \mathbb{C}^\times \mid -\pi < \operatorname{Arz} z < \pi\}$. The inverse $\ln : \mathbb{C} \smallsetminus R_- \to E$ of the exponential function $\exp : \mathbb{C} \smallsetminus R_- \to E$ is called the natural logarithm. Therefore $\ln z = \ln |z| + \mathrm{i} \operatorname{Arz} z$, where the argument $\operatorname{Arz} z$ of the complex number $z \in \mathbb{C} \smallsetminus \mathbb{R}_-$ is chosen in the open interval $]-\pi, \pi[$. For arbitrary $\alpha \in \mathbb{C}$, $z^\alpha := e^{\alpha \ln z}$ for $z \in \mathbb{C} \smallsetminus \mathbb{R}_-$, defines the power-function on $\mathbb{C} \smallsetminus R_-$. — **Hint:** Use similar argument as in the part (a).)

†**S3.18** **(a)** (Quasi-polynomials) The functions $t^n e^{\alpha t}$, $n \in \mathbb{N}$, $\alpha \in \mathbb{C}$, are linearly independent in the $\mathbb{C}$-vector space $\mathbb{C}^D$ of $\mathbb{C}$-valued functions on a subset $D \subseteq \mathbb{C}$ which has a limit point in $\mathbb{C}$. (**Remark:** The $\mathbb{C}$-subspace generated by these functions is called the space of quasi-polynomials.)

(b) The quasi-polynomials are the solutions of the linear differential equations with constant coefficients $P(D)y = 0$, $P \in \mathbb{C}[X] \setminus \{0\}$. More precisely: Let $P = (X - \lambda_1)^{\alpha_1} \cdots (X - \lambda_r)^{\alpha_r} \in \mathbb{K}[X]$ be a polynomial with pairwise distinct zeros $\lambda_1, \ldots, \lambda_r \in \mathbb{K}$. Then

$$e^{\lambda_1 t}, \ldots, t^{\alpha_1 - 1} e^{\lambda_1 t}, \ldots, e^{\lambda_r t}, \ldots, t^{\alpha_r - 1} e^{\lambda_r t}$$

is a $\mathbb{K}$-basis of the solution space $\{y \in C^n_{\mathbb{K}}(I) \mid P(D)y = 0\}$ of the corresponding homogeneous differential equation $P(D)(y) = (D - \lambda_1)^{\alpha_1} \cdots (D - \lambda_r)^{\alpha_r}(y) = 0$ consisting $n := \alpha_1 + \cdots + \alpha_r = \deg P$ elements. (**Proof:** It is enough to prove that:

(b.1) *Every solution of the homogeneous differential equation $P(D)y = 0$ is of the form*

$$f_1 e^{\lambda_1 t} + \cdots + f_r(t) e^{\lambda_r t}$$

with uniquely determined polynomials $f_i \in \mathbb{C}[t]$ of degree $< n_i$, $i = 1, \ldots, r$.

We shall prove this by induction on $n = \deg P$. For this we need the following simple observation:

(b.2) *Let $\lambda, \mu \in \mathbb{C}$ and $f \in \mathbb{C}[t]$ be a polynomial of degree m. Then the differential equation*

$$(D - \lambda)y = \dot{y} - \lambda y = f(t) e^{\mu t}$$

has a solution of the form $h(t)e^{\mu t}$, where $h \in \mathbb{C}[t]$ is a polynomial of degree m if $\lambda \neq \mu$ and is a polynomial of degree $m+1$ if if $\lambda = \mu$. The polynomial h is uniquely determined if $\lambda \neq \mu$ and is uniquely determined up to the constant term if $\lambda = \mu$. **Proof:** Put $y(t) := h(t)e^{\mu t}$ with a polynomial $h(t) = \sum_{j=0}^{m+1} a_j t^j \in \mathbb{C}[t]$ and let $f(t) = \sum_{j=0}^{m} b_j t^j$. Then

$$\dot{y} - \lambda y = \big(\dot{h} + (\mu - \lambda)h\big)\, e^{\mu t}.$$

Therefore $y(t)$ is a solution of $\dot{y} - \lambda y = f(t)e^{\mu t}$ if and only if $\dot{h} + (\mu - \lambda)h = f$, i.e., the equality

$$\sum_{j=0}^{m} \big((j+1)a_{j+1} + (\mu - \lambda)a_j\big)\, t^j + (\mu - \lambda)\, a_{m+1} t^{m+1} = \sum_{j=0}^{m} b_j t^j$$

holds. If $\mu = \lambda$, then a_0 is arbitrary and $a_{j+1} = \frac{1}{j+1} b_j$ for $j = 1, \ldots, m$. If $\mu \neq \lambda$ then this equality holds if and only if $a_{m+1} = 0$ and $a_j = \big(b_j - (j+1)a_{j+1}\big)/(\mu - \lambda)$ for $j = 0, \ldots, m$. This proves the assertion (b.2).

Proof of (b.2): By induction on $n := \deg P$, we shall show that the given functions are precisely all solutions of the differential equation $P(D)y = 0$. For $n = 1$, this is trivial. Assume that $n > 1$. We may assume that $n_1 > 0$ and aplly the induction hypothesis to the equation $Q(D)z = 0$ with $Q := P/(X - \lambda_1)$.

[2] **Theorem** (Identity Theorem for Analytic Functions) Let D be an interval in $\mathbb{R}$ or a domain (for every point $a, b \in D$, a stretching or traverse-line $[a_0, a_1, \ldots, a_m] := \cup_{j=0}^{m-1} [a_j, a_{j+1}] \subseteq D$ with $a_0 = a$ and $a_m = b$) in $\mathbb{C}$. Two analytic functions on D coincide if and only if they already coincide on a subset of D which has at least one limit point in D.

Let y be a solution of $P(D)y=0$. Since $Q(D)z=Q(D)(D-\lambda_1)y=P(D)y=0$, by induction hypothesis $z:=\dot{y}-\lambda_1 y$ is of the form

$$z(t)=g_1(t)e^{\lambda_1 t}+\cdots+g_n(t)e^{\lambda_n t}$$

with polynomials g_i of degree $<n_i-1$ for $i=1$ and of degree $<n_i$ for $i\geq 2$. By (b.2) $\dot{u}-\lambda_1 u=g_\rho e^{\lambda_\rho t}$ has a solution of the form $u(t)=f_i(t)e^{\lambda_i t}$ with $\deg f_i<n_i$, and hence by substituting a solution of $\dot{y}-\lambda_1 y=z$, we get a solution as in the assertion. Since the solutions of the equation $\dot{y}-\lambda_1 y=z$ differ by a solution $ce^{\lambda_1 t}$ of the corresponding homogeneous equation, it follows that $y(t)$ is of the of the required form.

Conversely, suppose that $y=f_1(t)e^{\lambda_1 t}+\cdots+f_n(t)e^{\lambda_n t}$ with polynomial functions $f_i(t)$ of degree $<n_i$. Then, if $z(t)=(D-\lambda_1)y(t)$, then, since, $(D-\lambda_1)(f_i(t)e^{\lambda_i t})=g_\rho(t)e^{\lambda_\rho t}$ with polynomials $g_1:=\dot{f}_1$ (of degree $<n_1-1$) and $g_i=\dot{f}_i+(\lambda_i-\lambda_1)f_i$ (of degree $<n_i$), $i\geq 2$, by induction hypothesis, is a solution of $Q(D)z=0$. Therefore $P(D)y=Q(D)z=0$.

Finally, we still needs to show that the polynomial functions f_i in the above form of y are uniquely determined by y. By induction hypothesis, $g_1,\ldots,g_n$ in z are uniquely determined by z and hence uniquely determine by y. First, the uniqueness of $f_2,\ldots,f_n$ follows from the uniqueness assertion in (b.1) and f_1 is uniquely determined, up to a constant, by y. But, then f_1 is also unique. This completes the proof.)

(c) The functions

$$t^m e^{bt}\cos\beta t\,,\quad m\in\mathbb{N},\ b\in\mathbb{R},\ \beta\in\mathbb{R}_+\,;\qquad t^k e^{ct}\sin\gamma t\,,\quad k\in\mathbb{N},\ c\in\mathbb{R},\ \gamma\in\mathbb{R}_+^\times\,,$$

together form a basis of the $\mathbb{R}$-vector space of the real valued quasi-polynomials $\mathbb{R}\to\mathbb{R}$.

(d) Let Λ be the set of numbers $\lambda\in\mathbb{C}$ with $\operatorname{Re}\lambda>0$ or with $\operatorname{Re}\lambda=0$ and $\operatorname{Im}\lambda>0$. Then the functions

$$t^n\,,\quad n\in\mathbb{N}\,;\qquad t^m\cos\beta t\,,\quad m\in\mathbb{N}\,,\ \beta\in\Lambda\,;\qquad t^k\sin\gamma t\,,\quad k\in\mathbb{N}\,,\ \gamma\in\Lambda\,,$$

together form a basis of the $\mathbb{C}$-vector space of the quasi-polynomials $\mathbb{R}\to\mathbb{C}$.