E0 219 Linear Algebra and Applications / August-December 2016

(ME, MSc. Ph. D. Programmes)

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Lectures : Monday and Wednesday ; 11:00–12:30 **Venue:** CSA, Lecture Hall (Room No. 117)

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Midterms : 1-st Midterm : Saturday, September 17, 2016; 15:00 – 17:00 **2-nd Midterm :** Sunday, October 23, 2016; 15:00 – 17:00

Final Examination : Thursday, December 08, 2016, 09:00 – -12:00

Evaluation Weightage : Assignments : 20% **Midterms (Two) :** 30% **Final Examination :** 50%

Range of Marks for Grades (Total 100 Marks)						
	Grade S	**Grade A**	**Grade B**	**Grade C**	**Grade D**	**Grade F**
Marks-Range	> 90	76 — 90	61 — 75	46 — 60	35 — 45	< 35

	Grade A$^+$	**Grade A**	**Grade B$^+$**	**Grade B**	**Grade C**	**Grade D**	**Grade F**
Marks-Range	> 90	81 — 90	71 — 80	61 — 70	51 — 60	40 — 50	< 40

Supplement 11

Eigenvalues, Characteristic Polynomials and Minimal Polynomials

To understand and appreciate the Supplements which are marked with the symbol † one may possibly require more mathematical maturity than one may have! These are steps towards applications to various other branches of mathematics, especially to analysis, number theory and Affine and Projective Geometry.

Participants may ignore these Supplements — altogether or in the first reading!!

S11.1 Let $n \in \mathbb{N}^+$ and let $V := \mathbb{K}[t]_n$. For the linear operators $D := d/dt : V \to V$ defined by $P \mapsto P' := d/dt(P)$ and $f : V \to V$ defined by $P \mapsto P(t+1)$ compute the characteristic polynomial, minimal polynomial, eigenvalues and eigenspaces. (**Ans :** $\chi_D = X^n = \mu_D$ and $\chi_f = (X-1)^n = \mu_f$.

Hint : The matrix $\mathfrak{A} = \mathfrak{M}_{\mathfrak{t}}^{\mathfrak{t}}(D)$ (respectively $\mathfrak{B} = \mathfrak{M}_{\mathfrak{t}}^{\mathfrak{t}}(f)$) of the operator D (respectively f) with respect to the basis $\mathfrak{t} := (1, t, \ldots, t^{n-1})$ of $V = K[t]_n$ are

$$\mathfrak{A} := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & i & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & i+1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & n-1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathfrak{B} := \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ 0 & 1 & 2 & \cdots & j-1 & j & \cdots & n-1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{i}{i-1} & \binom{i+1}{i-1} & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 & \binom{i+1}{i} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & n-1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \end{pmatrix}$$

Therefore $\chi_D = \operatorname{Det}(X\mathfrak{E} - \mathfrak{A}) = X^n$ and e-Spec $(D) = \mathrm{Z}_{\mathbb{K}}(\chi_D) = \{0\}$. Further, since $\deg P' = \deg P - 1$ for every non-constant $P \in \mathbb{K}[t]_n$. It follows that the eigenspace $V_D(0) = \operatorname{Ker} D = \mathbb{K}$ (=the space of constant polynomials) and since $D^{n-1}(t^{n-1}) = (n-1)! \neq 0$. Therefore $D^{n-1} \neq 0$ and hence $\mu_D = X^n = \chi_D$, since μ_D divides χ_D. Further, $\chi_f = \operatorname{Det}(X\mathfrak{E} - \mathfrak{B}) = (X-1)^n$, e-Spec $(D) = \mathrm{Z}_{\mathbb{K}}(\chi_f) = \{1\}$ and since $(t+1)^j - t^j = jt^{j-1} + \cdots$,, we have $\deg(f - \mathrm{id})(P) = \deg(P(t+1) - P(t)) = \deg P(t) - 1$ for every non-constant $P \in \mathbb{K}[t]_n$. It follows that the eigenspace $V_f(1) = \operatorname{Ker}(f - \mathrm{id}) = \mathbb{K}$ (= the space of constant polynomials) and since $(f - \mathrm{id})^{n-1}(t^{n-1}) = (n-1)! \neq 0$. Therefore $(f - \mathrm{id})^{n-1} \neq 0$ and hence $\mu_f = (X-1)^n = \chi_f$, since μ_f divides χ_f.)

S11.2 Let D be the differentiation operator $f \mapsto f'$ on the vector space $\mathrm{C}_{\mathbb{K}}^{\infty}(\mathbb{R})$ of infinitely many times differentiable $\mathbb{K}$-valued functions on $\mathbb{R}$. Compute the eigenvalues, spectral-values and eigenspaces for D. (**Ans :** e-Spec $(D) = \operatorname{Spec} D = \mathbb{K}$ and $\mathrm{V}_D(\lambda) = \mathbb{K}e^{\lambda x}$ is the eigenspace of $\lambda \in \mathbb{K}$.

Hint : Every $\lambda \in \mathbb{K}$ is an eigenvalue of D, since the solutions of linear differential equation $f' - \lambda f = 0$ are $ce^{\lambda x}$, $c \in \mathbb{K}$. Therefore, the corresponding eigenspace if 1-dimensional with basis $e^{\lambda x}$. In particular, every $\lambda \in \mathbb{K}$ is also a spectral value of D.)

S11.3 For $k \in \mathbb{N} \cup \{\infty\}$, let S denote the integration operator $f \longmapsto \left(t \mapsto \int_0^t f(\tau)\,d\tau\right)$ on the vector space $\mathrm{C}^k_{\mathbb{K}}(\mathbb{R})$ of the k-times continuously differentiable $\mathbb{K}$-valued functions on $\mathbb{R}$. Then S has no eigenvalue and 0 is the only spectral value S, i. e. e-Spec$(S) = \emptyset$ and Spec $S = \{0\}$. (**Hint :** From $S(f) = 0$, $f \in \mathrm{C}^k_{\mathbb{K}}(\mathbb{R})$, it follows that $f = 0$ by differentiating with respect to the upper limit of the integral. Therefore S is injective and hence 0 is not an eigenvalue of S. The operator S is not surjective, since from $S(f) = g$, $f, g \in \mathrm{C}^k_{\mathbb{K}}(\mathbb{R})$, it follows that $g(0) = \int_0^0 f(\tau)d\tau = 0$, and hence no g with $g(0) \neq 0$ can belong to Im(S). It is more difficult to prove (and need analysis!) to show that $\lambda \in \mathbb{K}^\times$ is neither an eigenvalue of S nor a spectral value of S.)

S11.4 Let $P = X^n + a_{n-1}X^{n-1} + \cdots a_1X + a_0 = (X-\lambda_1)^{r_1}\cdots(X-\lambda_m)^{r_m}$ be a monic polynomial with coefficients $a_0, \ldots, a_{n-1} \in \mathbb{C}$ and pairwise distinct zeros $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$ of multiplicities $r_1, \ldots, r_m > 0$, respectively. Let $V := \{y \in \mathrm{C}^n_{\mathbb{C}}(\mathbb{C}) \mid P(D)y = 0\}$ be the $\mathbb{C}$-vector space of the complex-valued solutions of the homogeneous linear differential equation of n-th order $P(D)y = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots a_1y' + a_0y = 0$. Show that the differentiation $D : V \to V$, $y \mapsto Dy = y'$ is a $\mathbb{C}$-linear operator on V and compute its minimal polynomial, characteristic polynomials, e-Spec D and the eigenspaces. (**Hint :** By construction $V = \mathrm{Ker}\,P(D)$ and $\mathrm{Dim}_{\mathbb{C}}V = r_1 + \cdots + r_m == n = \deg P$. Since $P(D)y = 0$, it follows that $P(D)(Dy) = D(P(D)y) = 0$ and hence D induces an operator on V. Further, since $P(D) = 0$ on V, the minimal polynomial μ_D divides P by the definition of minimal polynomial. Since $V \subseteq \mathrm{Ker}\,\mu_D(D)$, it follows that $\deg P = \mathrm{Dim}_{\mathbb{C}}V \leq \mathrm{Dim}_{\mathbb{C}}\mathrm{Ker}\,\mu_D(D) = \deg \mu_D$ and hence $\mu_D = P$. Moreover, by Cayley-Hamilton Theorem 11.A.7, $\chi_D = \mu_D = P$. The eigenspectrum e-Spec$(D) = \mathrm{Z}(\chi_D) = \{\lambda_1, \ldots, \lambda_m\}$ and the corresponding eigenspaces $\mathrm{V}_D(\lambda_i) = \mathrm{Ker}\,(\lambda_i\mathrm{id} - D) = \mathbb{C}e^{\lambda_i t}$, $i = 1, \ldots, m$, since $y \in \mathrm{Ker}\,(\lambda_i\mathrm{id} - D)$ if and only if y is a solution of the differential equation $y' - \lambda_i y = 0$.)

S11.5 Show that the characteristic polynomial of the diagonal matrix $\mathfrak{D} = \mathrm{Diag}(a_1, \ldots, a_n) \in \mathrm{M}_n(K)$ is $\chi_{\mathfrak{D}} = \prod_{i=1}^n = (X - a_i)$ and the minimal polynomial $\mu_{\mathfrak{D}} = \prod_\rho^r (X - a_{i_\rho})$, where $a_{i_1}, \ldots, a_{i_r}$ are the distinct elements among $a_1, \ldots, a_n$. Further, show that $\mathfrak{D}$ is cyclic (see Exercise 11.8 (d)) if and only if $a_1, \ldots, a_n$ are distinct. Moreover, in this case $x_1 + \cdots + x_n$ is a cyclic vector (see Exercise 11.8 (d)) for every operator $f : V \to V$ whose matrix with respect to a basis $x_1, \ldots, x_n$ of a K-vector space V is $\mathfrak{D}$.

S11.6 Let $\mathfrak{E}_\sigma \in \mathrm{M}_n(K)$ be the matrix of the permutation $\sigma \in \mathfrak{S}_n$, i. e., $\mathfrak{E}_\sigma = \left(\delta_{i\sigma(j)}\right)$. Suppose that $\nu(\sigma) = (\nu_1, \ldots, \nu_n)$ be the cycle type of σ. Then show that the characteristic polynomial and the minimal polynomial of $\mathfrak{E}_\sigma$ are, resp. :

$$\chi_{\mathfrak{E}_\sigma} = \prod_{i=1}^n (X^i - 1)^{\nu_i} \quad \text{and} \quad \mu_{\mathfrak{E}_\sigma} = \mathrm{lcm}\,(X^{i_1} - 1, \ldots, X^{i_r} - 1)\,,$$

where $\mathrm{Supp}\,(\nu(\sigma)) = \{i_1, \ldots, i_r\}$. Moreover, $\mathfrak{E}_\sigma$ is cyclic (see Exercise 11.8 (d)) if and only if σ is a cycle of order n. (**Hint :** Use the following two observations : (1) Note that for non-zero elements $a_1, \ldots, a_n$ in any *unique factorisation domain* A,

$$\mathrm{lcm}(a_1, \ldots, a_n) = \prod_{J \in \mathfrak{P}(\mathbb{N}_n^*),\, J \neq \emptyset} g(J)^{\mathrm{s}(J)} \qquad \text{and} \qquad \gcd(a_1, \ldots, a_n) = \prod_{J \in \mathfrak{P}(\mathbb{N}_n^*),\, J \neq \emptyset} \ell(J)^{\mathrm{s}(J)}\,,$$

where for a subset $J \in \mathfrak{P}(\mathbb{N}_n^*)$, $g(J) := \gcd(a_j \mid j \in J)$, $\ell(J) := \mathrm{lcm}(a_j \mid j \in J)$ and $\mathrm{s}(J) := -(-a)^{|J|}$. (prove these formulae by using p-exponents). (2) For polynomials $X^m - 1, X^n - 1 \in K[X]$, where K is arbitrary field, $\gcd(X^m - 1, X^n - 1) = X^{\gcd(m,n)} - 1$. Prove this by going to the field extension $L|K$ such that both $X^m - 1$ and $X^n - 1$ splits into linear factors in $L[X]$. See also Supplement S11.14.)

S11.7 Let f be an operator on the n-dimensional K-vector space V. Suppose that the degree of the minimal polynomial μ_f is m. Then show that

(a) $\chi_{f+a\,\mathrm{id}}(X) = \chi_f(X - a)$ and $\mu_{f+a\,\mathrm{id}}(X) = \mu_f(X - a)$, $a \in K$.

(b) $\chi_{af}(X) = a^n\chi_f(X/a)$ and $\mu_{af}(X) = a^m\mu_f(X/a)$, $a \in K^\times$.

(c) If f is invertible, then $\chi_{f^{-1}}(X) = \dfrac{(-1)^n}{\mathrm{Det}\,f}X^n\chi_f(1/X)$ and $\mu_{f^{-1}}(X) = \dfrac{1}{\mu_f(0)}X^m\mu_f(1/X)$, Further, deduce that : $f^{-1} = \dfrac{\mu - \mu(0)}{\mu(0)X}(f)$ and that eigenvalue s of f are all non-zero and $0 \neq \lambda \in K$ is an eigenvalue of f if and only if λ^{-1} is an eigenvalue of f^{-1}. (**Remark:** Let $\mathfrak{A} \in \mathrm{M}_n(K)$, $n \in \mathbb{N}^*$, be an invertible matrix with adjoint $\operatorname{Adj}\mathfrak{A} = (\operatorname{Det}\mathfrak{A})^{-1}\mathfrak{A}^{-1}$ see Theorem 9.D.13. If $\chi_{\mathfrak{A}} = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0$ is the characteristic polynomial of $\mathfrak{A}$, then $\operatorname{Det}\mathfrak{A} = (-1)^n a_0$ and $\chi_{\mathfrak{A}}(\mathfrak{A}) = 0$ by Cayley-Hamilton Theorem 11.A.7. It follows that $\operatorname{Adj}\mathfrak{A} = (-1)^n a_0 \mathfrak{A}^{-1} = (-1)^{n-1}(a_1\mathfrak{E}_n + a_2\mathfrak{A} + \cdots + a_{n-1}\mathfrak{A}^{n-2} + \mathfrak{A}^{n-1})$. *Moreover, the equation*

$$\operatorname{Adj}\mathfrak{A} = (-1)^{n-1}\Delta_{\mathfrak{A}}(\mathfrak{A})\,, \qquad \textit{where} \quad \Delta_{\mathfrak{A}} := (\chi_{\mathfrak{A}} - \chi_{\mathfrak{A}}(0))/X\,,$$

also hold for arbitrary $n\times n$-matrices, even if $\mathfrak{A}$ is not invertible. Further, by Supplement S11.?? the coefficient $(-1)^{n-1}a_1$ is the trace of $\operatorname{Adj}\mathfrak{A}$. — For a **Proof** of this equation, we use the Kronecker's method of indeterminates and consider the invertible matrix $\mathfrak{B} := Z\mathfrak{E}_n + \mathfrak{A} \in \mathrm{M}_n(K(Z))$, where Z is an indeterminate over K, with the characteristic polynomial $\chi_{\mathfrak{B}}(X) = \chi_{\mathfrak{A}}(X-Z) = \sum_{k\geq 0} F_k(Z)X^k \in \big(K[Z]\big)[X]$, see Supplement S11.6 (a), and $\Delta_{\mathfrak{B}} = \big(\chi_{\mathfrak{A}}(X-Z) - \chi_{\mathfrak{A}}(-Z)\big)/X = \sum_{k\geq 1} F_k(Z)X^{k-1}$. It is $\Delta_{\mathfrak{A}} = \Delta_{\mathfrak{A}}(X) = \sum_{k\geq 1} F_k(0)X^{k-1}$. By the above proof for the adjoint, we have

$$\operatorname{Adj}\mathfrak{B} = \operatorname{Adj}(Z\mathfrak{E}_n + \mathfrak{A}) = (-1)^{n-1}\Delta_{\mathfrak{B}}(Z\mathfrak{E}_n + \mathfrak{A}) = (-1)^{n-1}\textstyle\sum_{k\geq 1} F_k(Z)(Z\mathfrak{E}_n + \mathfrak{A})^{k-1}.$$

Now, substituting $Z = 0$, we get $\operatorname{Adj}\mathfrak{A} = (-1)^{n-1}\sum_{k\geq 1} F_k(0)\,\mathfrak{A}^{k-1} = (-1)^{n-1}\Delta_{\mathfrak{A}}(\mathfrak{A})$. •

More generally, we put $\Delta_{\mathfrak{A},\lambda} := (\chi_{\mathfrak{A}} - \chi_{\mathfrak{A}}(\lambda))/(X-\lambda) \in K[X]$. Then

$$\operatorname{Adj}(\mathfrak{A} - \lambda\mathfrak{E}_n) = (-1)^{n-1}\Delta_{\mathfrak{A},\lambda}(\mathfrak{A})\,.$$

By Supplement S11.6 (a) we have $\Delta_{\mathfrak{A}-\lambda\mathfrak{E}_n} = (\chi_{\mathfrak{A}}(X+\lambda) - \chi_{\mathfrak{A}}(\lambda))/((X+\lambda)-\lambda) = \Delta_{\mathfrak{A},\lambda}(X+\lambda)$ and it follows that $\operatorname{Adj}(\mathfrak{A} - \lambda\mathfrak{E}_n) = (-1)^{n-1}\Delta_{\mathfrak{A}-\lambda\mathfrak{E}_n}(\mathfrak{A} - \lambda\,\mathfrak{E}_n) = \Delta_{\mathfrak{A},\lambda}(\mathfrak{A})$.

Note that if $\mathfrak{A}$ is not invertible, then the matrices $\operatorname{Adj}\mathfrak{A}$ and hence $\Delta_{\mathfrak{A}}(\mathfrak{A})$ have rank 1 if $\operatorname{Rank}\mathfrak{A} = n-1$, and are equal to 0 if $\operatorname{Rank}\mathfrak{A} < n-1$, see Supplement S9.??. Therefore, if $\operatorname{Dim}\operatorname{Ker}\mathfrak{A} = 1$, then the factor X in the minimal polynomial $\mu_{\mathfrak{A}}$ have the same multiplicity as in the characteristic polynomial $\chi_{\mathfrak{A}}$. If $\operatorname{Dim}\operatorname{Ker}\mathfrak{A} > 1$, then the multiplicity of X in $\mu_{\mathfrak{A}}$ is smaller than that in $\chi_{\mathfrak{A}}$. Correspondingly, we have for $\operatorname{Adj}(\mathfrak{A} - \lambda\mathfrak{E}_n)$ resp. $\Delta_{\mathfrak{A},\lambda}(\mathfrak{A})$ and the multiplicities of the factor $X-\lambda$ in $\mu_{\mathfrak{A}}$ resp. in $\chi_{\mathfrak{A}}$, if λ is an eigenvalue of $\mathfrak{A}$. For example, a nilpotent $n\times n$-matrix has the minimal polynomial is X^n if and only if its rank is $n-1$, see Exercise 11.2.)

S11.8 Let V be a K-vector space and let $f: V \to V$ be a linear operator. Show that

(a) f is a projection if and only if μ_f is a divisor of $X(X-1) = X^2 - X$.

(b) f is an involution if and only if μ_f is a divisor of $(X+1)(X-1) = X^2 - 1$.

(c) For a projection (resp. involution) on a finite dimensional vector space find the characteristic polynomial. (**Hint:** For involutions the case $1+1=0$ in K, i. e., $\operatorname{Char}K = 2$ needs to be treated carefully!. — **Ans:** $\chi_f = (X-a)^r \cdot X^r$ with $r = \operatorname{Rank} f$; in particular, $\operatorname{Tr} f = \operatorname{Rank} f$ (resp. $\chi_f = (X+1)^r(X-1)^{n-r}$ if $\operatorname{Char}K \neq 2$ (since $\frac{1}{2}(\mathrm{id}_V - f)$ is a projection) and $\chi_f = (X-1)^n$ if $\operatorname{Char}K = 2$.))

S11.9 Let $f: V \to V$ be an operator of rank r on the n-dimensional K-vector space V.

(a) χ_f is divisible by X^{n-r}. **(b)** μ_f has degree $\leq r+1$.

(**Hint:** Note that $\operatorname{Ker} f$ is an f-invariant subspace of f of dimension $n-r$ by the Rank-Theorem, $f\,|\operatorname{Ker} f = 0$ and hence $\chi_{f|\_\operatorname{Ker} f} = X^{n-r}$, $\mu_{f|_{\operatorname{Ker} f}} = X$ and $\deg \mu_{\overline{f}} \leq \deg \chi_{\overline{f}} = \operatorname{Dim}_K \overline{V} = \operatorname{Dim}_K V - \operatorname{Dim}_K \operatorname{Ker} f = \operatorname{Rank} f = r$, where $\overline{f}: \overline{V} \to \overline{V}$ is the operator induced by f on the quotient space $\overline{V} := V/\operatorname{Ker} f$. Therefore by 11.A.8 $\chi_f = \chi_{f|_{\operatorname{Ker} f}} \cdot \chi_{\overline{f}} = X^{n-r} \cdot \chi_{\overline{f}}$ and μ_f divides $\mu_{f|_{\operatorname{Ker} f}} \cdot \mu_{\overline{f}} = X \cdot \mu_{\overline{f}}$, in particular, χ_f is divisible by X^{n-r} and $\deg \mu_f \leq r+1$. See also Supplement S11.14. — **Remark:** If $f: V \to V$ is a K-linear operator of rank ≥ 1 and $n = \operatorname{Dim}_K V \geq 2$, then χ_f is of the form $\chi_f = X^n + a_{n-1}X^{n-1} = X^n - (\operatorname{Tr} f)X^{n-1} = X^{n-1}(X - \operatorname{Tr} f)$ and $\operatorname{Tr} f = -a_{n-1}$ is the only $\neq 0$ eigenvalue of f. Then the minimal polynomial $\mu_f = X(X - \operatorname{Tr} f)$, since f is not homothecy. In particular, an operator is a projection onto an 1 dimensional subspace if and only if its rank is 1 and its trace is 1.)

S11.10 **(a)** The characteristic polynomial of the $n \times n$-matrix

$$\mathfrak{A} = \begin{pmatrix} a & b & \cdots & b \\ b & a & \cdots & b \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \cdots & a \end{pmatrix}$$

is $(X+b-a)^{n-1}(X-a-(n-1)b)$. Compute its minimal polynomial. determine the conditions on a and b so that $\mathfrak{A}$ is invertible, moreover, in these cases, compute the inverse of this matrix. (**Hint :** See also Supplement S10.52-(a).)

(b) Let $\mathfrak{A}_0,\ldots,\mathfrak{A}_{n-1}\in \mathrm{M}_m(K)$. The characteristic polynomial of the $mn\times mn$-matrix

$$\mathfrak{B}:=\begin{pmatrix} 0 & 0 & \cdots & 0 & -\mathfrak{A}_0\\ \mathfrak{E}_m & 0 & \cdots & 0 & -\mathfrak{A}_1\\ 0 & \mathfrak{E}_m & \cdots & 0 & -\mathfrak{A}_2\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & \mathfrak{E}_m & -\mathfrak{A}_{n-1}\end{pmatrix}$$

is $\mathrm{Det}\,(X^n\mathfrak{E}_m+X^{n-1}\mathfrak{A}_{n-1}+\cdots+X\mathfrak{A}_1+\mathfrak{A}_0)$.

(c) Let $\mathfrak{A}=\mathrm{Diag}\,(a_1,\ldots,a_n)\in \mathrm{M}_n(K)$ be a diagonal matrix and let $\mathfrak{B}=(b_{ij})\in \mathrm{M}_n(K)$ be a matrix of rank ≤ 1. Then

$$\chi_{\mathfrak{A}-\mathfrak{B}}=\prod_{i=1}^n(X-a_i)+\sum_{j=1}^n b_{jj}\prod_{i\neq j}(X-a_i)\,.$$

If $\mathfrak{A}$ is invertible, then $\mathfrak{A}-\mathfrak{B}$ is invertible if and only if $c:=\sum_{j=1}^n b_{jj}a_j^{-1}\neq 1$. Further, in this case

$$(\mathfrak{A}-\mathfrak{B})^{-1}=\frac{1}{1-c}\big((1-c)\,a_i^{-1}\delta_{ij}+a_i^{-1}b_{ij}a_j^{-1}\big)_{1\leq i,j\leq n}\,.$$

S11.11 Let f be a linear operator on the K-vector space V. In the parts (c) and (d) below assume that $\mathrm{Dim}_K V=n\in\mathbb{N}$. Show that

(a) f is nilpotent if and only if μ_f is a power of X. Deduce that: if f is nilpotent, then $\mathrm{Tr}(f)=0$ and $\mathrm{Det}\,(f)=0$.

(b) f is unipotent, i. e. $f-\mathrm{id}$ is nilpotent if and only if μ_f is a power of $X-1$. Deduce that: if f is nilpotent, then $\mathrm{Tr}(f)=n$ and $\mathrm{Det}\,(f)=1$.

(c) f is nilpotent if and only if $\chi_f=X^n$. (**Hint :** Use Cayley-Hamilton Theorem 11.A.7.)

(d) f is unipotent if and only if $\chi_f=(X-1)^n$.

S11.12 Let $K\subseteq L$ be a field extension and let $\mathfrak{A}\in \mathrm{M}_n(K)\subseteq \mathrm{M}_n(L)$. For the minimal- as well as the characteristic polynomial of $\mathfrak{A}$ are independent if the matrix $\mathfrak{A}$ is considered over K or over L. (**Hint :** For the minimal polynomial use the Supplement S7.36.)

S11.13 Let f and g be two commuting operators on the K-vector space V and assume that the operator g is nilpotent. Then $\chi_{f+g}=\chi_f$ and in particular, $\mathrm{Det}\,(f+g)=\mathrm{Det}\,f$ and $\mathrm{Tr}\,(f+g)=\mathrm{Tr}\,f$. (**Hint:** It is enough to prove the assertion for matrices. First note that the matrix $X\mathfrak{E}_I-\mathfrak{A}$ is invertible in $\mathrm{M}_I(K(X))$. Since $\mathfrak{A}\mathfrak{B}=\mathfrak{B}\mathfrak{A}$ and $\mathfrak{B}$ is nilpotent, $(X\mathfrak{E}_I-\mathfrak{A})^{-1}\mathfrak{B}$ is also nilpotent and hence $\mathrm{Det}\,\big(\mathfrak{E}_I-(X\mathfrak{E}_I-\mathfrak{A})^{-1}\mathfrak{B}\big)=1$ by Supplement S11.11 (b). Therefore from the equality $X\mathfrak{E}_I-(\mathfrak{A}+\mathfrak{B})=(X\mathfrak{E}_I-\mathfrak{A})\,\big(\mathfrak{E}_I-(X\mathfrak{E}_I-\mathfrak{A})^{-1}\mathfrak{B}\big)$, it follows that $\chi_{\mathfrak{A}+\mathfrak{B}}=\chi_{\mathfrak{A}}$.)

S11.14 Suppose that the K-vector space V is the sum of invariant subspaces U and W under the K-linear operator $f\colon V\to V$. Then f is algebraic if and only if $f|U$ and $f|W$ are algebraic. Further, in this case $\mu_f=\mathrm{lcm}\,(\mu_{f|U}\,,\mu_{f|W})$. (**Remark :** See Exercise 10.8-(c) for an application. — **Hint :** Since $\mu_f(f\upharpoonright U)=\mu_f(f)\upharpoonright U=0$ and $\mu_f(f\upharpoonright W)\upharpoonright=\mu_f(f)\upharpoonright W=0$, clearly (by definition of minimal polynomial), $\mu_{f|U}$ and $\mu_{f|W}$ both divide μ_f. On the other hand put $\mu:=\mathrm{lcm}\,(\mu_{f|U}\,,\mu_{f|W})$. Then $\mu(f)\upharpoonright U=\mu(f\upharpoonright U)=0$ and $\mu(f)\upharpoonright W=\mu(f\upharpoonright W)=0$, since μ is a multiple of both $\mu_{f|U}$ and $\mu_{f|W}$. Now, since $V=U+W$, it follows that $\mu(f)=0$. Therefore (by definition of μ_f) μ_f divides μ.)

S11.15 Let $f:V\to V$ be an operator and let μ be the minimal polynomial of the restriction of f on $\mathrm{im}\,f$. Then either μ or $X\cdot\mu$ is the minimal polynomial of f. In particular, an operator f of finite rank r is algebraic and the degree of its minimal polynomial is $\leq r+1$. (Note that for the minimal polynomial μ_f of f, the operator $\mu_f(f)=0$ and hence $\mu_f(f\upharpoonright \mathrm{Im}\,f)=\mu_f(f)\upharpoonright \mathrm{Im}\,f=0$. Therefore $\mu=\mu_{f|\mathrm{Im}\,f}$ divides μ_f. On the other hand $(X\cdot\mu)(f)=f\circ\mu(f)=\mu(f)\circ f=0$, since $\mu(f)\upharpoonright \mathrm{Im}\,f=0$. This proves that μ_f divides $X\cdot\mu$ and hence the only possibilities are either $\mu_f=\mu$ or $\mu_f=X\cdot\mu$.)

S11.16 Let f be an invertible operator on the K-vector space V. Show that $\lambda \in K$ is an eigenvalue (resp. a spectral-value) of f if and only if $1/\lambda$ is an eigenvalue (resp. spectral-value) of f^{-1}, i. e., $\text{e-Spec}\,(f^{-1}) = (\text{e-Spec}\,f)^{-1} := \{\lambda^{-1} \mid \lambda \in \text{e-Spec}\,f\}$ and $\text{Spec}\,(f^{-1}) = (\text{Spec}\,f)^{-1} := \{\lambda^{-1} \mid \lambda \in \text{Spec}\,f\}$.

S11.17 Let f and g be operators on the K-vector space V. Then show that

(a) The non-zero eigenvalue s of fg and gf are same.

(b) The non-zero spectral-values of fg and gf are same. (**Hint:** For $a \in K^\times$, $fg - a\,\text{id}$ is invertible if and only if $gf - a\,\text{id}$ invertible. In this case $(gf - a\,\text{id})^{-1} = a^{-1}\big(g\,(fg - a\,\text{id})^{-1} f - \text{id}\big)$.)

(c) Given an example such that the eigenvalue s (resp. spectral-values) of fg and gf are not same. (**Hint:** Let $f, g : V := K[X] \to V = K[X]$ be the K-linear operators on the K-vector space $V = K[X]$ of polynomials over K (with basis X^n, $n \in \mathbb{N}$) defined by $f(X^n) := X^{n+1}$, $n \in \mathbb{N}$ and $g(X^n) := X^{n-1}$, for $n \geq 1$ and $g(X^0) = g(1) = 0$, i. e. $f := \lambda_X$ is the left multiplication by X and $g(P) := (P - P(0))/X$ for $PinK[X]$. Then 0 is an eigenvalue (and hence a spectral-value) of fg, since $(fg)(1) = f(0) = 0 = 0 \cdot 1$, but 0 is not an eigenvalue (and moreover, not a sspectral-value) of gf, since $0 \cdot \text{id}_V - gf = gf = \text{id}_V$ because $(gf)(X^n) = g(X^{n+1}) = X^n$ for all $n \in \mathbb{N}$.)

S11.18 Let $f : V \to V$ be a K-linear operator on the K-vector space V and let $U \subseteq V$ be an f-invariant subspace of V. Further, let $\overline{f} : V/U \to V/U$ be the operator on V/U induced by f. Then

(a) Show that every eigenvalue of $f|U$ is an eigenvalue of f and every eigenvalue of f is aneigenvalue of $f|U$ or of $\overline{f}$.

(b) The same statement as in the part (a) for the spectral-values, i. e.,

$$\text{Spec}\,f|U \subseteq \text{Spec}\,f \subseteq \text{Spec}\,(f|U) \cup \text{Spec}\,\overline{f}\,.$$

(c) If f is algebraic, then $\text{Spec}\,f = \text{Spec}\,(f|U) \cup \text{Spec}\,\bar{f}$.

S11.19 Let $f : V \to V$ be a K-linear operator and let V be the direct sum of the f-invariant subspaces V_i, $i \in I$. Show that

(a) The set of all eigenvalues of f is the union of the set of all eigenvalue s of $f|V_i$, $i \in I$, i. e.,

$$\text{e-Spec}\,(f) = \bigcup_{i \in I} \text{e-Spec}\,(f|V_i)\,.$$

(b) For the spectral-values the analogous statement as in the part (a) holds, i. e.,

$$\text{Spec}\,f = \bigcup_{i \in I} \text{Spec}\,(f|V_i)\,.$$

(c) Let λ_X denote the multiplication by the indeterminate X on the K-vectors space

(i) $V = K[X]$ of polynomials over K, then $\text{e-Spec}\,(\lambda_X) = \emptyset$ and $\text{Spec}\,(\lambda_X) = K$.
(ii) $V = K(X)$ of rational functions over K, then $\text{e-Spec}\,(\lambda_X) = \text{Spec}\,(\lambda_X) = \emptyset$.
(iii) $V = \{P/Q \in K(X) \mid P, Q \in K[X], Q(0) \neq 0\}$, then $\text{e-Spec}\,(\lambda_X) = \emptyset$ and $\text{Spec}\,(\lambda_X) = \{0\}$.
(iv) $V = K[\![X]\!]$ of formal power series K, then $\text{e-Spec}\,(\lambda_X) = \emptyset$ and $\text{Spec}\,(\lambda_X) = \{0\}$.

S11.20 Let $f : V \to V$ be an operator on the K-vector space V and let $P \in K[X]$ be a non-constant polynomial. Then show that

(a) If λ is an eigenvalue (resp. spectral-value) of f, then $P(\lambda)$ is an eigenvalue (resp. a spectral-value) of $P(f)$, i. e. $P(\text{e-Spec}\,(f)) \subseteq \text{e-Spec}\,P(f)$ and $P(\text{Spec}\,(f)) \subseteq \text{Spec}\,P(f)$. (**Hint:** Let $\lambda \in K$. Then λ is a zero of the polynomial $P(X) - P(\lambda) \in K[X]$ and hence $P(X) - P(\lambda) = (X - \lambda) \cdot Q(X)$ for some $Q \in K[X]$. Therefore $P(\lambda)\text{id}_V - P(f) = (\lambda\text{id}_V - f) \circ Q(f) = Q(f) \circ (\lambda\text{id}_V - f)$ and hence if $(\lambda\text{id}_V - f)$ is not injective (resp. not surjective), then $P(\lambda)\text{id}_V - P(f)$ is not injective (resp. not surjective).)

(b) If K is *algebraically closed*,[1] then every eigenvalue (resp. every spectral-value) of $P(f)$ of the form $P(\lambda)$ with an eigenvalue (resp. a spectral-value) λ of f, i. e., $P(\text{e-Spec}\,(f)) =$

[1] A field K is called an a l g e b r a i c a l l y c l o s e d if every non-constant polynomial $P \in K[X]$ has a zero in K. For example, by the Fundamental Theorem of Algebra (see Footnote 2) the field $\mathbb{C}$ of complex numbers is algebraically closed. But the fields $\mathbb{Q}$, $\mathbb{R}$ and finite fields are not algebraically closed.

e-Spec $P(f)$ and $P(\operatorname{Spec}(f)) \subseteq \operatorname{Spec} P(f)$. **Hint:** Let $\mu \in K$ and let $P(X)-\mu = c(X-\lambda_1)\cdots(X-\lambda_n)$ with $c, \lambda_1, \ldots \lambda_n \in K$ (since K is algebraically closed. Therefore $\mu \operatorname{id}_V - P(f) = (-1)^{n-1}c(\lambda_1 \operatorname{id}_V - f)\circ\cdots\circ(\lambda_n \operatorname{id}_V - f)$ and hence if $\lambda_i \notin$ e-Spec f (resp. $\lambda_i \notin \operatorname{Spec} f$), then $\mu \notin$ e-Spec $P(f)$ (resp. $\mu \notin \operatorname{Spec} P(f)$).)

S11.21 Let f and g be operators on the K-vector space V with $[f,g] := fg - gf = a\operatorname{id}_V$ and let $a \neq 0$ in K. Show that if λ is an eigenvalue of gf with the eigenvector $x \in V$, then $gf(g^n(x)) = (\lambda + na)\,g^n(x)$, $n \in \mathbb{N}$. In particular, if $g^n(x) \neq 0$, then $\lambda + na$ is also an eigenvalue of gf. Moreover, if g is invertible, then $\lambda + na$ is an eigenvalue of gf with the eigenvector $g^n(x)$ for $n \in \mathbb{Z}$. (**Hint:** By the way the relation $fg - gf = a\operatorname{id}_V$ with $a \neq 0$ is possible only in the case of a field characteristic 0 and only if V is either 0 or infinite dimensional. Otherwise, $(\operatorname{Dim} V)\cdot a = \operatorname{Tr}(a\operatorname{id}_V) = \operatorname{Tr}(fg) - \operatorname{Tr}(gf) = 0$ is a contradiction. It follows that there is no finite dimensional subspace $0 \neq U \subseteq V$ which is invariant under both f as well as g. In particular, f and g have no common eigenvectors.)

S11.22 Let $f : V \to V$ be an operator on the K-vector space with the dual operator $f^* : V^* \to V^*$. Then show that

(a) A subspace U of V is f-invariant if and only if U° is f^*-invariant. (**Hint:** Suppose that $f(U) \subseteq U$ and $e \in U^\circ$. Then $e(x) = 0$ for all $x \in U$ and hence $(f^*(e))(x) = e(f(x)) = 0$ for all $x \in U$, since $f(x) \in U$ for all $x \in U$, i. e. $f^*(e) \in U^\circ$. This proves that $f^*(U^\circ) \subseteq U^\circ$. Conversely, suppose that $f^*(U^\circ) \subseteq U^\circ$ and let $x \in U$. For every $e \in U^\circ$, we have $f^*(e) \in U^\circ$ and hence $e(f(x)) = (f^*(e))(x) = 0$. Therefore every $e \in V^*$ which vanish on U also vanish on $f(x)$ and hence $f(x) \in U$ by Theorem 5.G.7. This proves that $f(U) \subseteq U$.)

(b) If a subspace W of V^* is f^*-invariant, then ${}^\circ W$ is f-invariant. If V is finite dimensional, then the converse hold. (**Hint:** Suppose that $f^*(W) \subseteq W$ and let $x \in {}^\circ W$. Then for every $e \in W$, we have $f^*(e) \in W$ and hence $e(f(x)) = (f^*(e))(x) = 0$, since $x \in {}^\circ W$. Therefore $f({}^\circ W) \subseteq {}^\circ W$. Conversely, suppose that V is finite dimensional and $f({}^\circ W) \subseteq {}^\circ W$. Then by Theorem 5.G.10 $({}^\circ W)^\circ = W$ and hence by the part (a) $f^*(W) = f^*(({}^\circ W)^\circ) \subseteq ({}^\circ W)^\circ = W$.)

(c) $\operatorname{Spec} f^* = \operatorname{Spec} f$ and in general e-Spec $f^* \neq$ e-Spec f (Example?).

S11.23 Let V be a n-dimensional vector space over a field K and let $\Delta \in \operatorname{Alt}_K(n,V)$ be an n-alternating linear form $V^n \to K$. For $f \in \operatorname{End}_K(V)$ and $x_1, \ldots, x_n \in V$, show that

$$\operatorname{Tr}(f)\cdot\Delta(x_1,\ldots,x_n) = \sum_{i=1}^{n} \Delta(x_1,\ldots,x_{i-1}, f(x_i), x_{i+1},\ldots,x_n)\,.$$

S11.24 Let $f : V \to V$ be an operator on the finite dimensional K-vector space V and U be an f-invariant subspace of V. Then show that

$$\operatorname{Tr} f = \operatorname{Tr}(f \mid U) + \operatorname{Tr}\overline{f}\,,$$

where $\overline{f}$ is the operator $V/U \to V/U$ induced by f. In particular,

$$\operatorname{Tr} f = \operatorname{Tr}(f \mid \operatorname{Im} f) + \operatorname{Tr}(\overline{f}) \quad \text{with} \quad \overline{f} : V/\operatorname{Ker} f \longrightarrow V/\operatorname{Ker} f\,.$$

(**Hint:** By 11.A.8 we have $\chi_f = \chi_{f|U}\cdot\chi_{\overline{f}}$. – **Remark:** The last equation is used to define trace of an operator of finite rank on not necessary on finite dimensional vector spaces.)

S11.25 Let $f : V \to V$ be an operator on the finite dimensional K-vector space $V \neq 0$. Show that the following statements are equivalent:

(i) χ_f is a prime polynomial in $K[X]$.

(ii) 0 and V are the only f-invariant subspaces of V.

(iii) Every non-zero $x \in V$ is a cyclic vector (see Exercise 10.8-(d)) for f.

(**Hint:** If U is an f-invariant subspace of V with $0 < m := \operatorname{Dim}_K U < \operatorname{Dim}_K V$, then $\chi_f = \chi_{f|U}\cdot\chi_{\overline{f}}$ by 11.A.8 and $\deg \chi_{f|U} = \operatorname{Dim}_K U = m$ and hence $\chi_{f|U}$ is a proper divisor of χ_f, in particular, χ_f cannot be a prime polynomial. Conversely, if χ_f is not a prime polynomial and if P is a proper prime divisor of χ_f, then by 11.A.12 there exists an f-invariant subspace U of V of dimension $\operatorname{Dim}_K U = \deg P < \deg \chi_f = \operatorname{Dim}_K V$.)

S11.26 Let $f : V \to V$ be an operator on the finite dimensional K-vector space V. Show that

(a) If f is cyclic (see Exercise 10.8-(d)) with the characteristic polynomial $\chi := \chi_f$, then V has exactly

$$\prod_{\pi \in \mathbb{P}(K[X])} (\mathrm{v}_\pi(\chi) + 1)$$

f-invariant subspaces and restrictions of f to each one of these subspaces is again a cyclic operator, where $\mathbb{P}(K[X])$ denote the set of all monic prime polynomials in $K[X]$ and v_π denote the π-exponents.

(b) If K is infinite and if V has only finitely many f-invariant subspaces, then f is a cyclic operator. (**Hint :** Use Exercise 2.2.)

S11.27 Let $f:V \to V$ be a cyclic operator (see Exercise 10.8-(d)) on the finite dimensional K-vector space V of dimension n with the cyclic vector $x \in V$. Then the dual operator $f^*: V^* \to V^*$ is also a cyclic operator on the dual space V^* with a cyclic vector $(f^{n-1}(x))^*$, where $(f^{n-1}(x))^*$ belong to the dual basis of V^* with respect to the basis $x, f(x), \ldots, f^{n-1}(x)$ of V.

S11.28 Let $f:V \to V$ be an operator on the finite dimensional K-vector space V.

(a) Let v_i, $i \in I$ be a K-basis of V. Show that $\operatorname{Tr} f = \sum_{i\in I} v_i^*(f(v_i))$. (**Hint :** Let $\mathfrak{M}_{\mathfrak{v}}^{\mathfrak{v}}(f) = (a_{ij})_{(i,j)\in I\times I}$ be the matrix of f with respect to the basis $\mathfrak{v} = \{v_i \mid i \in I\}$, i. e. $f(v_j) = \sum_{i\in I} a_{ij} v_i$. Therefore $v_j^*(f(v_j)) = v_j^*(\sum_{i\in I} a_{ij} v_i) = \sum_{i\in I} a_{ij} v_j^*(v_i) = \sum_{i\in I} a_{ij}\delta_{ij} = a_{jj}$ and $\sum_{j\in I} v_j^*(f(v_j)) = \sum_{j\in I} a_{jj} = \operatorname{Tr}(f)$.)

(b) If Rank $f \le 1$, then show that f is nilpotent if and only if $\operatorname{Tr} f = 0$. (**Hint :** By Test-Exercise T10.9 the characteristic polynomial $\chi_f = X^{n-1}(X - \operatorname{Tr}(f))$.)

S11.29 Let K be a field and let $n \in \mathbb{N}^*$. Then

(a) Show that the c o m m u t a t o r s $[\mathfrak{A}, \mathfrak{B}] := \mathfrak{A}\mathfrak{B} - \mathfrak{B}\mathfrak{A}$, $\mathfrak{A}, \mathfrak{B} \in \mathrm{M}_n(K)$, generate a subspace of codimension 1 in $\mathrm{M}_n(K)$. This subspace is the kernel of the trace function $\operatorname{Tr}: \mathrm{M}_n(K) \to K$.

(b) Show that every K-linear form $h: \mathrm{M}_n(K) \to K$ with $h(\mathfrak{A}\mathfrak{B}) = h(\mathfrak{B}\mathfrak{A})$ for all $\mathfrak{A}, \mathfrak{B} \in \mathrm{M}_n(K)$ is a scalar multiple of the trace function on $\mathrm{M}_n(K)$.

S11.30 Let $n \in \mathbb{N}$ and let K be a field with $k1_K \neq 0$ for $k = 1, \ldots, n$.

(a) For every operator $f:V \to V$ with $\operatorname{Tr} f = 0$ on a n-dimensional K-vector space V, show that there exists a basis $v_1, \ldots, v_n$ of V with $v_i^*(f(v_i)) = 0$, $i = 1, \ldots, n$. (**Hint:** By induction on k show that : there exist linearly independent vectors $v_1, \ldots, v_k$ and a subspace W_k of V such that

$$Kv_1 \oplus \cdots \oplus Kv_k \oplus W_k = V \quad \text{and} \quad f(v_i) \in \sum_{j\neq i} Kv_j + W_k\,.$$

Suppose that $k = 1$. If every element of V is an eigenvector of f, then by Exercise 10.3 f is the homothecy $a\mathrm{id}_V$, $a \in K$ and it follows that $0 = \operatorname{Tr} f = n \cdot a$. Therefore $a = 0$ and $f = 0$, in this case the assertion is trivial. Otherwise, there exists a vector $v_1 \in V$ with $f(v_1) \notin Kv_1$. We extend $v_1, f(v_1)$ to a basis $v_1, f(v_1), w_1, \ldots, w_{n-2}$ of V and take W_1 the subspace of V generated by $f(v_1), w_1, \ldots, w_{n-2}$. With this the required assertion holds.

For the inductive step rom k to $k+1$, consider the map $p \circ f|W_k$, where p projection onto W_k along $\sum_{j=1}^k Kv_j$. Extend $v_1, \ldots, v_k$ to a basis $v_1, \ldots, v_k, w_1, \ldots, w_{n-k}$. Then removing the first k rows and first k columns from the matrix of f with respect to this basis, we obtain the matrix of $p \circ f|W_k$ with respect to the basis $w_1, \ldots, w_{n-k}$. Since the first k digonal elements of the matrix of f are 0 by construction and since $\operatorname{Tr} f = 0$, it follows that $\operatorname{Tr}(p \circ f|W_k) = 0$.

If every non-zero element of W_k is an eigenvector of $p \circ f|W_k$, then by Exercise 10.3 $p \circ f|W_k$ is a homothecy $a \cdot \mathrm{id}_{W_k}$, $a \in K$ and it follows that $0 = \operatorname{Tr}(p \circ f|W_k) = (n-k) \cdot a$ and hence $a = 0$ by hypothesis on K. Therefore $p \circ f|W_k = 0$,i. e. $f(W_k) \subseteq Kv_1 \oplus \cdots \oplus Kv_k$. We can take arbitrary non-zero $v_{k+1} \in W_k$ and W_{k+1} a complement of Kv_{k+1} in W_k.

Otherwise there exists $v_{k+1} \in W_k$ such that $(p \circ f|W_k)(v_{k+1}) \notin Kv_{k+1}$ and so $f(v_{k+1}) \notin Kv_1 \oplus \cdots \oplus Kv_k \oplus Kv_{k+1}$. We extend $v_1, \ldots, v_k, v_{k+1}, f(v_{k+1})$ to a basis $v_1, \ldots, v_k, v_{k+1}, f(v_{k+1}), w_1, \ldots, w_{n-k-1}$ of V and take W_{k+1} the subspace of W_k generated by $f(v_{k+1}), w_1, \ldots, w_{n-k-1}$. With this the required assertion holds.

Now, in the case $k = n$, $W_n = 0$ and hence $v_1, \ldots, v_n$ is a basis of V such that $f(v_j) = \sum_{j\neq i} a_{ij} v_j$, i. e. the diagonal elements of the matrix of f with respect to this basis are all 0.)

(b) Show that every matrix $\mathfrak{A} \in \mathrm{M}_n(K)$ with $\operatorname{Tr}\mathfrak{A} = 0$ is a commutator, i.e. is of the form $[\mathfrak{B}, \mathfrak{C}] = \mathfrak{B}\mathfrak{C} - \mathfrak{C}\mathfrak{B}$. (**Hint :** By part (a) above the matrix $\mathfrak{A}$ is similar to the matrix $\mathfrak{A}'$ whose diagonal entries are all 0, i. e. there exists an invertible matrix $\mathfrak{D} \in \mathrm{M}_n(K)$ such that $\mathfrak{A} = \mathfrak{D}\mathfrak{A}'\mathfrak{D}^{-1}$. It is enough to show that there are matrices $\mathfrak{B}, \mathfrak{C} \in \mathrm{M}_n(K)$ such that $[\mathfrak{B}, \mathfrak{C}] = \mathfrak{A}'$. For, then $\mathfrak{A} = \mathfrak{D}\mathfrak{A}'\mathfrak{D}^{-1} = \mathfrak{D}(\mathfrak{B}\mathfrak{C} - \mathfrak{C}\mathfrak{B})\mathfrak{D}^{-1} = (\mathfrak{D}\mathfrak{B}\mathfrak{D}^{-1})(\mathfrak{D}\mathfrak{C}\mathfrak{D}^{-1}) - (\mathfrak{D}\mathfrak{C}\mathfrak{D}^{-1})(\mathfrak{D}\mathfrak{B}\mathfrak{D}^{-1}) = [\mathfrak{D}\mathfrak{B}\mathfrak{D}^{-1}, \mathfrak{D}\mathfrak{C}\mathfrak{D}^{-1}]$. Therefore, without loss of generality assume that all main-diagonal entries of $\mathfrak{A} = (a_{ij})$ are 0. Since $\#K > n$ by hypothesis on K, there exists distinct elements $b_1, \ldots, b_n \in K$. Then for the diagonal matrix $\mathfrak{B} = \operatorname{Diag}(b_1, \ldots, b_n)$, and an arbitrary matrix $\mathfrak{C} = (c_{ij}) \in \mathrm{M}_n(K)$, we have

$$\begin{pmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \cdot \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix} - \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix} \cdot \begin{pmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$
$$= \begin{pmatrix} b_1c_{11} & b_1c_{12} & \cdots & b_1c_{1n} \\ b_2c_{21} & b_2c_{22} & \cdots & b_2c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_nc_{n1} & b_nc_{n2} & \cdots & b_nc_{nn} \end{pmatrix} - \begin{pmatrix} b_1c_{11} & b_2c_{12} & \cdots & b_nc_{1n} \\ b_1c_{21} & b_2c_{22} & \cdots & b_nc_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_1c_{n1} & b_2c_{n2} & \cdots & b_nc_{nn} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & (b_1-b_2)c_{12} & \cdots & (b_1-b_n)c_{1n} \\ (b_2-b_1)c_{21} & 0 & \cdots & (b_2-b_n)c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ (b_n-b_1)c_{n1} & b_nc_{n2} & \cdots & 0 \end{pmatrix}.$$

Now, one can take $c_{ij} := a_{ij}/(b_i - b_j)$ for $i \neq j$ and $c_{ii} = 0$, so that the equation $[\mathfrak{B}, \mathfrak{C}] = \mathfrak{A}$ holds.)

S11.31 Let V be a finite dimensional K-vector space.

(a) For a projection p of V, show that $\operatorname{Tr} p = \operatorname{Rank} p \,(= (\operatorname{Rank} p) 1_K)$. (**Hint:** Use Test-Exercise T8.9-(a).)

(b) Suppose that $m \cdot 1_K \neq 0$ for $1 \leq m \leq \operatorname{Dim}_K V$. Further, let $p_1, \ldots, p_r$ be projections of V with $p_1 + \cdots + p_r = \mathrm{id}_V$. Further, suppose that either $\operatorname{Char} K = 0$ or $\sum_{i=1}^r \operatorname{Rank} p_i - \operatorname{Dim}_K V < \operatorname{Char} K$, if $\operatorname{Char} K > p$. Then show that $p_i p_j = \delta_{ij} p_i$ for $1 \leq i, j \leq r$ and in particular, V is the direct sum of the subspaces $\operatorname{Im} p_i$, $i = 1, \ldots, r$. (**Hint:** Since $p_1 + \cdots + p_r = \mathrm{id}_V$, we have $\operatorname{Im} p_1 + \cdots + \operatorname{Im} p_r = V$ and hence $\operatorname{Dim}_K V = \operatorname{Tr}(\mathrm{id}_V) = \operatorname{Tr}(p_1) + \cdots + \operatorname{Tr} p_r == \operatorname{Rank} p_1 + \cdots + \operatorname{Rank} p_r$. Therefore by the assumption on the characteristic of K, the equality $\operatorname{Dim}_K V = \operatorname{Rank} p_1 + \cdots + \operatorname{Rank} p_r$ also hold in $\mathbb{N}$ and hence the sum $V = \operatorname{Im}_1 \oplus \cdots \oplus \operatorname{Im} p_r$ is direct. Therefore $\operatorname{Im} p_j \subseteq \operatorname{Ker} p_i$ for all $i \neq j$ and hence $p_i \circ p_j = 0$ for all $i, j, i \neq j$. Further, $p_i \circ p_i = p_i$, since p_i is a projection, for all $i = 1, \ldots, r$.)

(c) Suppose that a finite group G operates on V as the group of K- automorphisms and that $|G| \cdot 1_K \neq 0$ in K. Then show that:
$$\frac{1}{|G|} \sum_{\sigma \in G} \sigma$$
is a projection of V onto $\operatorname{Fix}_G V$ (see also Example 6.E.10) and the equality (in K)
$$\operatorname{Dim}_K \operatorname{Fix}_G V = \frac{1}{|G|} \sum_{\sigma \in G} \operatorname{Tr} \sigma.$$

(**Hint:** For a fixed $\tau \in G$, note that $G = \{\tau\sigma \mid \sigma \in G\}$. Therefore for $p := \frac{1}{\#G} \sum_{\sigma \in G} \sigma$, we have
$$p^2 = \frac{1}{(\#G)^2} \sum_{\sigma \in G} \sigma \sum_{\tau \in G} \tau\sigma = \frac{\#G}{(\#, G)^2} \sum_{\sigma \in G} \sigma = \frac{1}{\#G} \sum_{\sigma \in G} \sigma = p.$$
Therefore p is a projection of V. For a $x \in \operatorname{Fix}_G V$, $\sigma(x) = x$ for all $\sigma \in G$ and hence $p(x) = \frac{1}{\#G} \sum_{\sigma \in G} x = x$. Conversely, for $y = p(x) \in \operatorname{Im} p$, it is immediate that $\tau(y) = \frac{1}{\#G} \sum_{\sigma \in G} \tau\sigma(x) = \frac{1}{\#G} \sum_{\sigma \in G} \sigma(x) = p(x) = y$ for all $\tau \in G$. Therefore $\operatorname{Dim}_K \operatorname{Fix}_G V = \operatorname{Dim}_K \operatorname{Im} p = \operatorname{Rank} p = \operatorname{Tr} p = \frac{1}{\#G} \sum_{\sigma \in G} \operatorname{Tr} \sigma$.)

S11.32 (J a c o b s o n - L e m m a) Let f, g be operators on the n-dimensional K-vector space V with $[f, [f, g]] = 0$. Suppose that $m \cdot 1_K \neq 0$ for $1 \leq m \leq \operatorname{Dim}_K V$. Then $[f, g]$ nilpotent. (**Hint:** The condition $[f, [f, g]] = 0$ is equivalent with $f[f, g] = [f, g]f$ and so f commute with the powers $[f, g]^n$, $n \in \mathbb{N}$. It follows that $[f, g]^n = (fg - gf)[f, g]^{n-1} = fg[f, g]^{n-1} - gf[f, g]^{n-1} = fg[f, g]^{n-1} - g[f, g]^{n-1} f = [f, g[f, g]^{n-1}]$. Now, since $[f, g]^{n-1}$ are also commutators, they have trace 0 and hence $[f, g]$ is nilpotent by Exercise 10.5-(a).)

S11.33 Let $\mathfrak{A}$ be a $n \times n$-matrix over the field K. Suppose that the sum of elements of every row of $\mathfrak{A}$ is equal to $\lambda \in K$. Then show that λ is an eigenvalue of $\mathfrak{A}$ with the eigenvector ${}^t(1, 1, \ldots, 1) \in K^n$. If all the column-sum of $\mathfrak{A}$ are equal to λ, then λ is an eigenvalue of $\mathfrak{A}$. (**Hint:** Clearly, $\mathfrak{A}\,{}^{\mathrm{tr}}(1, \ldots, 1) = {}^{\mathrm{tr}}(\lambda, \ldots, \lambda) = \lambda\,{}^{\mathrm{tr}}(1, \ldots, 1)$, i. e., λ is an eigenvalue of $\mathfrak{A}$. — **Remark:** An eigenvector corresponding to this eigenvalue is, in general, no so easy to give explicitly.)

S11.34 Let $\mathfrak{A} \in \mathrm{M}_{m,n}(K)$ and $\mathfrak{B} \in \mathrm{M}_{n,m}(K)$, $m \geq n$. Show that $\chi_{\mathfrak{A}\mathfrak{B}} = X^{m-n}\chi_{\mathfrak{B}\mathfrak{A}}$. (**Hint:** Fill the matrices $\mathfrak{A}$ and $\mathfrak{B}$ with zeroes to get square $m \times m$-matrices. $(\mathfrak{A}\ 0)\begin{pmatrix}\mathfrak{B}\\0\end{pmatrix} = \mathfrak{A}\mathfrak{B}$ and $\begin{pmatrix}\mathfrak{B}\\0\end{pmatrix}(\mathfrak{A}\ 0) = \begin{pmatrix}\mathfrak{B}\mathfrak{A} & 0\\0 & 0\end{pmatrix}$. Therefore the characteristic polynomial $\chi_{\mathfrak{A}\mathfrak{B}}$ is equal to that of $(\mathfrak{A}\ 0)\begin{pmatrix}\mathfrak{B}\\0\end{pmatrix}$ and hence the characteristic polynomial of $\begin{pmatrix}\mathfrak{B}\\0\end{pmatrix}(\mathfrak{A}\ 0)$ is equal to $\mathrm{Det}\begin{pmatrix}X\mathfrak{E}_n - \mathfrak{B}\mathfrak{A} & 0\\0 & X\mathfrak{E}_{m-n}\end{pmatrix} = X^{m-n}\mathrm{Det}(X\mathfrak{E}_n - \mathfrak{B}\mathfrak{A}) = X^{m-n}\chi_{\mathfrak{b}\mathfrak{A}}$ by Exercise 10.7-(b).)

S11.35 **(a)** Let V be a finite dimensional vector space over a field K and let $f \in \mathrm{End}_K V$. Further, let $\mathrm{L}_f : \mathrm{End}_K V \to \mathrm{End}_K V$, $g \mapsto fg$ (respectively $\mathrm{R}_f : \mathrm{End}_K V \to \mathrm{End}_K V$, $g \mapsto gf$ be the left-translation by f. Show that

$$\chi_{\mathrm{L}(f)} = \chi_{\mathrm{R}(f)} = \big(\chi_f\big)^n\,,\ \mathrm{Tr}\,\mathrm{L}(f) = \mathrm{Tr}\,\mathrm{R}(f) = n\cdot\mathrm{Tr}\,f \text{ and } \mathrm{Det}\,\mathrm{L}(f) = \mathrm{Det}\,\mathrm{R}(f) = (\mathrm{Det}\,f)^n\,.$$

(See also Example 11.A.27).

(b) Show that the characteristic polynomial of a complex number z as an element of the $\mathbb{R}$-algebra $\mathbb{C}$ is $\chi_z = (X-z)(X-\bar{z})$. In particular, $\mathrm{N}_{\mathbb{R}}^{\mathbb{C}} z = z\bar{z} = |z|^2$ and $\mathrm{Tr}_{\mathbb{R}}^{\mathbb{C}} z = z + \bar{z} = 2\,\mathrm{Re}\,z$.

S11.36 Let f be an operator on a finite dimensional K-vector space and let $P \in K[X]$ be a polynomial. Show that $P(f)$ is invertible if and only if P and μ_f (or also P and χ_f) are relatively prime. (**Hint :** Let $Q := \gcd(P, \mu_f)$. If $Q = 1$, then $SP + T\mu_f = 1$ for some polynomials $S, T \in K[X]$ and hence $\mathrm{id} = S(f)P(f) + T(f)\mu_f(f) = S(f)P(f)$, i. e. $P(f)$ is invertible with inverse $S(f)$. Conversely, if $Q \neq 1$, then $\mu_f = R \cdot Q$, $P = P' \cdot Q$ with $R, P' \in K[X]$ and $\deg R < \deg \mu_f$ and hence $R(f) \neq 0$ and $Q(f) \neq 0$, but $0 = \mu_f(f) = R(f) \circ Q(f) = Q(f) \circ R(f)$. Therefore $Q(f)$ is not injective and hence $P(f) = P'(f) \circ Q(f)$ is also not injective. In particular, $P(f)$ is not invertible.)

S11.37 Let K be a field.

(a) Let P and Q be monic polynomials over the field K. Suppose that $\deg P = n$, Q is a divisor of P and moreover that P and Q have the same prime factors in $K[X]$. Then show that on every n-dimensional K-vector space V there exists an operator $f \in \mathrm{End}_K V$ with characteristic polynomial $\chi_f = P$ and minimal polynomial $\mu_f = Q$.

(b) Let S and S' be subsets of K with $S \subseteq S'$. Show that there exists a K-linear operator $f : V \to V$ on a K-vector space V such that $\text{e-Spec}\,f = S$ and $\mathrm{Spec}\,f = S'$. (**Hint :** For each $a \in K$, let $g_a = -\lambda_a$ and $h_a := \lambda_{X-a}$ be operators on the K-vector space $K[\![X]\!]$. Then $\text{e-Spec}\,g_a = \{a\} = \mathrm{Spec}\,g_a$, $\text{e-Spec}\,h_a = \emptyset$ and $\mathrm{Spec}\,h_a = \{a\}$, see Test-Exercise T10.19-(c). Let $g := (\oplus_{a\in S} g_a) : K^{(S)} \to K^{(S)}$ and $h := (\oplus_{a\in S'\setminus S} h_a) : K^{(S'\setminus S)} \to K^{(S'\setminus S)}$ be the direct sum of operators g_a, $a \in S$ and h_a, $a \in S' \setminus S$ respectively. Now it is easy to check that the operator $f := g \oplus h$ have the required properties. See Test-Exercise T10.19 also.)

S11.38 Show that an operator f on a $\mathbb{R}$-vector space has exactly one real eigenvalue if and only if f^2 has an eigenvalue ≥ 0. (**Hint :** $f^2 - a^2\mathrm{id} = (f - a\,\mathrm{id})(f + a\,\mathrm{id})$.)

S11.39 Let f be a $\mathbb{C}$-linear operator on the finite dimensional $\mathbb{C}$-vector space V, which we consider as $\mathbb{R}$-vector space. Then show that f is also $\mathbb{R}$-linear and

$$\chi_{f,\mathbb{R}} = \chi_{f,\mathbb{C}} \cdot \overline{\chi}_{f,\mathbb{C}}\,. \quad \text{(for a polynomial } P = \textstyle\sum a_i X^i \in \mathbb{C}[X]\text{, we put } \overline{P} := \textstyle\sum \overline{a}_i X^i\text{)}$$

Further, for the minimal polynomials show that $\mu_{f,\mathbb{R}} = \mathrm{LCM}\left(\mu_{f,\mathbb{C}}, \overline{\mu}_{f,\mathbb{C}}\right)$.

S11.40 Let $\mathfrak{A} = (a_{ij}) \in \mathrm{M}_n(K)$ be a $n \times n$-matrix over the field K. Then

(a) Let $X_1, \ldots, X_n$ be indeterminates over K. For $1 \leq i_1 < \cdots < i_r \leq n$, show that the coefficient of $X_{i_1} \cdots X_{i_r}$ in the polynomial

$$\begin{vmatrix} a_{11} + X_1 & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} + X_n \end{vmatrix} \in K[X_1, \ldots, X_n]$$

is equal to the diagonal minor of $\mathfrak{A}$ obtained by removing the rows and columns numbered by $i_1, \ldots, i_r$. (**Hint :** Expand the determinant successively using the rows $i_1, \ldots, i_r$.)

(b) For $r=1,\ldots,n$, show that the coefficient a_r of X^r in the characteristic polynomial $\chi_{\mathfrak{A}}$ of $\mathfrak{A}$ is $(-1)^{n-r}$-times the sum of the diagonal minors of the order $n-r$ of $\mathfrak{A}$.

S11.41 Let $K\subseteq L$ be a field extension and let $\mathfrak{A}\in \mathrm{M}_n(K)\subseteq \mathrm{M}_n(L)$ be a matrix with an eigenvalue $\lambda\in L-K$. Then there exists an eigenvector $\mathfrak{x}\neq 0$ in L^n of $\mathfrak{A}$, i. e. $\mathfrak{A}\mathfrak{x}=\lambda\mathfrak{x}$; but there is no eigenvector in K^n, i. e. $K^n\cap \mathrm{Ker}\,(\lambda\,\mathfrak{E}_n-\mathfrak{A})=0$.

S11.42 (Jacobi's Matrix) For $k=0,\ldots,n$, let

$$\mathfrak{D}_k := \begin{pmatrix} a_1 & b_1 & 0 & \cdots & 0 & 0\\ c_1 & a_2 & b_2 & \cdots & 0 & 0\\ 0 & c_2 & a_3 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & a_{k-1} & b_{k-1}\\ 0 & 0 & 0 & \cdots & c_{k-1} & a_k \end{pmatrix} \in \mathrm{M}_k(K)$$

and let $\mathrm{D}_k := \mathrm{Det}\,(\mathfrak{D}_k)$ (see exercise (13.30)). Put $\chi_k := \chi_{\mathfrak{D}_k}$. Show that

(a) $\chi_0=1\,,\chi_1=X-a_1\,,\chi_k=(X-a_k)\chi_{k-1}-b_{k-1}c_{k-1}\chi_{k-2}$ for all $k=2,\ldots,n$.

(b) If $K=\mathbb{R}$ and $b_kc_k>0$ for all $k=1,\ldots,n$, then χ_n has n-distinct real roots and the number of positive roots of χ_n is the number of changes in the sign of the sequence $1,-\mathrm{D}_1,\ldots,(-1)^n\mathrm{D}_n$.

S11.43 Let $\mathfrak{A}=(a_{ij})_{1\le i,j\le n}\in \mathrm{M}_n(K)$. Show that

$$\chi_{\mathfrak{A}} = X^n - s_1X^{n-1}+s_2X^{n-2}-\cdots+(-1)^n s_n\,,$$

where s_k is the sum of $\binom{n}{k}$ minors $\mathrm{Det}\,(\mathfrak{A}(i_1,i_2,\ldots,i_k))$, $1\le i_1<i_2<\cdots<i_k\le n$.

S11.44 Let $a,b,c\in\mathbb{C}$ with $bc\neq 0$ and let

$$\mathfrak{T}_n := \begin{pmatrix} a & b & 0 & \cdots & 0 & 0\\ c & a & b & \cdots & 0 & 0\\ 0 & c & a & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & a & b\\ 0 & 0 & 0 & \cdots & c & a \end{pmatrix} \in \mathrm{M}_k(K)\,,\ \text{for } k=0,\ldots,n.$$

Show that :

(a) $\lambda_k = a+2\sqrt{bc}\,\cos\left(\frac{\pi k}{n+1}\right)$, $k=1,\ldots,n$ are eigenvalues of $\mathfrak{T}_n$.

(b) For $k=1,\ldots,n$, the vector with i-th components $\left(\sqrt{\frac{c}{b}}\right)^{i-1}\sin\left(\frac{\pi k}{n+1}\right)$ $i=1,\ldots,n$, is an eigenvector corresponding to the eigenvalue λ_k. (**Hint :** We may assume that $a=0$. Let $\mu\in\mathbb{C}$ with $\mu^2\neq bc$ and let $T_n(\mu):=\mathrm{Det}\,(\mu\,\mathfrak{E}_n-\mathfrak{T}_n)$. Then show that $T_0(\mu)=1, T_1(\mu)=\mu$ and $T_{k+2}(\mu)=\mu T_{k+1}(\mu)-bcT_k(\mu)$ for all $k\ge 0$. Therefore by Test-Exercise T10.42 $T_n(\mu)=\dfrac{(\mu_1^{n+1}-\mu_2^{n+1})}{(\mu_1-\mu_2)}$ where μ_1 and μ_2 are distinct roots of the quadratic $X^2-\mu X+bc$. Now, determine μ so that $\mu_1^{n+1}=\mu_2^{n+1}$.)

S11.45 Let V be a n-dimensional vector space over a field K and let $f\in \mathrm{End}_K(V)$.

(a) If $\mathrm{Char}(K)=p>0$ then, show that $\chi_{f^p}(X^p)=\left(\chi_f\right)^p$. In particular, $\mathrm{Tr}(f^p)=(\mathrm{Tr}(f))^p$. (**Hint :** For $\mathfrak{A}\in \mathrm{M}_n(A)$ we have $(X\mathfrak{E}_n-\mathfrak{A})^p = X^p\mathfrak{E}_n-\mathfrak{A}^p$. — This is a special case of the following more general exercise in part (b) below.)

(b) For $r\in\mathbb{N}^+$, prove that

$$\chi_{f^r}(X^r) = (-1)^{n(r-1)}\prod_{i=1}^{r}\chi_f(\zeta_i X)\,,$$

where $\zeta_i\,, i=1,\ldots r$ are the r-th roots of unity, i.e. $X^r-1=\prod_{i=1}^r(X-\zeta_i)$. Deduce that $\chi_{f^2}(X^2)=(-1)^n\chi_f(X)\chi_f(-X)$.

S11.46 Let $\mathfrak{A}\in \mathrm{M}_n(K)$ and let $\chi_{\mathfrak{A}}=X^n+a_{n-1}X^{n-1}+\cdots+a_1X+a_0$. Show that

(a) $\operatorname{Adj}(\mathfrak{A}) = (-1)^{n+1}(\mathfrak{A}^{n-1} + a_{n-1}\mathfrak{A}^{n-2} + \cdots + a_1\mathfrak{E}_n)$.

(b) $\chi_{\operatorname{Adj}(\mathfrak{A})} = X^n + (-1)^n \sum_{i=1}^n a_i \,(\operatorname{Det}(\mathfrak{A}))^{i-1} X^{n-i}$, where $a_n := 1$.

S11.47 Let I be a finite indexed set. Let $R := K[X_{ij} \mid i,j \in I]$ (respectively, $Q := K(X_{ij} \mid i,j \in I\})$ be a polynomial algebra (respectively the field of rational functions) over a field K and let $\mathfrak{A} = \big(X_{ij}\big) \in \mathrm{M}_I(Q)$. Then the characteristic polynomial $\chi_{\mathfrak{A}} \in R[X]$ is a prime polynomial in $R[X]$.

S11.48 Let f, g be operators on a finite dimensional K-vector space V such that $\chi_f = \chi_g$. Then show that $\chi_{P(f)} = \chi_{P(g)}$ for every polynomial $P \in K[X]$. (**Hint :** It is enough to show that: if $\mathfrak{A} \in \mathrm{M}_n(K)$ and if $\mathfrak{B}$ is the companion matric of the polynomial $\chi_{\mathfrak{A}}$, then $\chi_{P(\mathfrak{A})} = \chi_{P(\mathfrak{B})}$ for all $P \in K[X]$. For this we may take $R := K[X_{ij}, Y_k \mid i,j \in I, k = 0,\ldots,m]$ (respectively, $Q := K(X_{ij}, Y_k \mid i,j \in I, k = 0,\ldots,m\})$ the polynomial algebra (respectively the field of rational functions) over K, $\mathfrak{A} := (X_{ij}) \in \mathrm{M}_I(Q)$ and $P = Y_0 + Y_1X + \cdots + Y_mX^m$. Now $\mathfrak{A}$ is similar to the companion matrix of $\mathfrak{A}$ by Test-Exercises T10.?? and T10.??.)

S11.49 Let $\mathfrak{A} \in \mathrm{M}_n(K)$. Show that the following equality holds in the field of rational functions $K(X)$ over K:

$$\operatorname{Tr}\left((X\mathfrak{E}_n - \mathfrak{A})\right) = \frac{\chi'_{\mathfrak{A}}}{\chi_{\mathfrak{A}}}, \quad \text{where} \quad \chi'_{\mathfrak{A}} = \frac{d}{dX}\chi_{\mathfrak{A}}\,.$$