MA 312 Commutative Algebra / January-April 2015

(Int PhD. and Ph. D. Programmes)

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Lectures : Monday and Thursday ; 11:00–12:30 **Venue:** MA LH-3 (**if LH-1 is not free**) / LH-1

Midterms : **Quizzes :** (**Wed-Lect**)

Final Examination :

Evaluation Weightage : Quizzes : 10% **Seminar :** 10% **Midterms :** 30% **Final Examination :** 50%

Range of Marks for Grades (Total 100 Marks)						
	Grade S	Grade A	Grade B	Grade C	Grade D	Grade F
Marks-Range	> 90	76–90	61–75	46–60	35–45	< 35

2. Modules[1] and Submodules

2.1 Let A be a ring and let $V\neq 0$ be an A-module. If V, does not have maximal submodules, then V does not have a minimal generating system. (**Hint :** If $x_i, i\in I$ is a minimal generating system for V, then $I\neq\emptyset$. Let $i_0\in I$ and $W:=\sum_{i\in I\setminus\{i_0\}}Ax_i$. Then W is a cofinite submodule of V and hence V has maximal submodules.)

2.2 The $\mathbb{Z}$–module $\mathbb{Q}$ does not have minimal generating system. (**Hint :** In fact the additive group $(\mathbb{Q},+)$ does not have a subgroup of finite index $\neq 1$. This follows from the fact that the group $(\mathbb{Q},+)$ is divisible[2] and hence every quotient group of $(\mathbb{Q},+)$ is also divisible. Further, *If H finitely generated divisible abelian group, then $H=0$.*)

2.3 Let A be an integral domain. If the set of all non-zero ideals in A have a minimal element (with respect to the inclusion). Show that A is a field. In particular, an integral domain such that the set of all ideals is an artinian ordered set (with respect to inclusion), is a field. (An ordered set $(X,\leq)$ is called artinian if every non-empty subset of X has a minimal element. For example finite ordered sets are artinian. An ordered set is well ordered if it is totally ordered and artinian. The prototype of the well ordered set is the set $\mathbb{N}$ of natural numbers with its natural order.)

2.4 Let A be a non-zero ring and let I be an *infinite* indexed set. For every $i\in I$, let e_i be the I-tuple $(\delta_{ij})_{j\in I}\in A^I$ with $\delta_{ij}=1$ for $j=i$ and $\delta_{ij}=0$ for $j\neq i$.

(a) The family e_i, $i\in I$, is a minimal generating system for the left-ideal $A^{(I)}$ in the ring A^I. In particular, $A^{(I)}$ is not finitely generated ideal. (**Remark :** Submodules of finitely generated modules need not be finitely generated!)

(b) There exists a generating system for $A^{(I)}$ as an A^I–module that does not contain any minimal generating system. (**Hint :** First consider the case $I=\mathbb{N}$ and the tuples $e_0+\cdots+e_n$, $n\in\mathbb{N}$.)

†**2.5** Let $K_i, i\in I$, be a family of fields. For $a=(a_i)_{i\in I}\in A:=\prod_{i\in I}K_i$, let $\mathrm{V}(a)\subseteq I$ denote the zero set $\{i\in I\mid a_i=0\}$ of a and $\mathrm{D}(a):=I\smallsetminus \mathrm{V}(a)=\{i\in I\mid a_i\neq 0\}$ its complement. Furthermore, set $\mathfrak{F}(\mathfrak{a}):=\{\mathrm{V}(a)\mid a\in\mathfrak{a}\}\subseteq\mathfrak{P}(I)$ for an ideal $\mathfrak{a}\subseteq A$. Then show that:

[1] The concept of a module seems to have made its first appearance in Algebra in *Algebraic Number Theory*– in studying subsets of *rings of algebraic integers*. Modules first became an important tool in Algebra in late 1920's largely due to the insight of Emmy Noether, who was the first to realize the potential of the module concept. In particular, she observed that this concept could be used to bridge the gap between two important developments in Algebra that had been going on side by side and independently:the theory of representations (=homomorphisms) of finite groups by matrices due to Frobenius, Burnside, Schur et al and the structure theory of algebras due to Molien, Cartan, Wedderburn et al.

[2] **Divisible abelian groups.** An abelian (additively written) group H is divisible if for every $n\in\mathbb{Z}$, the group homomorphism $\lambda_n: H\to H$, defined by $a\mapsto na$ is surjective. For example, the group $(\mathbb{Q},+)$ is divisible, the group $(\mathbb{Z},+)$ and finite groups are not divisible. Further, *quotient of a divisible group is also divisible. Free abelian groups of finite rank are not divisible.*

(a) For $a \in A$, $Aa = Ae_{\mathrm{D}(a)}$ and $\mathfrak{F}(Aa) = \{J \subseteq I \mid \mathrm{V}(a) \subseteq J\}$. ($e_J \in A$ denotes the indicator function of a subset $J \subseteq I$.)

(b) The map $\mathfrak{a} \mapsto \mathfrak{F}(\mathfrak{a})$ is an isomorphism of the lattice[3] of the ideals of A onto the lattice of the filters[4] defined on the set I.

(c) The ideal $\mathfrak{a}$ is maximal if and only if $\mathfrak{F}(\mathfrak{a})$ is an ultra filter[5] on I. Hence, the maximal spectrum $\mathrm{Spm} A$ of A can be identified with the the set of ultra filters on the index set I.

(d) Deduce the following assertion from the Theorem of Krull: If $X \neq \emptyset$, then the set of of filters on X different from $\mathfrak{P}(X)$ is inductively ordered with respect to the inclusion and that every filter on X different from $\mathfrak{P}(X)$ is contained in an ultra-filter on X.

(e) The principal ultra filters $\mathfrak{F}(i) = \{J \subseteq I \mid i \in J\}$, where $i \in I$ is fixed, correspond to the principal maximal ideals $Ae_{I \setminus \{i\}}$, $i \in I$. If I is finite, then these are all maximal ideals, indeed, in this case A is a principal ideal ring.

(f) If I is infinite, then there are non-principal maximal ideals. More precisely, the non-principal maximal ideals of A are exactly the maximal ideals which contain the direct sum ideal $\mathfrak{s} := \bigoplus_{i \in I} K_i \subseteq A$. Which filter is $\mathfrak{F}(\mathfrak{s})$ in case I is infinite? (It is the *Fréchet filter* on I. — For an infinite set X, the complements of the finite subsets of X form a free filter $\neq \mathfrak{P}(X)$ which is also known as *Fréchet filter* on X.)

(g) The sum and the intersection of finitely many principal ideals in A is principal, too. More precisely, $Ae_L + Ae_M = Ae_{L \cup M}$ and $Ae_L \cap Ae_M = Ae_L e_M = Ae_{L \cap M}$ for arbitrary subsets $L, M \subseteq I$.

(h) Every finitely generated ideal in A is a principal ideal.

Below one can see some Supplements / Test-Exercises to the results proved in the class.

[3]**Lattice.** A partially ordered set $(X, \leq)$ is called a *lattice* if for every two elements $x, y \in X$, the supremum $\sup\{x, y\}$ and the infimum $\inf\{x, y\}$ exist. For example, the set of all ideals in a ring form a lattice with respect to the inclusion. What are $\sup\{\mathfrak{a}, \mathfrak{b}\}$ and $\inf\{\mathfrak{a}, \mathfrak{b}\}$ for ideals $\mathfrak{a}, \mathfrak{b}$ in A?

[4]**Filter on a set.** Let X be any set and let $\mathfrak{P}(X)$ denote the power st of X. A *filter* on X is a subset $\mathfrak{F}$ of $\mathfrak{P}(X)$ such that : (1) $\mathfrak{F}$ is closed under finite intersections, i.e. intersection of finitely many elements of $\mathfrak{F}$ belongs to $\mathfrak{F}$. (In particular, the empty intersection $= X \in \mathfrak{F}$). (2) If $Y \in \mathfrak{F}$ and $Y \subseteq Z$, then $Z \in \mathfrak{F}$. Note that $\mathfrak{F} = \mathfrak{P}(X)$ if and only if $\emptyset \in \mathfrak{F}$. A filter $\mathfrak{F}$ on X is called *fixed* if the intersection $\cap_{F \in \mathfrak{F}} F \neq \emptyset$, otherwise it is called *free*. For a subset $A \subset X$, the subset $\mathfrak{F}(A) := \{F \in \mathfrak{P}(X) \mid A \subseteq F\} \subseteq \mathfrak{P}(A)$ is a filter on X called the *principal filter generated* by A.

[5]**Ultra-filters on a set.** The set of filters on a set X is ordered by inclusion and it forms a lattice. Maximal elements in the set of filters on X different from $\mathfrak{P}(X)$ are called *ultra-filters* on X.

Test-Exercises

T2.1 Let V be an A-module and let $a \in A$ be a unit. Then the homothecy $\vartheta_a : V \to V \;\; x \mapsto ax$ is bijective. Give an example of a non-zero A-module and a non-unit $a \in A$ such that the homothecy ϑ_a is bijective. (**Hint :** Consider $\mathbb{Z}$-modules — Finite abelain groups.)

T2.2 Let U, W, U', W' be submodules of an A-module V. Then :

(a) (Modular Law) If $U \subseteq W$, then $W \cap (U + U') = U + (W \cap U')$.

(b) If $U \cap W = U' \cap W'$, then U is the intersection of $U + (W \cap U')$ and $U + (W \cap W')$.

T2.3 Let A be a ring and let V_i, $i \in I$, be an infinite family of non-zero A-modules. Prove that $W := \bigoplus_{i \in I} V_i$ is not a finite A-module.

T2.4 Let K be a field and let A be a subring of K such that every element of K can be expressed as a quotient a/b with $a, b \in A$, $b \neq 0$. (i. e. K is the quotient field of A). If K is a finite A-module, then prove that $A = K$. In particular, $\mathbb{Q}$ is not a finite $\mathbb{Z}$–module. (**Hint :** Suppose $K = Ax_1 + \cdots + Ax_n$ and $b \in A$, $b \neq 0$, with $bx_i \in A$ for $i = 1, \ldots, n$. Now, try to express $1/b^2$ as a linear combination of x_i, $i = 1, \ldots, n$.)

T2.5 Let A be an integral domain with quotient field K. Then :

(a) If V is a torsion module over A, then $\operatorname{Hom}_A(V, A) = 0$.

(b) $\operatorname{Hom}_A(K, A) \neq 0$ if and only if $A = K$. In particular, $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) = 0$. (**Hint :** Every element $f \in \operatorname{Hom}_A(K, A)$ is a homothecy of K by the element $f(1)$.)

(c) If K is finite module, then $A = K$. (**Hint :** See Exercise T2.4. — Moreover, if K is a A-submodule of a arbitrary direct sum of finite A-modules, then $A = K$.)

T2.6 Let K be a field and let V be a K-vector space. Suppose that $V_1, \ldots, V_n$ be distinct K-subspaces of V. If K has at least n elements (in particular, if K is infinite), then $V_1 \cup \cdots \cup V_n \neq V$. (**Hint :** By induction we may assume that $V_n \not\subseteq V_1 \cup \cdots \cup V_{n-1}$. Then there exist an elements $x \in V_n$, $x \notin V_1 \cup \cdots \cup V_{n-1}$ and $y \in V$, $y \notin V_n$. Now, consider the linear combinations $ax + y$, $a \in K$.)